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New type of inequalities involving differentiable h-convexity with applications

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Abstract

In this article, we explore the type of integral inequalities using Hölder inequality together with the class of h-convexity. Also, we present some trapezoid-type inequalities for the class of mappings whose second derivative in absolute value at certain power is h-convex. The applications to special means of some results obtained are equally discussed. In some selected cases, several previous results were generalized.

Keywords: h-Convex function, integral inequality, Hölder inequality, power-mean inequality.

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1. Introduction

Theory of inequalities plays a vital role in many disciplines including applied mathematics, information theory, and control theory. Different inequalities exist in the literature, such as Lyapunov, Hardy, and Minkowski. One can consult the following references for further understanding of inequalities [1, 4, 6, 17, 22], some of which have been widely studied like those of Hölder's type. Due to its wide range of applications, Hölder inequality is considered as the most important tool in both pure and applied sciences. The inequality has been reported to solve many problems in social and natural sciences. Thus, the Hölder inequality for integrals is given as follows.

Theorem 1.1 ([11]). *Suppose that* $p^{-1}+q^{-1}=1$ *and* $p>1$. *If* $\mathfrak{U}$ *and* $\mathfrak{M}$ *are real functions on* $[z,r]$ *and* $|\mathfrak{U}|^p$ *and* $|\mathfrak{M}|^q$ *are integrable functions on* $[z,r]$, *then*

$$\int_z^r |\mathfrak{U}(\eta)\mathfrak{M}(\eta)|\,d\eta \leqslant \left(\int_z^r |\mathfrak{U}(\eta)|^p\,d\eta\right)^{1/p}\left(\int_z^r |\mathfrak{M}(\eta)|^q\,d\eta\right)^{1/q}$$

with equality if and only if $H|\mathfrak{U}(\eta)|^p = E|\mathfrak{M}(\eta)|^q$ *almost everywhere for some constants* H *and* E.

One other version of Hölders inequality is the power mean integral inequality whose applications can also be seen in different branches of mathematical analysis. This inequality is given as follows.

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Theorem 1.2. *Suppose that* $q \geqslant 1$*. If the real functions* $\mathfrak{U}$ *and* $\mathfrak{M}$ *defined on* $[z, r]$ *and* $|\mathfrak{U}|, |\mathfrak{U}||\mathfrak{M}|^q$ *are integrable on* $[z, r]$*, then*

$$\int_z^b |\mathfrak{U}(\eta)\mathfrak{M}(\eta)|d\eta \leqslant \left(\int_z^r |\mathfrak{U}(\eta)|d\eta\right)^{1-\frac{1}{q}} \left(\int_z^r |\mathfrak{U}(\eta)||\mathfrak{M}(\eta)|^q d\eta\right)^{\frac{1}{q}}. \tag{1.1}$$

Using the Hölder's inequality, many scientists generalized and refined different interesting results through the theory of convexity. This theory (also widely studied) has impact on many real-world problems ranging from medicine to engineering. Convexities have equally been reported to have solved problems with constrained control as well as those dealing with estimation. Now we present the definition of classical convexity as follows.

Definition 1.3. A function $\mathfrak{U} : V \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on interval V if

$$\mathfrak{U}(\gamma z + (1-\gamma)r) \leqslant \gamma\mathfrak{U}(z) + (1-\gamma)\mathfrak{U}(r)$$

for all $z, r \in V$ and $\gamma \in [0, 1]$.

Connected with inequalities, convexities have recently undergone through rapid developments due to their wide range of applications in both science and engineering. Approximate solutions of many mathematical problems (whose analytical results could not be easily found) can be determined through the applications of inequalities. A well known inequality for convexity exists in the literature is the Hermite-Hadamard type and is given as follows.

Theorem 1.4. *Let* $\mathfrak{U} : V \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ *be a convex function on* V *and* $z, r \in V$ *with* $z < r$*,*

$$\mathfrak{U}\left(\frac{z+r}{2}\right) \leqslant \frac{1}{r-z}\int_z^r \mathfrak{U}(\eta)d\eta \leqslant \frac{\mathfrak{U}(z)+\mathfrak{U}(r)}{2}. \tag{1.2}$$

Interesting result related to (1.2) was thoroughly studied by Dragomir and Agarwal [8] via differentiable convexity as follows.

Theorem 1.5. *Suppose that* $\mathfrak{U} : V^\circ \subseteq \mathbb{R} \to \mathbb{R}$ *is a differentiable mapping on* $V^\circ, z, r \in V^\circ$*, with* $z < r$*. If* $|\mathfrak{U}'|$ *is convex on* $[z, r]$*, then*

$$\left|\frac{\mathfrak{U}(z)+\mathfrak{U}(r)}{2} - \frac{1}{r-z}\int_z^r \mathfrak{U}(\eta)d\eta\right| \leqslant \frac{(r-z)\left(|\mathfrak{U}'(z)|+|\mathfrak{U}'(r)|\right)}{8} \tag{1.3}$$

holds.

Inequalities (1.2) and (1.3) have been reported in many studies through different types of convexities, one of such examples is the work of Sarkaya et al. [16] which generalized the inequalities of Hadamard type via h-convexity, which is defined as follows.

Definition 1.6 ([18])**.** Suppose that $h : J \to \mathbb{R}$ is a positive mapping, $h \neq 0$, if

$$\mathfrak{U}(\gamma z + (1-\gamma)r) \leqslant h(\gamma)\mathfrak{U}(z) + h(1-\gamma)\mathfrak{U}(r) \tag{1.4}$$

for all $z, r \in V$, $\gamma \in [0, 1]$, then we called $\mathfrak{U} : V \to \mathbb{R}$ h-convex function, or $\mathfrak{U}$ is belonging to the class $SX(h, V)$, if it is non negative.

Reserving inequality (1.4), we say that $\mathfrak{U}$ is h-concave, i.e., $\mathfrak{U} \in SV(h, V)$. Certainly, if $h(\lambda) = \lambda$, all non negative convex functions are belonging to $SX(h, V)$. Meanwhile, all non negative concave functions are belonging to $SV(h, V)$. When $h(\gamma) = \frac{1}{\gamma}$, $SX(h, V) = Q(V)$, if $h(\gamma) = 1$, then $SX(h, V) \supseteq P(V)$. Also, for $h(\gamma) = \gamma^s$, with $s \in (0, 1)$, $SX(h, V) \supseteq K_s^2$. For more recent studies involving h-convex functions, one can consult the following references [5, 9–11, 15, 21]. We now present Sarkaya et al. result (mentioned above) as follows.

Theorem 1.7 ([16]). *Let $\mathfrak{U} \in SX(h, V), z, r \in V$ with $z < r$ and $\mathfrak{U} \in L_1[z, r]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\mathfrak{U}\left(\frac{z+r}{2}\right) \leqslant \frac{1}{r-z}\int_z^r \mathfrak{U}(\eta)d\eta \leqslant [\mathfrak{U}(z)+\mathfrak{U}(r)]\int_0^1 h(\gamma)d\gamma. \tag{1.5}$$

Numerous studies grabbing the attention of many researchers exist in the literature to improve and extend inequalities (1.2), (1.3), and (1.5). See references for quantum integrals [2], generalized fractional integrals [12], strongly log-convexity [7], (h,m)-convexity[13], modified (h,m)-convexity [3], (h,s,m)-convexity [14].

Using a new identity presented in the following lemma, Işcan et al. [9] have recently established new generalized inequalities for convexity.

Lemma 1.8. *Let $\mathfrak{U} : V^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on V°, where $z, r \in V^\circ$, with $z < r$. If $\mathfrak{U}' \in L[z, r]$, then,*

$$\begin{aligned}\Omega_k(\mathfrak{U}, z, r) &= \sum_{\rho=0}^{k-1}\frac{1}{2k}\left[\mathfrak{U}\left(\frac{(k-\rho)z+\rho r}{k}\right)+\mathfrak{U}\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right]-\frac{1}{r-z}\int_z^r\mathfrak{U}(\eta)d\eta \\ &= \sum_{\rho=0}^{k-1}\frac{r-z}{2n^2}\left[\int_0^1(1-2\gamma)\mathfrak{U}'\left(\gamma\frac{(k-\rho)z+\rho r}{k}+(1-\gamma)\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)d\gamma\right].\end{aligned} \tag{1.6}$$

holds.

Yildiz et al. [19] established a new inequalities for s-convexity through identity (1.6).

Theorem 1.9. *Let $\mathfrak{U} : V \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on V°, where $z, r \in V^\circ$, with $z < r$. If $|\mathfrak{U}'|^q$ is s-convex on $[z, r]$ for some fixed $q > 1$, then the following inequality*

$$|\Omega_k(\mathfrak{U}, z, r)| \leqslant \sum_{\rho=0}^{k-1}\frac{r-z}{2k^2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left|\mathfrak{U}'\left(\frac{(k-\rho)z+\rho r}{k}\right)\right|^q+\left|\mathfrak{U}'\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|^q\right]^{\frac{1}{q}}$$

holds, where $\frac{1}{p}+\frac{1}{q}=1$.

In [20], the authors presented the following integral identity including the second-order derivative of $\mathfrak{U}$ to establish some new generalization of (1.2) for exponential type convexity.

Lemma 1.10 ([20]). *Let $\mathfrak{U} : V^\circ \subset \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function on V°, where $z, r \in V^\circ$, with $z < r$ and $k \in \mathbb{N}$. If $\mathfrak{U}'' \in L[z, r]$, then the following holds,*

$$\begin{aligned}\Omega\left(\mathfrak{U}_k, [z, r]\right) &= \sum_{\rho=0}^{k-1}\frac{1}{2k}\left[\mathfrak{U}\left(\frac{(k-\rho)z+\rho r}{k}\right)+\mathfrak{U}\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right]-\frac{1}{r-z}\int_z^r\mathfrak{U}(\eta)d\eta \\ &= \sum_{\rho=0}^{k-1}\frac{(r-z)^2}{2k^3}\left[\int_0^1\gamma(1-\gamma)\mathfrak{U}''\left(\gamma\frac{(k-\rho)z+\rho r}{k}+(1-\gamma)\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)d\gamma\right].\end{aligned} \tag{1.7}$$

Motivated by the above references, we present new generalized inequalities through the class of h-convexity and Hölder's inequality. We obtain some refinements of trapezoid type inequalities for twice-differentiable mappings whose derivatives in absolute value are h-convex. Some special cases for obtaining new and previous results were discussed in this study as well. Some applications to special means were thoroughly studied. Furthermore, the other parts of this study are organized as follows. Section 2 presents new inequalities via h-convexity for differentiable mappings. Applications to special means are discussed in Section 3. Section 4 concludes the findings of this studies.

2. Main results

In this section, we establish a new inequalities for differentiable mappings belonging to the space $SX(h, V)$. Thus, we present results involving first-order derivative as follows.

Theorem 2.1. *Let* $h : \Xi \subset \mathbb{R} \to \mathbb{R}$ *and* $\mathfrak{U} : [z, r] \to \mathbb{R}$ *be positive functions with* $0 \leqslant z < r$ *and* $h^q \in L_1[0, 1]$, $\mathfrak{U} \in L_1[z, r]$. *If* $|\mathfrak{U}'|$ *is an* h*-convex mapping on* $[z, r]$, *then the following equality*

$$|\Omega_k(\mathfrak{U}, z, r)| = \sum_{\rho=0}^{k-1} \frac{r-z}{2k^2}\left[\left(\frac{1}{2}\right)^{\frac{q-1}{q}} \left(\int_0^1 |1-2\gamma| h(\gamma) \left|\mathfrak{U}'\left(\frac{(k-\rho)z+\rho r}{k}\right)\right|^q d\gamma \right.\right.$$
$$\left.\left. + \int_0^1 |1-2\gamma| h(1-\gamma) \left|\mathfrak{U}'\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right]$$

holds, where $p > 1$ *and* $p^{-1} + q^{-1} = 1$.

Proof. Let $q \geqslant 1$. Using identity (1.6) and the power-mean inequality, we get

$$\begin{aligned}|\Omega_k(\mathfrak{U}, z, r)| &\leqslant \sum_{\rho=0}^{k-1} \frac{r-z}{2k^2}\left(\int_0^1 \left|(1-2\gamma)\mathfrak{U}'\left(\gamma\frac{(k-\rho)z+\rho r}{k} + (1-\gamma)\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right| d\gamma\right)\\
&\leqslant \sum_{\rho=0}^{k-1} \frac{r-z}{2k^2}\left(\int_0^1 |1-2\gamma| d\gamma\right)^{1-\frac{1}{q}}\\
&\quad\times \left(\int_0^1 |1-2\gamma| \left|\mathfrak{U}'\left(\gamma\frac{(k-\rho)z+\rho r}{k} + (1-\gamma)\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|^q d\gamma\right)^{\frac{1}{q}}.\end{aligned}$$

From the h-convexity of $|\mathfrak{U}'|^q$, we have

$$\begin{aligned}|\Omega_k(\mathfrak{U}, z, r)| &\leqslant \sum_{\rho=0}^{k-1} \frac{r-z}{2k^2}\left[\int_0^1 |1-2\gamma| d\gamma\right]^{1-\frac{1}{q}} \left[\int_0^1 |1-2\gamma| \left(h(\gamma)\left|\mathfrak{U}'\left(\frac{(k-\rho)z+\rho r}{k}\right)\right|^q\right.\right.\\
&\quad \left.\left. + h(1-\gamma)\cdot\left|\mathfrak{U}'\left(\frac{(k-i-1)z+(\rho+1)r}{k}\right)\right|^q\right) d\gamma\right]^{\frac{1}{q}}\\
&= \sum_{\rho=0}^{k-1} \frac{r-z}{2k^2}\left(\int_0^1 |1-2\gamma| d\gamma\right)^{1-\frac{1}{q}} \left(\int_0^1 |1-2\gamma| h(\gamma)\left|\mathfrak{U}'\left(\frac{(k-\rho)z+\rho r}{k}\right)\right|^q d\gamma\right.\\
&\quad \left. + \int_0^1 |1-2\gamma| h(1-\gamma)\left|\mathfrak{U}'\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|^q d\gamma\right)^{\frac{1}{q}}\\
&= \sum_{\rho=0}^{k-1} \frac{r-z}{2k^2}\left[\left(\frac{1}{2}\right)^{\frac{q-1}{q}} \left(\int_0^1 |1-2\gamma| h(\gamma). \left|\mathfrak{U}'\left(\frac{(k-\rho)z+\rho r}{k}\right)\right|^q d\gamma\right.\right.\\
&\quad \left.\left. + \int_0^1 |1-2\gamma| h(1-\gamma)\left|\mathfrak{U}'\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right].\end{aligned}$$

□

From Theorem 2.1, we present the following corollaries for $h(\gamma) = \gamma$ and $h(\gamma) = \gamma^s$.

Corollary 2.2. *In Theorem 2.1, we apply* $h(\gamma)=\gamma$ *to get*

$$|\Omega_k(\mathfrak{U},z,r)| = \sum_{\rho=0}^{k-1}\frac{r-z}{k^2(2)^{2+\frac{1}{q}}}\left(\left|\mathfrak{U}'\left(\frac{(k-\rho)z+\rho r}{z}\right)\right|^q+\left|\mathfrak{U}'\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|^q\right)^{\frac{1}{q}},$$

which has been proved by İşcan et al. in [9].

Corollary 2.3. *We set* $h(\gamma)=\gamma^s$ *in Theorem 2.1 to have the following*

$$\begin{aligned}|\Omega_k(\mathfrak{U},z,r)| \leqslant \sum_{\varepsilon=0}^{k-1}\frac{r-z}{k^2 2^{2-\frac{1}{q}}}&\left(\frac{1}{2^s(s+1)(s+2)}+\frac{s}{(s+1)(s+2)}\right)^{\frac{1}{q}}\\ &\times\left[\left|\mathfrak{U}'\left(\frac{(k-\varepsilon)z+\varepsilon r}{k}\right)\right|^q+\left|\mathfrak{U}'\left(\frac{(k-\varepsilon-1)z+(\varepsilon+1)r}{k}\right)\right|^q\right]^{\frac{1}{q}},\end{aligned}$$

which has been established by Yildiz et al. in [19].

We now present a new generalized inequality for first-order derivative by applying the Hölder inequality.

Theorem 2.4. *Let* $h:\Xi\subset\mathbb{R}\to\mathbb{R}$ *and* $\mathfrak{U}:[z,r]\to\mathbb{R}$ *be positive functions with* $0\leqslant z<b$ *and* $h^q\in L_1[0,1]$, $\mathfrak{U}\in L_1[z,r]$. *If* $|\mathfrak{U}'|$ *is an* h*-convex mapping on* $[z,r]$, *then the following inequality*

$$\begin{aligned}|\Omega_k(\mathfrak{U},z,r)| \leqslant \sum_{\rho=0}^{k-1}\frac{r-z}{2k^2}\Bigg[\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\Bigg(\int_0^1\Bigg(h(\gamma)&\left|\mathfrak{U}'\left(\frac{(k-\rho)z+\rho r}{k}\right)\right|^q\\ &+h(1-\gamma)\left|\mathfrak{U}'\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|^q\Bigg)d\gamma\Bigg)^{\frac{1}{q}}\Bigg]\end{aligned}\tag{2.1}$$

holds, where $q^{-1}+p^{-1}=1$.

Proof. Suppose that $p>1$. Applying Lemma 1.8 and the Hölder inequality, we have

$$\begin{aligned}|\Omega_k(\mathfrak{U},z,r)| &\leqslant \sum_{\rho=0}^{k-1}\frac{r-z}{2k^2}\left[\left(\int_0^1\left|(1-2\gamma)\mathfrak{U}'\left(t\frac{(k-\rho)z+\rho r}{k}+(1-\gamma)\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|d\gamma\right)\right]\\ &\leqslant \sum_{\rho=0}^{k-1}\frac{r-z}{2k^2}\Bigg[\left(\int_0^1|1-2\gamma|^p d\gamma\right)^{\frac{1}{p}}\\ &\quad\times\left(\int_0^1\left|\mathfrak{U}'\left(\gamma\frac{(k-\rho)z+\rho b}{k}+(1-\gamma)\frac{(k-\rho-1)z+(\rho+1)b}{k}\right)\right|^q d\gamma\right)^{\frac{1}{q}}\Bigg].\end{aligned}$$

Since $|\mathfrak{U}'|^q$ is h-convex, then we observe that

$$\begin{aligned}&\int_0^1\left|\mathfrak{U}'\left(\gamma\frac{(k-\rho)z+\rho r}{k}+(1-\gamma)\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|d\gamma\\ &\quad\leqslant\int_0^1\left(h(\gamma)\left|\mathfrak{U}'\left(\frac{(k-\rho)z+\rho r}{k}\right)\right|^q+h(1-\gamma)\left|\mathfrak{U}'\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|\right)d\gamma.\end{aligned}$$

Therefore, we deduce

$$|\Omega_k(\mathfrak{U},z,r)| \leqslant \sum_{\rho=0}^{k-1}\frac{r-z}{2k^2}\left[\left(\int_0^1|1-2\gamma|^p d\gamma\right)^{\frac{1}{p}}\left(\int_0^1\left(h(\gamma)\left|\mathfrak{U}'\left(\frac{(k-\rho)z+\rho r}{k}\right)\right|^q\right.\right.\right.$$
$$\left.\left.\left.+h(1-\gamma)\left|\mathfrak{U}'\left(\frac{(k-\rho-1)z+(i+1)r}{k}\right)\right|^q\right)d\gamma\right)^{\frac{1}{q}}\right]$$
$$\leqslant \sum_{\rho=0}^{k-1}\frac{r-z}{2k^2}\left[\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left(\int_0^1\left(h(\gamma)\left|\mathfrak{U}'\left(\frac{(k-\rho)z+\rho r}{k}\right)\right|^q\right.\right.\right.$$
$$\left.\left.\left.+h(1-\gamma)\left|\mathfrak{U}'\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|^q\right)d\gamma\right)^{\frac{1}{q}}\right]. \qquad \square$$

Remark 2.5. Setting $h(\gamma)=\gamma^s$ in Theorem 2.4, we have Theorem 6 in [19].

Remark 2.6. As special cases of Theorem 2.1 and Theorem 2.4, one can get more interesting inequalities. For example, choosing $h(\gamma)=\frac{1}{\gamma}$ and $h(\gamma)=1$, we get inequalities for Godunova-Levin functions and p convexity, respectively.

Meanwhile, for twice-differentiable h-convexity, we begin by presenting the following theorem.

Theorem 2.7. *Let* $h:\Xi\subset\mathbb{R}\to\mathbb{R}$ *and* $\mathfrak{U}:V\to\mathbb{R}$ *be a continuously differentiable function on* $V^\circ, z, r\in V^\circ$ *with* $z<r$, *and* $\mathfrak{U}''\in L[z,r]$. *If* $|\mathfrak{U}''|$ *is an* h*-convex function on* $[z,r]$, *then the following inequality*

$$|\Omega(\mathfrak{U}_k,z,r)| \leqslant \sum_{\rho=0}^{k-1}\frac{(r-z)^2}{2k^3}\left[\left|\mathfrak{U}''\left(\frac{(k-\rho)z+\rho r}{k}\right)\right|\int_0^1\left(\gamma-\gamma^2\right)h(\gamma)\,d\gamma\right.$$
$$\left.+\left|\mathfrak{U}''\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|\int_0^1\left(\gamma-\gamma^2\right)h(1-\gamma)\,d\gamma\right]$$

holds.

Proof. From identity (1.7) and h-convexity of $|\mathfrak{U}''|$, we get

$$|\Omega(\mathfrak{U}_k,[z,r])| \leqslant \sum_{\rho=0}^{k-1}\frac{(r-z)^2}{2k^3}\left[\int_0^1\left|\gamma-\gamma^2\right|\left|\mathfrak{U}''\left(\gamma\frac{(k-\rho)z+\rho r}{k}+(1-\gamma)\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|d\gamma\right]$$
$$\leqslant \sum_{\rho=0}^{k-1}\frac{(r-z)^2}{2k^3}\int_0^1\left|\gamma-\gamma^2\right|\left\{h(\gamma)\left|\mathfrak{U}''\left(\frac{(k-\rho)z+\rho r}{k}\right)\right|\right.$$
$$\left.+h(1-\gamma)\left|\mathfrak{U}''\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|\right\}d\gamma$$
$$=\sum_{i=0}^{k-1}\frac{(r-z)^2}{2k^3}\left[\left|\mathfrak{U}''\left(\frac{(k-\rho)z+\rho r}{k}\right)\right|\int_0^1\left(\gamma-\gamma^2\right)h(\gamma)\,d\gamma\right.$$
$$\left.+\left|\mathfrak{U}''\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|\int_0^1\left(\gamma-\gamma^2\right)h(1-\gamma)\,d\gamma\right]. \qquad \square$$

Theorem 2.8. *Let* $h:\Xi\subset\mathbb{R}\to\mathbb{R}$ *and* $\mathfrak{U}:I\to\mathbb{R}$ *be a continuously differentiable function on* $V^\circ, z, r\in V^\circ$ *with*

$z < r$, and $\mathfrak{U}'' \in L[z,r]$. If $|\mathfrak{U}''|$ is an h-convex function on $[z,r]$, then the following inequality

$$|\Omega(\mathfrak{U}_k, z, r)| \sum_{\rho=0}^{k-1} \frac{(r-z)^2}{2k^3} \beta^{\frac{1}{p}}(p+1, p+1) \times \left\{ \left| \mathfrak{U}''\left(\frac{(k-\rho)z+\rho r}{k}\right) \right| \int_0^1 h(\gamma)\, d\gamma + \left| \mathfrak{U}''\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right) \right| \int_0^1 h(1-\gamma)\, d\gamma \right\}^{\frac{1}{q}}. \tag{2.2}$$

holds, where $\beta^{\frac{1}{p}}(p+1,p+1)$ is beta function.

Proof. From Lemma 1.10 and Hölder's integral inequality, we get

$$\begin{aligned} |\Omega(\mathfrak{U}_k, z, r)| &\leqslant \sum_{\rho=0}^{k-1} \frac{(r-z)^2}{2k^3} \left[\int_0^1 \left|\gamma-\gamma^2\right| \left| \mathfrak{U}''\left(\gamma \frac{(k-\rho)z+\rho r}{k} + (1-\gamma)\frac{(k-\rho-1)z+(\rho+1)r}{k}\right) \right| d\gamma \right] \\ &\leqslant \sum_{\rho=0}^{k-1} \frac{(r-z)^2}{2k^3} \left\{ \int_0^1 \left|\gamma-\gamma^2\right|^p d\gamma \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \int_0^1 \left| \mathfrak{U}''\left(\gamma \frac{(k-\rho)z+\rho r}{k} + (1-\gamma)\frac{(k-\rho-1)z+(\rho+1)r}{k}\right) \right|^q d\gamma \right\}^{\frac{1}{q}}. \end{aligned} \tag{2.3}$$

Applying h-convexity on inequality (2.3), we get

$$\begin{aligned} |\Omega(\mathfrak{U}_k, z, r)| &\leqslant \sum_{\rho=0}^{k-1} \frac{(r-z)^2}{2k^3} \left\{ \int_0^1 \gamma^p (1-\gamma)^p d\gamma \right\}^{\frac{1}{p}} \left\{ \int_0^1 \left[h(\gamma) \left| \mathfrak{U}''\left(\frac{(k-\rho)z+\rho r}{k}\right) \right|^q \right.\right. \\ &\quad \left.\left. + h(1-\gamma) \left| \mathfrak{U}''\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right) \right|^q \right] d\gamma \right\}^{\frac{1}{q}} \\ &= \sum_{\rho=0}^{k-1} \frac{(r-z)^2}{2k^3} \beta^{\frac{1}{p}}(p+1,p+1) \left\{ \left| \mathfrak{U}''\left(\frac{(k-\rho)z+\rho r}{k}\right) \right| \int_0^1 h(\gamma)\, d\gamma \right. \\ &\quad \left. + \left| \mathfrak{U}''\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right) \right| \int_0^1 h(1-\gamma)\, d\gamma \right\}^{\frac{1}{q}}. \end{aligned}$$

□

Theorem 2.9. *Let $h : \Xi \subset \mathbb{R} \to \mathbb{R}$ and $\mathfrak{U} : V \to \mathbb{R}$ be a continuously differentiable function on V°, $z, r \in V^\circ$ with $z < r$, and $\mathfrak{U}'' \in L[z,r]$. If $|\mathfrak{U}''|$ is an h-convex function on $[z,r]$, then the following inequality holds,*

$$\begin{aligned} |\Omega(\mathfrak{U}_k, z, r)| &\leqslant \sum_{\rho=0}^{k-1} \frac{(r-z)^2}{2k^3} \left(\frac{1}{6}\right)^{1-\frac{1}{q}} \left(\left| \mathfrak{U}''\left(\frac{(k-\rho)z+\rho r}{k}\right) \right|^q \int_0^1 \left(\gamma-\gamma^2\right) h(\gamma)\, d\gamma \right. \\ &\quad \left. + \left| \mathfrak{U}''\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right) \right|^q \int_0^1 \left(\gamma-\gamma^2\right) h(1-\gamma)\, d\gamma \right)^{\frac{1}{q}}. \end{aligned}$$

Proof. Using Lemma 1.10, identity (1.1), and the property of the h-convexity of $|\mathfrak{U}''|^q$, we obtain

$$|\Omega(\mathfrak{U}_k, z, r)| \leqslant \sum_{\rho=0}^{k-1} \frac{(r-z)^2}{2k^3} \left[\int_0^1 \left|\gamma-\gamma^2\right| \left| \mathfrak{U}''\left(\gamma \frac{(k-\rho)z+\rho r}{k} + (1-\gamma)\frac{(k-\rho-1)z+(\rho+1)r}{k}\right) \right| d\gamma \right]$$

$$\leqslant \sum_{\rho=0}^{k-1} \frac{(r-z)^2}{2k^3} \left\{ \int_0^1 \left|\gamma-\gamma^2\right| d\gamma \right\}^{1-\frac{1}{q}}$$

$$\times \left\{ \int_0^1 \left|\gamma-\gamma^2\right| \left|\mathfrak{U}''\left(\gamma\frac{(k-\rho)z+\rho r}{k} + (1-\gamma)\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|^q d\gamma \right\}^{\frac{1}{q}}$$

$$\leqslant \sum_{\rho=0}^{k-1} \frac{(r-z)^2}{2n^3} \left\{ \int_0^1 \left(\gamma-\gamma^2\right) d\gamma \right\}^{1-\frac{1}{q}} \left\{ \int_0^1 \left(\gamma-\gamma^2\right) \left[h(\gamma) \left|\mathfrak{U}''\left(\frac{(k-\rho)z+\rho r}{k}\right)\right|^q \right.\right.$$

$$\left.\left. + h(1-\gamma) \left|\mathfrak{U}''\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|^q \right] d\gamma \right\}^{\frac{1}{q}}$$

$$= \sum_{\rho=0}^{k-1} \frac{(r-z)^2}{2n^3} \left(\frac{1}{6}\right)^{1-\frac{1}{q}} \left(\left|\mathfrak{U}''\left(\frac{(k-\rho)z+\rho r}{k}\right)\right|^q \int_0^1 \left(\gamma-\gamma^2\right) h(\gamma)\, d\gamma \right.$$

$$\left. + \left|\mathfrak{U}''\left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)\right|^q \int_0^1 \left(\gamma-\gamma^2\right) h(1-\gamma)\, d\gamma \right)^{\frac{1}{q}}. \qquad \square$$

3. Applications to special means

In this section, some special means can be applied to our results presented in Section 2. Let $z, r \in \mathbb{R}$,

1. the arithmetic mean is defined by $A = A(z,r) := \frac{z+r}{2}, z, r \geqslant 0$;
2. the harmonic mean is defined by $H = H(z,r) := \frac{2zr}{z+r}, z, r > 0$;
3. the logarithmic mean is defined by $L = L(z,r) := \begin{cases} a, & \text{if } z = r, \\ \frac{r-z}{\ln r - \ln z}, & \text{if } z \neq r, \end{cases} \quad z, r > 0.$;
4. the p-logarithmic mean is defined as

$$L_p = L_p(z,r) := \begin{cases} z, & \text{if } z = r, \\ \left[\frac{r^{p+1}-z^{p+1}}{(p+1)(r-z)}\right]^{\frac{1}{p}}, & \text{if } z \neq r, \end{cases} \quad p \in \mathbb{R}\backslash\{-1,0\}, z, r > 0.$$

Proposition 3.1. *Let $z, r \in \mathbb{R}, 0 < z < r$, and $m \in \mathbb{N}, m \geqslant 2$. Then,*

$$\left| \sum_{\rho=0}^{k-1} \frac{1}{k\rho} A\left(\left(\frac{(k-\rho)z+\rho r}{k}\right)^m, \left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)^m \right) - L_m^m(z,r) \right|$$

$$\leqslant \sum_{\rho=0}^{k-1} \frac{(r-z)m}{2^{2-\frac{1}{q}}k^2} \left[\left(\int_0^1 |1-2\gamma| h(\gamma) d\gamma \right) \left(\frac{(k-\rho)z+\rho r}{k}\right)^{(m-1)q} \right.$$

$$\left. + \left(\int_0^1 |1-2\gamma| h(1-\gamma) d\gamma \right) \left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)^{(m-1)q} \right) \right]^{\frac{1}{q}}$$

holds, for all $q \geqslant 1$.

Proof. The proof of this proposition follows from (2.1) in Theorem 2.1 with $\mathfrak{U}(\eta) = \eta^m, \eta \in [z,r], m \in \mathbb{N}, m \geqslant 2$. $\square$

Proposition 3.2. *Let $z, r \in \mathbb{R}, 0 < z < r$, and $m \in \mathbb{N}, m \geqslant 2$. Then, the following*

$$\left| \sum_{\rho=0}^{k-1} \frac{1}{k} A\left(\left(\frac{(k-\rho)z+\rho r}{k}\right)^m, \left(\frac{(k-\rho-1)z+(\rho+1)r}{k}\right)^m \right) - L_m^m(z,r) \right|$$

$$\leqslant \sum_{\rho=0}^{k-1} \frac{(r-z)m}{2^{2-\frac{1}{q}}k^2} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left[\left(\int_0^1 h(\gamma) d\gamma \right) \left(\frac{(k-\rho)z+\rho r}{k} \right)^{(m-1)q} + \left(\int_0^1 h(1-\gamma) d\gamma \right) \left(\frac{(k-\rho-1)z+(\rho+1)r}{k} \right)^{(m-1)q} \right]^{\frac{1}{q}}$$

holds, for all $q > 1$.

Proof. The proof of this proposition follows from (2.1) in Theorem 2.4 with $\mathfrak{U}(\eta) = \eta^m, \eta \in [z, r], m \in \mathbb{N}, m \geqslant 2$. □

Proposition 3.3. *Let* $z, r \in \mathbb{R}, 0 < z < r$, *and* $\gamma \in \mathbb{N}, \gamma \geqslant 3$. *Then,*

$$\left| \sum_{\rho=0}^{k-1} \frac{1}{k} A \left(\left(\frac{(k-\rho)z+\rho r}{k} \right)^{\gamma}, \left(\frac{(k-\rho-1)z+(\rho+1)r}{k} \right)^{\gamma} \right) - L_{\gamma}^{\gamma}(z,r) \right|$$
$$\leqslant \sum_{\rho=0}^{k-1} \frac{(r-z)^2}{2k^3} \left[\int_0^1 \left(\gamma-\gamma^2\right) h\left(\gamma\right) d\gamma \left(\frac{(k-\rho)z+\rho r}{k} \right)^{\gamma-2} + \int_0^1 \left(\gamma-\gamma^2\right) h\left(1-\gamma\right) d\gamma \left(\frac{(k-\rho-1)z+(\rho+1)r}{k} \right)^{\gamma-2} \right].$$

holds.

Proof. The proof of this proposition follows from Theorem 2.7 with $\mathfrak{U}(\eta) = \eta^{\gamma}, \eta \in [z, r], \eta \in \mathbb{N}, \eta \geqslant 3$. □

4. Conclusion

Important results are often reported through inequalities and convexities, two of which are correlated. Thus, h-convexity and Hölder's inequality (plying important rules in optimization theory) are used here to establish new integral inequalities. This study generalizes inequalities applicable to the class of h-convexity of first and second order derivatives. Certain applications of our results are equality reported in this study. The findings of this study could generalize interesting and new results for inequalities through variant types of convexities.

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