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Existence and uniqueness theorems for nonlinear coupled boundary value problem of the $\mathcal{ABC}$ fractional differential equation

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Abstract

In this paper, we study existence and uniqueness of solutions for a new systems of nonlinear coupled boundary value problem of the $\mathcal{ABC}$ fractional differential equation. By applying the Banach's fixed point theorem and Krasnoselskii's fixed point theorem, the existence of solutions is obtained. Further, the provided problem Ulam-Hyers ($\mathcal{UH}$) and generalized Ulam-Hyers ($\mathcal{GUH}$) stability are both investigated. The result obtained in this work are well illustrated with the aid of examples.

Keywords: $\mathcal{ABC}$ fractional differential equation, boundary value problem, existence and uniqueness, $\mathcal{UH}$ stability, generalized $\mathcal{UH}$ stability, fixed point theorem.

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1. Introduction

Fractional calculus is widely used in research and industry to represent complex processes [5, 11, 24, 27, 31, 33, 35, 36]. Academics are exploring unique fractional derivatives ($\mathcal{FD}$s) with singular or nonsingular kernels to model real-world scenarios across scientific and engineering disciplines. New fractional operators have shown to be highly successful tools for professionals and academics due to their contributions to physical phenomena and real-world applications. Prior to 2015, all fractional derivatives were represented as solitary kernels. It is difficult to simulate physical occurrences that contain singularities.

In 2015, Caputo and Fabrizio [20] identified a new $\mathcal{FD}$ in the exponential kernel. Losada and Nieto discussed the properties of this unique kind in [34]. In [15], Atangana and Baleanu ($\mathcal{AB}$) investigated Mittag-Leffler kernels to create a novel and intriguing FD. Abdeljawad expanded the type investigated by AB in [1], resulting in related sequencing integral operators spanning from $(0, 1)$ to more random orders. In addition, he investigated the existence and uniqueness theorems for the Riemann-type ($\mathcal{ABR}$) and Caputo-type ($\mathcal{ABC}$) FDs, additionally higher-order initial value issues. The distinct types of new

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operators was explored by Abdeljawad and Baleanu [2, 3]. We propose reading articles [8, 16, 32] for theoretical research on AB fractional differential equations ($\mathcal{FDE}$s).

Baleanu et al. [18] solved four fractional integro differential equations ($\mathcal{FIDE}$s), with numerical solutions reported in [12–14, 22] under appropriate conditions. Under some situations, the solution set for the second FIDE appears to be unlimited in length. The Caputo-Fabrizio fractional derivative [23] was utilized by Aydogan et al. [17] to create the CFD and DCF, two new high-order derivatives. They looked into two high-order FIDEs' solutions.

Guoa et al. [25, 26] used semigroups and Mönch fixed-point techniques to study the Hyers-Ulam ($\mathcal{HU}$) stability of FDEs in fractional and inclination are used in nonlocal contexts. Researchers have studied the $\mathcal{UH}$ stability of solutions for $\mathcal{FDE}$s with starting or boundary conditions in many publications [6, 7, 9, 10, 28, 29].

Abdo et al. [4] studied $\mathcal{ABC}$-type pantograph $\mathcal{FDE}$s with nonlocal circumstances. The following systems of coupled nonlinear Atangana Baleanu type fractional differential equations were explored by Hammad et al. [30] as follows:

$$\begin{cases} {}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\vartheta(\pi) = \mathfrak{g}(\pi, \vartheta(\pi), \kappa(\pi)), \ \pi \in [\wp, \ell], \delta \in (2,3], \\ {}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\kappa(\pi) = \mathfrak{g}(\pi, \kappa(\pi), \vartheta(\pi)), \ \pi \in [\wp, \ell], \delta \in (2,3], \\ \vartheta(\wp) = 0, \vartheta(\ell) = {}^{\mathcal{AB}}\ \mathcal{I}^{\zeta}_{\wp^+}\vartheta(\aleph), \ \aleph \in (\wp, \ell), \\ \kappa(\wp) = 0, \kappa(\ell) = {}^{\mathcal{AB}}\ \mathcal{I}^{\zeta}_{\wp^+}\kappa(\aleph), \aleph \in (\wp, \ell), \end{cases}$$

$$\begin{cases} {}^{\mathcal{ABC}}\mathcal{D}^{\zeta}_{\wp^+}\vartheta(\pi) = \mathfrak{g}(\pi, \vartheta(\pi), \kappa(\pi)), \ \pi \in [\wp, \ell], \zeta \in (1,2], \\ {}^{\mathcal{ABC}}\mathcal{D}^{\zeta}_{\wp^+}\kappa(\pi) = \mathfrak{g}(\pi, \kappa(\pi), \vartheta(\pi)), \ \pi \in [\wp, \ell], \zeta \in (1,2], \\ \vartheta(\wp) = 0, \vartheta(\ell) = {}^{\mathcal{AB}}\ \mathcal{I}^{\zeta}_{\wp^+}\vartheta(\aleph), \ \aleph \in (\wp, \ell), \\ \kappa(\wp) = 0, \kappa(\ell) = {}^{\mathcal{AB}}\ \mathcal{I}^{\zeta}_{\wp^+}\kappa(\aleph), \aleph \in (\wp, \ell), \end{cases}$$

where ${}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\vartheta(\pi)$ and ${}^{\mathcal{ABC}}\mathcal{D}^{\zeta}_{\wp^+}\vartheta(\pi)$ show the $\mathcal{ABR}$ and $\mathcal{ABC}$ fractional derivatives of order $\delta \in (2,3]$ and $\zeta \in (1,2]$, correspondingly, $0 < \zeta \leqslant 1$ and $\mathfrak{g} : [\wp, \ell] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous.

Motivated by the prior results, in this paper, we prove the existence and uniqueness theorems for new systems of nonlinear coupled boundary value problem (BVP) of the $\mathcal{ABC}$ fractional differential equation defined as

$$\begin{cases} {}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\vartheta(\pi) = \mathfrak{g}_1(\pi, \vartheta(\pi), \vartheta(r\pi), \kappa(\pi)), \ \pi \in [\wp, \ell], \delta \in (2,3], \\ {}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\kappa(\pi) = \mathfrak{g}_2(\pi, \vartheta(\pi), \kappa(\pi), \kappa(r\pi)), \ \pi \in [\wp, \ell], \delta \in (2,3], \\ \vartheta(\wp) = 0, \vartheta(\ell) = \sigma\vartheta(\aleph), \ \aleph \in (\wp, \ell), \\ \kappa(\wp) = 0, \kappa(\ell) = \lambda\kappa(\aleph_1), \ \aleph_1 \in (\wp, \ell), \end{cases} \tag{1.1}$$

$$\begin{cases} {}^{\mathcal{ABC}}\mathcal{D}^{\zeta}_{\wp^+}\vartheta(\pi) = \mathfrak{g}_1(\pi, \vartheta(\pi), \vartheta(r\pi), \kappa(\pi)), \ \pi \in [\wp, \ell], \zeta \in (2,3], \\ {}^{\mathcal{ABC}}\mathcal{D}^{\zeta}_{\wp^+}\kappa(\pi) = \mathfrak{g}_2(\pi, \vartheta(\pi), \kappa(\pi), \kappa(r\pi)), \ \pi \in [\wp, \ell], \zeta \in (2,3], \\ \vartheta(\wp) = 0, \vartheta(\ell) = \sigma\vartheta(\aleph), \ \aleph \in (\wp, \ell), \\ \kappa(\wp) = 0, \kappa(\ell) = \lambda\kappa(\aleph_1), \ \aleph_1 \in (\wp, \ell), \end{cases} \tag{1.2}$$

where ${}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}$ and ${}^{\mathcal{ABC}}\mathcal{D}^{\zeta}_{\wp^+}$ represent the $\mathcal{ABR}$ and $\mathcal{ABC}$ fractional derivatives of order $\delta \in (2,3]$, $r, \aleph, \aleph_1, \sigma, \lambda \in (\wp, \ell)$ and $\zeta \in (1,2]$, respectively, and $\mathfrak{g}_1, \mathfrak{g}_2 : [\wp, \ell] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are nonlinear continuous.

2. Preliminaries

Assume that $\mathcal{C}(\mathcal{J}, \mathbb{R})$ is the space of all continuous functions $\vartheta : \mathcal{J} \to \mathbb{R}$ given by $\|\vartheta\| = \max\{|\vartheta(\pi)| : \pi \in \mathcal{J}\}$, where $\mathcal{J} = [\wp, \ell]$. Clearly, $(\mathcal{C}(\mathcal{J}, \mathbb{R}), \|.\|)$ is a Banach space.

Definition 2.1 ([15])**.** Let us consider $\delta \in (0,1]$. Assume a function ϑ the left-sided $\mathcal{ABC}$ and $\mathcal{ABR}$ fractional derivatives of order δ defined as

$$^{\mathcal{ABC}}\mathcal{D}^{\zeta}_{\wp^+}\vartheta(\pi) = \frac{\phi(\delta)}{1-\delta}\int_{\wp}^{\pi} \delta_{\delta}\left(\frac{\delta}{\delta-1}(\pi-\chi)^{\delta}\right)\vartheta'(\chi)d\chi,\ \pi > \wp,$$
$$^{\mathcal{ABR}}\mathcal{D}^{\zeta}_{\wp^+}\vartheta(\pi) = \frac{\phi(\delta)}{1-\delta}\frac{d}{d\pi}\int_{\wp}^{\pi} \delta_{\delta}\left(\frac{\delta}{\delta-1}(\pi-\chi)^{\delta}\right)\vartheta'(\chi)d\chi,\ \pi > \wp,$$

repectively, where $\phi(\delta)$ refer in the normalizing function, such that $\phi(0) = 1 = \phi(1)$ and δ_{δ} shows the Mittag-Leffler function, defined as:

$$\delta_{\delta}(\vartheta) = \sum_{j=0}^{\infty}\frac{\vartheta^{j}}{\Gamma(1+j\delta)},\ \mathrm{Re}(\delta) > 0, \vartheta \in \mathbb{C}.$$

Furthermore, the analogous fractional integral of $\mathcal{AB}$ is given as

$$^{\mathcal{AB}}\mathcal{J}^{\delta}_{\wp^+}\vartheta(\pi) = \frac{1-\delta}{\phi(\delta)}\vartheta(\pi) + \frac{\delta}{\phi(\delta)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}(\chi)d\chi.$$

Lemma 2.2 ([2])**.** *Let us assume $\delta \in (0,1]$. If the $\mathcal{ABC}$ fractional derivatives occurs, we obtain*

$$^{\mathcal{AB}}\mathcal{J}^{\delta}_{\wp^+}\,^{\mathcal{ABC}}\mathcal{D}^{\delta}_{\wp^+}\vartheta(\pi) = \vartheta(\pi) - \vartheta(\wp).$$

Definition 2.3 ([1])**.** The relation between the $\mathcal{ABC}$ and $\mathcal{ABR}$ fractional derivatives is described as

$$^{\mathcal{ABC}}\mathcal{D}^{\delta}_{\wp^+}\vartheta(\pi) =^{\mathcal{ABR}} \mathcal{D}^{\delta}_{\wp^+}\vartheta(\pi) - \frac{\phi(\delta)}{1-\delta}\vartheta(\wp)\delta_{\delta}\left(\frac{\delta}{\delta-1}(\pi-\chi)^{\delta}\right). \tag{2.1}$$

Remark 2.4 ([30]). If we take $^{\mathcal{AB}}\mathcal{J}^{\delta}_{\wp^+}\vartheta(\pi)$ throughout the whole equation (2.1) and by Lemma 2.2, we have

$$^{\mathcal{AB}}\mathcal{J}^{\delta}_{\wp^+}\,^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\vartheta(\pi) = \vartheta(\pi).$$

Lemma 2.5 ([15])**.** *Let $\vartheta > 0$, $^{\mathcal{AB}}\mathcal{J}^{\delta}_{\wp^+} : \mathcal{C}(\mathcal{J},\mathbb{R}) \to \mathcal{C}(\mathcal{J},\mathbb{R})$ is bounded.*

Definition 2.6 ([1])**.** Assume that $\delta \in (\mathfrak{n}, \mathfrak{n}+1]$ and ϑ be such that $\vartheta^{(k)} \in \mathcal{H}^1(\wp, \ell)$. We take $\eta = \delta - k, \forall \eta \in (0,1]$, and it is defined as

$$(^{\mathcal{ABC}}\mathcal{D}^{\delta}_{\wp^+}\vartheta)(\pi) = (^{\mathcal{ABC}}\mathcal{D}^{\eta}_{\wp^+}\vartheta^{k})(\pi) \quad \text{and} \quad (^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\vartheta)(\pi) = (^{\mathcal{ABR}}\mathcal{D}^{\eta}_{\wp^+}\vartheta^{k})(\pi).$$

Further, the $\mathcal{AB}$ fractinal integral is described as

$$(^{\mathcal{AB}}\mathcal{J}^{\delta}_{\wp^+}\vartheta)(\pi) = (\mathcal{J}^{k}_{\wp^+}\,^{\eta}\mathcal{J}^{\delta}_{\wp^+}\vartheta)(\pi).$$

Theorem 2.7 ([21])**.** *If $\mathcal{Q}$ be a closed subspace of a Banach space $\mathcal{X}$ and $\mathcal{M} : \mathcal{Q} \to \mathcal{Q}$ be a contraction mapping, then $\mathcal{M}$ has a unique fixed point in $\mathcal{Q}$.*

Theorem 2.8 ([19, Krasnoselskii's fixed point theorem])**.** *Let $\mathcal{B}$ be a nonempty closed convex subset of a Banach space $(\mathcal{X}, \|.\|)$. Suppose that $\mathcal{P}$ and $\mathcal{Q}$ map $\mathcal{B}$ into $\mathcal{X}$ such that*

(A1) *$\mathcal{P}\vartheta + \mathcal{Q}\kappa \in \mathcal{B}$ whenever $\vartheta, \kappa \in \mathcal{B}$;*

(A2) *$\mathcal{P}$ is a contraction mapping;*

(A3) *$\mathcal{Q}$ is a continuous and compact.*

Then there exists $z \in \mathcal{B}$ such that $z = \mathcal{P}z + \mathcal{Q}z$.

Lemma 2.9 ([1]). *In $\mathcal{J}$ define a $\vartheta(\pi)$ and $\delta \in (\mathfrak{n}, \mathfrak{n}+1]$, for all $\mathfrak{n} \in \mathbb{N}$, the following conditions are hold:*

(i) $({}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\vartheta)(\pi) = \vartheta(\pi)$;

(ii) $({}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\,{}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\vartheta)(\pi) = \vartheta(\pi) - \sum_{j=0}^{\mathfrak{n}-1} \frac{\vartheta^{(k)}(\wp)}{k!}(\pi-\wp)^j$;

(iii) $({}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\,{}^{\mathcal{ABC}}\mathcal{D}^{\delta}_{\wp^+}\vartheta)(\pi) = \vartheta(\pi) - \sum_{j=0}^{\mathfrak{n}-1} \frac{\vartheta^{(k)}(\wp)}{k!}(\pi-\wp)^j$.

Lemma 2.10 ([30]). *Let us assume that $\delta \in (\mathfrak{n}, \mathfrak{n}+1]$. Then for $\eta = 1, 2, 3, \ldots, \mathfrak{n}-1$, ${}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}(\pi-\wp)^{\eta} = 0$.*

Lemma 2.11 ([1]). *Consider that $\zeta \in (2,3]$ and $\mathfrak{g} \in \mathcal{C}(\mathcal{J}, \mathbb{R})$. The equation defined as*

$$ {}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\vartheta(\pi) = \mathfrak{g}(\pi), \quad \pi \in [\wp, \ell], \vartheta(\wp) = \wp_1, \vartheta(\ell) = \wp_2, $$

has a solution

$$ \vartheta(\pi) = \wp_1 + \wp_2(\pi-\wp) + {}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}(\pi) = \wp_1 + \wp_2(\pi-\wp) + \mathcal{I}^{2}_{\wp^+}({}^{\mathcal{AB}}\mathcal{I}^{\eta}_{\wp^+}\mathfrak{g}(\pi)), \quad (\delta-2) = \eta \in (0,1], $$

where

$$ \begin{aligned} {}^{\mathcal{AB}}\mathcal{I}^{\eta}_{\wp^+}\vartheta(\pi) &= \frac{1-\eta}{\phi(\eta)}\mathfrak{g}(\pi) + \frac{\eta}{\phi(\eta)\Gamma(\eta)}\int_{\wp}^{\pi}(\pi-\chi)^{\eta-1}\mathfrak{g}(\chi)d\chi \\ &= \frac{3-\delta}{\phi(\delta-2)}\mathfrak{g}(\pi) + \frac{\delta-2}{\phi(\delta-2)\Gamma(\delta-2)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-3}\mathfrak{g}(\chi)d\chi, \end{aligned} $$

and

$$ {}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\vartheta(\pi) = \mathcal{I}^{2}_{\wp^+}({}^{\mathcal{AB}}\mathcal{I}^{\eta}_{\wp^+}\mathfrak{g}(\pi)) = \frac{3-\delta}{\phi(\delta-2)}\int_{\wp}^{\pi}(\pi-\chi)\mathfrak{g}(\chi)d\chi + \frac{\delta-2}{\phi(\delta-2)\Gamma(\delta-2)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}(\chi)d\chi. $$

Lemma 2.12 ([1]). *Consider that $\zeta \in (2,3]$ and $\mathfrak{g} \in \mathcal{C}(\mathcal{J}, \mathbb{R})$. The equation defined as*

$$ {}^{\mathcal{ABC}}\mathcal{D}^{\delta}_{\wp^+}\vartheta(\pi) = \mathfrak{g}(\pi), \pi \in [\wp, \ell], \vartheta(\wp) = \wp_1, \vartheta(\ell) = \wp_2, $$

has a solution

$$ \vartheta(\pi) = \wp_1 + \wp_2(\pi-\wp) + {}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}(\pi), $$

where

$$ {}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}(\pi) = \frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}\mathfrak{g}(\chi)d\chi + \frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}(\chi)d\chi. \tag{2.2} $$

Lemma 2.13. *Assume that $\delta \in (2,3]$, $\mathfrak{q} = \sigma(\aleph-\wp) - (\ell-\wp) \neq 0$ and $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathcal{C}(\mathcal{J} \times \mathbb{R}^3, \mathbb{R})$. If the coupled (ϑ, κ) satisfies the following fractional integral equations, then the functions ϑ and κ constitute a solution to the coupled $\mathcal{ABR}$ problem (1.1) as follows:*

$$ \begin{aligned} \vartheta(\pi) &= \frac{(\pi-\wp)}{\mathfrak{q}}\left({}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}_1(\ell, \vartheta(\ell), \vartheta(r\ell), \kappa(\ell)) - \sigma\,{}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}_1(\aleph, \vartheta(\aleph), \vartheta(r\aleph), \kappa(\aleph))\right) \\ &\quad + {}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}_1(\pi, \vartheta(\pi), \vartheta(r\pi), \kappa(\pi)), \\ \kappa(\pi) &= \frac{(\pi-\wp)}{\mathfrak{q}}\left({}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}_2(\ell, \vartheta(\ell), \kappa(\ell), \kappa(r\ell)) - \lambda\,{}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}_2(\aleph, \vartheta(\aleph), \kappa(\aleph), \kappa(r\aleph))\right) \\ &\quad + {}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}_2(\pi, \vartheta(\pi), \kappa(\pi), \kappa(r\pi)). \end{aligned} \tag{2.3} $$

or

$$\begin{aligned}
\vartheta(\pi) = \frac{(\pi-\wp)}{\mathfrak{q}}\Bigg(&\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\ell}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi \\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\ell}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi-\sigma\Bigg(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\aleph}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi \\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\aleph}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\Bigg)\Bigg) \\
&+\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi \\
\kappa(\pi) = \frac{(\pi-\wp)}{\mathfrak{q}}\Bigg(&\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\ell}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi \\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\ell}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi-\lambda\Bigg(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\aleph}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi \\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\aleph}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi\Bigg)\Bigg) \\
&+\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi.
\end{aligned}$$

Proof. Let ϑ be a solution of equation (2.3). Then, by Lemma 2.11, we obtain

$$\vartheta(\pi)=\wp_1+\wp_2(\pi-\wp)+{}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}_1(\pi,\vartheta(\pi),\vartheta(r\pi),\kappa(\pi)). \tag{2.4}$$

By condition $\vartheta(\wp)=0$, we have $\wp_1=0$. Therefore, the equation (2.4) may be represented as

$$\vartheta(\pi)=\wp_2(\pi-\wp)+{}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}_1(\pi,\vartheta(\pi),\vartheta(r\pi),\kappa(\pi)).$$

It follows from the condition $\vartheta(\ell)=\sigma\vartheta(\aleph)$, we get

$$\wp_2\sigma(\aleph-\wp)+\sigma{}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}_1(\aleph,\vartheta(\aleph),\vartheta(r\aleph),\kappa(\aleph))=\wp_2(\ell-\wp)+{}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}_1(\ell,\vartheta(\ell),\vartheta(r\ell),\kappa(\ell)).$$

Consequently,

$$\wp_2=\frac{1}{\sigma(\aleph-\wp)-(\ell-\wp)}\left({}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}_1(\ell,\vartheta(\ell),\vartheta(r\ell),\kappa(\ell))-\sigma{}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}_1(\aleph,\vartheta(\aleph),\vartheta(r\aleph),\kappa(\aleph))\right).$$

From (2.4), we have

$$\begin{aligned}
\vartheta(\pi)=&\frac{(\pi-\wp)}{\mathfrak{q}}\left({}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}_1(\ell,\vartheta(\ell),\vartheta(r\ell),\kappa(\ell))-\sigma{}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}_1(\aleph,\vartheta(\aleph),\vartheta(r\aleph),\kappa(\aleph))\right) \\
&+{}^{\mathcal{AB}}\mathcal{I}^{\delta}_{\wp^+}\mathfrak{g}_1(\pi,\vartheta(\pi),\vartheta(r\pi),\kappa(\pi)),
\end{aligned} \tag{2.5}$$

which implies that

$$\begin{aligned}
\vartheta(\pi) = \frac{(\pi-\wp)}{\mathfrak{q}}\Bigg(&\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\ell}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi \\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\ell}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi-\sigma\Bigg(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\aleph}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi \\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\aleph}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\Bigg)\Bigg)
\end{aligned}$$

$$+\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi.$$

Similarly, we can prove that

$$\kappa(\pi)=\frac{(\pi-\wp)}{\mathfrak{q}}\left({}^{\mathcal{AB}}\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_2(\ell,\vartheta(\ell),\kappa(\ell),\kappa(r\ell))-\lambda{}^{\mathcal{AB}}\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_2(\aleph,\vartheta(\aleph),\kappa(\aleph),\kappa(r\aleph))\right)$$
$$+{}^{\mathcal{AB}}\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_2(\pi,\vartheta(\pi),\kappa(\pi),\kappa(r\pi)), \tag{2.6}$$

which implies that

$$\kappa(\pi)=\frac{(\pi-\wp)}{\mathfrak{q}}\left(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\ell}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi\right.$$
$$+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\ell}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi-\lambda\left(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\aleph}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi\right.$$
$$\left.\left.+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\aleph}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi\right)\right)$$
$$+\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi.$$

Conversely, suppose that ϑ satisfies equation (2.5). Applying ${}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp+}$ on equation (2.5) and assisting Lemma2.9 and Lemma 2.10, we have

$${}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp+}\vartheta(\pi)={}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp+}\frac{(\pi-\wp)}{\mathfrak{q}}\left({}^{\mathcal{AB}}\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_1(\ell,\vartheta(\ell),\vartheta(r\ell),\kappa(\ell))-\sigma{}^{\mathcal{AB}}\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_1(\aleph,\vartheta(\aleph),\vartheta(r\aleph),\kappa(\aleph))\right)$$
$$+{}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp+}{}^{\mathcal{AB}}\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_1(\pi,\vartheta(\pi),\vartheta(r\pi),\kappa(\pi))=\mathfrak{g}_1(\pi,\vartheta(\pi),\vartheta(r\pi),\kappa(\pi)).$$

In the same way, we have

$${}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp+}\kappa(\pi)={}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp+}\frac{(\pi-\wp)}{\mathfrak{q}}\left({}^{\mathcal{AB}}\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_2(\ell,\vartheta(\ell),\kappa(\ell),\kappa(r\ell))-\lambda{}^{\mathcal{AB}}\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_2(\aleph,\vartheta(\aleph),\kappa(\aleph),\kappa(r\aleph))\right)$$
$$+{}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp+}{}^{\mathcal{AB}}\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_2(\pi,\vartheta(\pi),\kappa(\pi),\kappa(r\pi))=\mathfrak{g}_2(\pi,\vartheta(\pi),\kappa(\pi),\kappa(r\pi)).$$

As $\pi\to\wp$, in (2.5) and (2.6), then $\vartheta(\wp)=0=\kappa(\wp)$. Next,

$$\sigma\vartheta(\aleph)=\frac{\sigma(\aleph-\wp)}{\mathfrak{q}}\left({}^{\mathcal{AB}}\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_1(\ell,\vartheta(\ell),\vartheta(r\ell),\kappa(\ell))\right.$$
$$\left.-\sigma{}^{\mathcal{AB}}\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_1(\aleph,\vartheta(\aleph),\vartheta(r\aleph),\kappa(\aleph))\right)+\sigma{}^{\mathcal{AB}}\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_1(\aleph,\vartheta(\aleph),\vartheta(r\aleph),\kappa(\aleph))$$
$$=\left(1+\frac{(\ell-\wp)}{\mathfrak{q}}\right)\left({}^{\mathcal{AB}}\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_1(\ell,\vartheta(\ell),\vartheta(r\ell),\kappa(\ell))\right.$$
$$\left.-\sigma{}^{\mathcal{AB}}\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_1(\aleph,\vartheta(\aleph),\vartheta(r\aleph),\kappa(\aleph))\right)+\sigma{}^{\mathcal{AB}}\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_1(\aleph,\vartheta(\aleph),\vartheta(r\aleph),\kappa(\aleph))$$
$$=\frac{(\ell-\wp)}{\mathfrak{q}}\left({}^{\mathcal{AB}}\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_1(\ell,\vartheta(\ell),\vartheta(r\ell),\kappa(\ell))\right.$$
$$\left.-\sigma{}^{\mathcal{AB}}\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_1(\aleph,\vartheta(\aleph),\vartheta(r\aleph),\kappa(\aleph))\right)+\mathfrak{I}^{\delta}_{\wp+}\mathfrak{g}_1(\ell,\vartheta(\ell),\vartheta(r\ell),\kappa(\ell))=\vartheta(\ell).$$

Similarly, we can prove that $\lambda\kappa(\aleph_1)=\kappa(\ell)$. □

3. Main results

We are prepared to discuss our key findings. Using Lemma 2.13, we define an operator $\mathcal{R} : \mathcal{C}(\mathcal{J}, \mathbb{R}) \times \mathcal{C}(\mathcal{J}, \mathbb{R}) \to \mathcal{C}(\mathcal{J}, \mathbb{R})$ by

$$\mathcal{R}(\vartheta, \kappa)(\pi) = (\mathcal{R}_1(\vartheta, \kappa)(\pi), \mathcal{R}_2(\vartheta, \kappa)(\pi)), \tag{3.1}$$

where

$$\begin{aligned}\mathcal{R}_1(\vartheta,\kappa)(\pi) = \frac{(\pi-\wp)}{\mathfrak{q}}\Bigg(&\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\ell}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\ &+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\ell}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\ &-\sigma\Bigg(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\aleph}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\ &+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\aleph}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\Bigg)\Bigg)\\ &+\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\ &+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\mathcal{R}_2(\vartheta,\kappa)(\pi) = \frac{(\pi-\wp)}{\mathfrak{q}}\Bigg(&\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\ell}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi\\ &+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\ell}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi\\ &-\lambda\Bigg(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\aleph}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi\\ &+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\aleph}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi\Bigg)\Bigg)\\ &+\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi\\ &+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi.\end{aligned} \tag{3.3}$$

So the existence of solution for system (1.1) is equivalent to the existence of the fixed point for the operator $\mathcal{R}$ defined by (3.1)-(3.3). We are discussing the existence and uniqueness of solutions for the coupled $\mathcal{ABR}$ type fractional derivatives (1.1).

Theorem 3.1. *Suppose that $\mathfrak{g}_1, \mathfrak{g}_2$ are given nonlinear continuous functions such that*

(B1) $\forall\, \vartheta, \bar{\vartheta}, \kappa, \bar{\kappa} \in \mathcal{C}(\mathcal{J}, \mathbb{R})$ *and* $\pi \in \mathcal{J}$, *there exists* $k_1, k_2 > 0$ *such that*

$$|\mathfrak{g}_1(\pi,\vartheta(\pi),\vartheta(r\pi),\kappa(\pi)) - \mathfrak{g}_1(\pi,\bar{\vartheta}(\pi),\bar{\vartheta}(r\pi),\bar{\kappa}(\pi))| \leqslant \frac{k_1}{6}(2|\vartheta(\pi)-\bar{\vartheta}(\pi)| + |\kappa(\pi)-\bar{\kappa}(\pi)|)$$

and

$$|\mathfrak{g}_2(\pi,\vartheta(\pi),\kappa(\pi),\kappa(r\pi),) - \mathfrak{g}_2(\pi,\bar{\vartheta}(\pi),\bar{\kappa}(\pi),\bar{\kappa}(r\pi))| \leqslant \frac{k_2}{6}(|\vartheta(\pi)-\bar{\vartheta}(\pi)| + 2|\kappa(\pi)-\bar{\kappa}(\pi)|),$$

(B2) *If* $\mathrm{U}+\mathrm{W}<1$ *and*

$$\left(\left(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}\right)+\mathfrak{q}_2\right)\left(\frac{k_1}{3}+\frac{k_2}{3}\right)<1,$$

where

$$\mathrm{U}=k_1\left(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}+\mathfrak{q}_2\right),\qquad \mathrm{W}=k_2\left(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}+\mathfrak{q}_2\right),$$

$$\begin{aligned}\mathfrak{q}_1&=\frac{2-\delta}{\phi(\delta-1)}(\ell-\wp)+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta+1)}\left((\pi-\ell)^{\delta}-(\pi-\wp)^{\delta}\right)-\sigma\bigg(\frac{2-\delta}{\phi(\delta-1)}(\aleph-\wp)\\&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta+1)}\left((\pi-\aleph)^{\delta}-(\pi-\wp)^{\delta}\right)\bigg),\\ \mathfrak{q}_2&=\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp)+\frac{1-\delta}{\phi(\delta-1)\Gamma(\delta+1)}(\pi-\wp)^{\delta}.\end{aligned}$$

Then the coupled $\mathcal{ABR}$ type fractional derivatives (1.1) has a unique solution.

Proof. Assume a closed ball $\mathcal{B}_\nu$ such as $\mathcal{B}_\nu=\{\theta\in\mathcal{C}(\mathcal{J},\mathbb{R}):\|\theta\|\leqslant\nu\}$ with radius $\nu\geqslant\frac{\mathrm{U}_1+\mathrm{W}_1}{1-\frac{\mathrm{U}+\mathrm{W}}{2}}$,

$$\mathrm{U}_1=\left(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}+\mathfrak{q}_2\right)\mathrm{h}_1,\qquad \mathrm{W}_1=\left(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}+\mathfrak{q}_2\right)\mathrm{h}_2,$$

and $\mathrm{h}_1=\sup_{\pi\in\mathcal{J}}|\mathfrak{g}_1(\pi,0,0,0)|$, $\mathrm{h}_2=\sup_{\pi\in\mathcal{J}}|\mathfrak{g}_2(\pi,0,0,0)|$. For each $\vartheta,\kappa\in\mathcal{C}(\mathcal{J},\mathbb{R})$, using (B1) and (B2), we have

$$\begin{aligned}&|\mathfrak{g}_1(\pi,\vartheta(\pi),\vartheta(r\pi),\kappa(\pi))|\\&\quad\leqslant|\mathfrak{g}_1(\pi,\vartheta(\pi),\vartheta(r\pi),\kappa(\pi))-\mathfrak{g}_1(\pi,0,0,0)|+|\mathfrak{g}_1(\pi,0,0,0)|\\&\quad\leqslant\frac{k_1}{6}(2|\vartheta(\pi)|+|\kappa(\pi)|)+|\mathfrak{g}_1(\pi,0,0,0)|\leqslant\frac{k_1}{6}(3\nu)+\mathrm{h}_1=\frac{k_1}{2}\nu+\mathrm{h}_1,\\&|\mathfrak{g}_2(\pi,\vartheta(\pi),\kappa(\pi),\kappa(r\pi))|\\&\quad\leqslant|\mathfrak{g}_2(\pi,\vartheta(\pi),\kappa(\pi),\kappa(r\pi))-\mathfrak{g}_2(\pi,0,0,0)|+|\mathfrak{g}_2(\pi,0,0,0)|\\&\quad\leqslant\frac{k_2}{6}(|\vartheta(\pi)|+2|\kappa(\pi)|)+|\mathfrak{g}_2(\pi,0,0,0)|\leqslant\frac{k_2}{6}(3\nu)+\mathrm{h}_2=\frac{k_2}{2}\nu+\mathrm{h}_2,\end{aligned}$$

$$\begin{aligned}|\mathcal{R}_1(\vartheta,\kappa)|&\leqslant\frac{(\pi-\wp)}{\mathfrak{q}}\bigg(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\ell}|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))|d\chi\\&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\ell}(\pi-\chi)^{\delta-1}|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))|d\chi\\&\quad-\sigma\bigg(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\aleph}|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))|d\chi\\&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\aleph}(\pi-\chi)^{\delta-1}|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))|d\chi\bigg)\bigg)\\&\quad+\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))|d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))|d\chi\\&\leqslant\frac{(\pi-\wp)}{\mathfrak{q}}\bigg(\frac{2-\delta}{\phi(\delta-1)}(\ell-\wp)(\frac{k_1}{2}\nu+\mathrm{h}_1)\\&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\bigg(\frac{(\pi-\ell)^{\delta}}{\delta}-\frac{(\pi-\wp)^{\delta}}{\delta}\bigg)(\frac{k_1}{2}\nu+\mathrm{h}_1)-\sigma\bigg(\frac{2-\delta}{\phi(\delta-1)}(\aleph-\wp)(\frac{k_1}{2}\nu+\mathrm{h}_1)\end{aligned}$$

$$
\begin{aligned}
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\Big(\frac{(\pi-\aleph)^{\delta}}{\delta}-\frac{(\pi-\wp)^{\delta}}{\delta}\Big)(\frac{k_1}{2}\nu+h_1)\Big)\Big)\\
&+\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp)(\frac{k_1}{2}\nu+h_1)+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\Big(-\frac{(\pi-\wp)^{\delta}}{\delta}\Big)(\frac{k_1}{2}\nu+h_1)\\
=&\frac{(\pi-\wp)}{\mathfrak{q}}\Big(\frac{2-\delta}{\phi(\delta-1)}(\ell-\wp)+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\Big(\frac{(\pi-\ell)^{\delta}}{\delta}-\frac{(\pi-\wp)^{\delta}}{\delta}\Big)-\sigma\Big(\frac{2-\delta}{\phi(\delta-1)}(\aleph-\wp)\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\Big(\frac{(\pi-\aleph)^{\delta}}{\delta}-\frac{(\pi-\wp)^{\delta}}{\delta}\Big)\Big)\Big)(\frac{k_1}{2}\nu+h_1)\\
&+\Big(\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp)+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\Big(-\frac{(\pi-\wp)^{\delta}}{\delta}\Big)\Big)(\frac{k_1}{2}\nu+h_1)\\
=&\frac{(\pi-\wp)}{\mathfrak{q}}q_1(\frac{k_1}{2}\nu+h_1)+q_2(\frac{k_1}{2}\nu+h_1)\\
=&\frac{k_1}{2}\Big(\frac{q_1(\pi-\wp)}{\mathfrak{q}}+q_2\Big)\nu+\Big(\frac{q_1(\pi-\wp)}{\mathfrak{q}}+q_2\Big)h_1=\frac{U}{2}\nu+U_1,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|\mathcal{R}_1(\vartheta,\kappa)\|\leqslant&\frac{U}{2}\nu+U_1,\\
|\mathcal{R}_2(\vartheta,\kappa)|\leqslant&\frac{(\pi-\wp)}{\mathfrak{q}}\Big(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\ell}|\mathfrak{g}_2(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))|d\chi\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\ell}(\pi-\chi)^{\delta-1}|\mathfrak{g}_2(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))|d\chi\\
&-\lambda\Big(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\aleph}|\mathfrak{g}_2(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))|d\chi\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\aleph}(\pi-\chi)^{\delta-1}|\mathfrak{g}_2(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))|d\chi\Big)\Big)\\
&+\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}|\mathfrak{g}_2(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))|d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}|\mathfrak{g}_2(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))|d\chi\\
\leqslant&\frac{(\pi-\wp)}{\mathfrak{q}}\Big(\frac{2-\delta}{\phi(\delta-1)}(\ell-\wp)(\frac{k_2}{2}\nu+h_2)\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\Big(\frac{(\pi-\ell)^{\delta}}{\delta}-\frac{(\pi-\wp)^{\delta}}{\delta}\Big)(\frac{k_2}{2}\nu+h_2)-\lambda\Big(\frac{2-\delta}{\phi(\delta-1)}(\aleph-\wp)(\frac{k_2}{2}\nu+h_2)\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\Big(\frac{(\pi-\aleph)^{\delta}}{\delta}-\frac{(\pi-\wp)^{\delta}}{\delta}\Big)(\frac{k_2}{2}\nu+h_2)\Big)\Big)+\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp)(\frac{k_2}{2}\nu+h_2)\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\Big(-\frac{(\pi-\wp)^{\delta}}{\delta}\Big)(\frac{k_2}{2}\nu+h_2)\\
=&\frac{(\pi-\wp)}{\mathfrak{q}}\Big(\frac{2-\delta}{\phi(\delta-1)}(\ell-\wp)+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\Big(\frac{(\pi-\ell)^{\delta}}{\delta}-\frac{(\pi-\wp)^{\delta}}{\delta}\Big)\\
&-\lambda\Big(\frac{2-\delta}{\phi(\delta-1)}(\aleph-\wp)+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\Big(\frac{(\pi-\aleph)^{\delta}}{\delta}-\frac{(\pi-\wp)^{\delta}}{\delta}\Big)\Big)\Big)(\frac{k_2}{2}\nu+h_2)\\
&+\Big(\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp)+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\Big(-\frac{(\pi-\wp)^{\delta}}{\delta}\Big)\Big)(\frac{k_2}{2}\nu+h_2)\\
=&\frac{(\pi-\wp)}{\mathfrak{q}}q_1(\frac{k_2}{2}\nu+h_2)+q_2(\frac{k_2}{2}\nu+h_2)
\end{aligned}
$$

$$= \frac{k_2}{2}\left(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}+\mathfrak{q}_2\right)\nu+\left(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}+\mathfrak{q}_2\right)\mathrm{h}_2=\frac{\mathrm{W}}{2}\nu+\mathrm{W}_1,$$

which implies that $\|\mathcal{R}_2(\vartheta,\kappa)\|\leqslant \frac{\mathrm{W}}{2}\nu+\mathrm{W}_1$. Now,

$$\|\mathcal{R}(\vartheta,\kappa)\|=\|\mathcal{R}_1(\vartheta,\kappa)\|+\|\mathcal{R}_2(\vartheta,\kappa)\|\leqslant\frac{\mathrm{U}}{2}\nu+\mathrm{U}_1+\frac{\mathrm{W}}{2}\nu+\mathrm{W}_1\leqslant\nu.$$

Hence, $\|\mathcal{R}(\vartheta,\kappa)\|\leqslant\nu$ and so $\mathcal{R}\mathcal{B}_\nu\subset\mathcal{B}_\nu$. For each $\vartheta,\bar{\vartheta},\kappa,\bar{\kappa}\in\mathcal{C}(\mathcal{J},\mathbb{R})$, we derive that

$$\begin{aligned}
&|\mathcal{R}_1(\vartheta,\kappa)(\pi)-\mathcal{R}_1(\bar{\vartheta},\bar{\kappa})(\pi)|\\
&=\Bigg|\frac{(\pi-\wp)}{\mathfrak{q}}\Bigg(\frac{2-\delta}{\phi(\delta-1)}\int_\wp^\ell \mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\
&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_\wp^\ell(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\
&\quad-\sigma\Bigg(\frac{2-\delta}{\phi(\delta-1)}\int_\wp^\aleph \mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\
&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_\wp^\aleph(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\Bigg)\Bigg)\\
&\quad+\frac{2-\delta}{\phi(\delta-1)}\int_\wp^\pi \mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_\wp^\pi(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\
&\quad-\frac{(\pi-\wp)}{\mathfrak{q}}\Bigg(\frac{2-\delta}{\phi(\delta-1)}\int_\wp^\ell \mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))d\chi\\
&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_\wp^\ell(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))d\chi-\sigma\Bigg(\frac{2-\delta}{\phi(\delta-1)}\int_\wp^\aleph \mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))d\chi\\
&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_\wp^\aleph(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))d\chi\Bigg)\Bigg)\\
&\quad-\frac{2-\delta}{\phi(\delta-1)}\int_\wp^\pi \mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))d\chi-\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_\wp^\pi(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))d\chi\Bigg|\\
&\leqslant\frac{(\pi-\wp)}{\mathfrak{q}}\Bigg(\frac{2-\delta}{\phi(\delta-1)}\int_\wp^\ell|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))-\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))|d\chi\\
&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_\wp^\ell(\pi-\chi)^{\delta-1}|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))-\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))|d\chi\\
&\quad-\sigma\Bigg(\frac{2-\delta}{\phi(\delta-1)}\int_\wp^\aleph|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))-\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))|d\chi\\
&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_\wp^\aleph(\pi-\chi)^{\delta-1}|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))-\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))|d\chi\Bigg)\Bigg)\\
&\quad+\frac{2-\delta}{\phi(\delta-1)}\int_\wp^\pi|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))-\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))|d\chi\\
&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_\wp^\pi(\pi-\chi)^{\delta-1}|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))-\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))|d\chi,\\
&|\mathcal{R}_1(\vartheta,\kappa)(\pi)-\mathcal{R}_1(\bar{\vartheta},\bar{\kappa})(\pi)|\\
&\leqslant\Bigg(\frac{(\pi-\wp)}{\mathfrak{q}}\Bigg(\frac{2-\delta}{\phi(\delta-1)}(\ell-\wp)+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\Bigg(\frac{(\pi-\ell)^\delta}{\delta}-\frac{(\pi-\wp)^\delta}{\delta}\Bigg)-\sigma\Bigg(\frac{2-\delta}{\phi(\delta-1)}(\aleph-\wp)\\
&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\Bigg(\frac{(\pi-\aleph)^\delta}{\delta}-\frac{(\pi-\wp)^\delta}{\delta}\Bigg)\Bigg)\Bigg)+\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp)+\frac{1-\delta}{\phi(\delta-1)\Gamma(\delta)}\Bigg(\frac{(\pi-\wp)^\delta}{\delta}\Bigg)\Bigg),
\end{aligned}$$

$$\frac{k_1}{6}(2|\vartheta(\chi)-\bar{\vartheta}(\chi)|+|\kappa(\chi)-\bar{\kappa}(\chi)|)$$
$$=\left(\frac{(\pi-\wp)}{\mathfrak{q}}\left(\frac{2-\delta}{\phi(\delta-1)}(\ell-\wp)+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta+1)}\left((\pi-\ell)^{\delta}-(\pi-\wp)^{\delta}\right)-\sigma\left(\frac{2-\delta}{\phi(\delta-1)}(\aleph-\wp)\right.\right.\right.$$
$$\left.\left.\left.+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta+1)}\left((\pi-\aleph)^{\delta}-(\pi-\wp)^{\delta}\right)\right)\right)+\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp)+\frac{1-\delta}{\phi(\delta-1)\Gamma(\delta+1)}(\pi-\wp)^{\delta}\right),$$
$$\frac{k_1}{6}(2|\vartheta(\chi)-\bar{\vartheta}(\chi)|+|\kappa(\chi)-\bar{\kappa}(\chi)|)=\left(\left(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}\right)+\mathfrak{q}_2\right)\frac{k_1}{6}(2|\vartheta(\chi)-\bar{\vartheta}(\chi)|+|\kappa(\chi)-\bar{\kappa}(\chi)|),$$

which implies that

$$\begin{aligned}\|\mathcal{R}_1(\vartheta,\kappa)(\pi)-\mathcal{R}_1(\bar{\vartheta},\bar{\kappa})(\pi)\| &\leqslant \left(\left(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}\right)+\mathfrak{q}_2\right)\frac{k_1}{6}(2\|\vartheta(\chi)-\bar{\vartheta}(\chi)\|+\|\kappa(\chi)-\bar{\kappa}(\chi)\|)\\ &\leqslant \left(\left(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}\right)+\mathfrak{q}_2\right)\frac{k_1}{3}(\|\vartheta(\chi)-\bar{\vartheta}(\chi)\|+\|\kappa(\chi)-\bar{\kappa}(\chi)\|).\end{aligned}$$

Similarly, we can prove that

$$\|\mathcal{R}_2(\vartheta,\kappa)(\pi)-\mathcal{R}_2(\bar{\vartheta},\bar{\kappa})(\pi)\| \leqslant \left(\left(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}\right)+\mathfrak{q}_2\right)\frac{k_2}{3}(\|\vartheta(\chi)-\bar{\vartheta}(\chi)\|+\|\kappa(\chi)-\bar{\kappa}(\chi)\|).$$

Hence,

$$\begin{aligned}\|\mathcal{R}(\vartheta,\kappa)(\pi)-\mathcal{R}(\bar{\vartheta},\bar{\kappa})(\pi)\| &\leqslant \|\mathcal{R}_1(\vartheta,\kappa)(\pi)-\mathcal{R}_1(\bar{\vartheta},\bar{\kappa})(\pi)\|+\|\mathcal{R}_2(\vartheta,\kappa)(\pi)-\mathcal{R}_2(\bar{\vartheta},\bar{\kappa})(\pi)\|\\ &= \left(\left(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}\right)+\mathfrak{q}_2\right)\left(\frac{k_1}{3}+\frac{k_2}{3}\right)(\|\vartheta(\chi)-\bar{\vartheta}(\chi)\|+\|\kappa(\chi)-\bar{\kappa}(\chi)\|)\\ &= \left(\left(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}\right)+\mathfrak{q}_2\right)\left(\frac{k_1}{3}+\frac{k_2}{3}\right)\|(\vartheta,\kappa)-(\bar{\vartheta},\bar{\kappa})\|.\end{aligned}$$

Since $\left(\left(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}\right)+\mathfrak{q}_2\right)\left(\frac{k_1}{3}+\frac{k_2}{3}\right)<1$, then $\mathcal{R}$ is a contraction mapping. According to Theorem 2.7, $\mathcal{R}$ has a unique fixed point. □

Theorem 3.2. *Assume that (B1)-(B2) hold. If the following conditions hold, then system (1.1) has at least one solution on* $(-\infty,\ell]$.

(C1) *If* $\left(\left(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}\right)+\mathfrak{q}_2\right)\left(\frac{k_1}{3}\right)<1$, *where*

$$\begin{aligned}\mathfrak{q}_1 &= \frac{2-\delta}{\phi(\delta-1)}(\ell-\wp)+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta+1)}\left((\pi-\ell)^{\delta}-(\pi-\wp)^{\delta}\right)-\sigma\left(\frac{2-\delta}{\phi(\delta-1)}(\aleph-\wp)\right.\\ &\quad\left.+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta+1)}\left((\pi-\aleph)^{\delta}-(\pi-\wp)^{\delta}\right)\right),\\ \mathfrak{q}_2 &= \frac{2-\delta}{\phi(\delta-1)}(\pi-\wp)+\frac{1-\delta}{\phi(\delta-1)\Gamma(\delta+1)}(\pi-\wp)^{\delta}.\end{aligned}$$

Proof. Define the operator $\mathcal{R}:\mathcal{C}(\mathcal{J},\mathbb{R})\times\mathcal{C}(\mathcal{J},\mathbb{R})\to\mathcal{C}(\mathcal{J},\mathbb{R})$. We split $\mathcal{R}=\mathcal{P}+\mathcal{Q}$ such that, for all $\pi\in\mathcal{J}$, $\vartheta,\kappa\in\mathcal{C}(\mathcal{J},\mathbb{R})$,

$$\mathcal{P}_1(\vartheta,\kappa)(\pi)=\frac{(\pi-\wp)}{\mathfrak{q}}\left(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\ell}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\right.$$

$$
\begin{aligned}
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\ell}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\
&-\sigma\bigg(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\aleph}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\aleph}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\bigg)\bigg)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}_2(\vartheta,\kappa)(\pi)=\frac{(\pi-\wp)}{\mathfrak{q}}\bigg(&\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\ell}\mathfrak{g}_2(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\ell}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\
&-\sigma\bigg(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\aleph}\mathfrak{g}_2(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\aleph}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\bigg)\bigg)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Q}_1(\vartheta,\kappa)(\pi)=&\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi
\end{aligned}
$$

$$
\mathcal{Q}_2(\vartheta,\kappa)(\pi)=\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}\mathfrak{g}_2(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi.
$$

By the proof of the Theorem 3.1, we derive that

$$
\|\mathcal{R}(\vartheta,\kappa)\|=\|\mathcal{R}_1(\vartheta,\kappa)\|+\|\mathcal{R}_2(\vartheta,\kappa)\|\leqslant\frac{U}{2}\nu+U_1+\frac{W}{2}\nu+W_1\leqslant\nu.
$$

Therefore, $\mathcal{P}(\vartheta,\kappa)+\mathcal{Q}(\vartheta,\kappa)\in\mathcal{B}_\nu$, we also prove that $\mathcal{P}$ and $\mathcal{Q}$ map $\mathcal{B}_\nu$ into $\mathcal{B}_\nu\subset\mathcal{C}(\mathcal{J},\mathbb{R})$ and $\mathcal{Q}$ is uniformly bounded. Now, we prove that $\mathcal{P}$ is a contraction mapping. For this, for all $\pi\in\mathcal{J}$, $\vartheta,\kappa\in\mathcal{C}(\mathcal{J},\mathbb{R})$,

$$
\begin{aligned}
&|\mathcal{P}(\vartheta,\kappa)(\pi)-\mathcal{P}(\bar{\vartheta},\bar{\kappa})(\pi)|\\
&=\bigg|\frac{(\pi-\wp)}{\mathfrak{q}}\bigg(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\ell}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\ell}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi-\sigma\bigg(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\aleph}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\aleph}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\bigg)\bigg)+\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi-\frac{(\pi-\wp)}{\mathfrak{q}}\bigg(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\ell}\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))d\chi\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\ell}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))d\chi-\sigma\bigg(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\aleph}\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))d\chi\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\aleph}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))d\chi\bigg)\bigg)\\
&-\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))d\chi-\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))d\chi\bigg|,\\
&|\mathcal{P}(\vartheta,\kappa)(\pi)-\mathcal{P}(\bar{\vartheta},\bar{\kappa})(\pi)|
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{(\pi-\wp)}{\mathfrak{q}}\bigg(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\ell}|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))-\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))|d\chi\\
&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\ell}(\pi-\chi)^{\delta-1}|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))-\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))|d\chi\\
&\quad-\sigma\bigg(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\aleph}|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))-\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))|d\chi\\
&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\aleph}(\pi-\chi)^{\delta-1}|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))-\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))|d\chi\bigg)\bigg)\\
&\quad+\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))-\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))|d\chi\\
&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))-\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))|d\chi\\
&\leqslant \bigg(\frac{(\pi-\wp)}{\mathfrak{q}}\bigg(\frac{2-\delta}{\phi(\delta-1)}(\ell-\wp)+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\bigg(\frac{(\pi-\ell)^{\delta}}{\delta}-\frac{(\pi-\wp)^{\delta}}{\delta}\bigg)\\
&\quad-\sigma\bigg(\frac{2-\delta}{\phi(\delta-1)}(\aleph-\wp)+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\bigg(\frac{(\pi-\aleph)^{\delta}}{\delta}-\frac{(\pi-\wp)^{\delta}}{\delta}\bigg)\bigg)\bigg)\\
&\quad+\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp)+\frac{1-\delta}{\phi(\delta-1)\Gamma(\delta)}\bigg(\frac{(\pi-\wp)^{\delta}}{\delta}\bigg)\bigg),\\
&\frac{k_1}{6}(2|\vartheta(\chi)-\bar{\vartheta}(\chi)|+|\kappa(\chi)-\bar{\kappa}(\chi)|)\\
&\quad=\bigg(\frac{(\pi-\wp)}{\mathfrak{q}}\bigg(\frac{2-\delta}{\phi(\delta-1)}(\ell-\wp)+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta+1)}\big((\pi-\ell)^{\delta}-(\pi-\wp)^{\delta}\big)-\sigma\bigg(\frac{2-\delta}{\phi(\delta-1)}(\aleph-\wp)\\
&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta+1)}\big((\pi-\aleph)^{\delta}-(\pi-\wp)^{\delta}\big)\bigg)\bigg)+\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp)+\frac{1-\delta}{\phi(\delta-1)\Gamma(\delta+1)}(\pi-\wp)^{\delta}\bigg),\\
&\frac{k_1}{6}(2|\vartheta(\chi)-\bar{\vartheta}(\chi)|+|\kappa(\chi)-\bar{\kappa}(\chi)|)=\bigg(\bigg(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}\bigg)+\mathfrak{q}_2\bigg)\frac{k_1}{6}(2|\vartheta(\chi)-\bar{\vartheta}(\chi)|+|\kappa(\chi)-\bar{\kappa}(\chi)|),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|\mathcal{P}(\vartheta,\kappa)(\pi)-\mathcal{P}(\bar{\vartheta},\bar{\kappa})(\pi)\| &\leqslant \bigg(\bigg(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}\bigg)+\mathfrak{q}_2\bigg)\frac{k_1}{6}(2\|\vartheta(\chi)-\bar{\vartheta}(\chi)\|+\|\kappa(\chi)-\bar{\kappa}(\chi)\|)\\
&=\bigg(\bigg(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}\bigg)+\mathfrak{q}_2\bigg)\frac{k_1}{3}(\|\vartheta(\chi)-\bar{\vartheta}(\chi)\|+\|\kappa(\chi)-\bar{\kappa}(\chi)\|)\\
&=\bigg(\bigg(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}\bigg)+\mathfrak{q}_2\bigg)\frac{k_1}{3}\|(\vartheta,\kappa)-(\bar{\vartheta},\bar{\kappa})\|.
\end{aligned}
$$

Since $\Big(\Big(\frac{\mathfrak{q}_1(\pi-\wp)}{\mathfrak{q}}\Big)+\mathfrak{q}_2\Big)\frac{k_1}{3}<1$, hence $\mathcal{P}$ is a contraction mapping. Next, we prove that $\mathcal{Q}$ is equicontinuous. For this, for all $\pi_1,\pi_2\in\mathcal{J}$, $\vartheta,\kappa\in\mathcal{C}(\mathcal{J},\mathbb{R})$ with $\pi_1<\pi_2$,

$$
\begin{aligned}
&|\mathcal{Q}_1(\vartheta,\kappa)(\pi_2)-\mathcal{Q}_1(\vartheta,\kappa)(\pi_1)|\\
&\leqslant \bigg|\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi_2}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi_2}(\pi_2-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\
&\quad-\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi_1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi-\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi_1}(\pi_1-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\bigg|\\
&\leqslant \bigg|\frac{2-\delta}{\phi(\delta-1)}\int_{\pi_1}^{\pi_2}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\pi_1}^{\pi_2}(\pi_2-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\bigg|
\end{aligned}
$$

$$\leqslant \frac{2-\delta}{\phi(\delta-1)}(\pi_2-\pi_1)\frac{k_1}{2}\nu+h_1+\frac{1-\delta}{\phi(\delta-1)\Gamma(\delta)}\frac{(\pi_2-\pi_1)^\delta}{\delta}\left(\frac{k_1}{2}\nu+h_1\right)\to 0 \text{ as } \pi_2\to\pi_1.$$

Similarly, we can prove that

$$\begin{aligned}&\|\mathcal{Q}_2(\vartheta,\kappa)(\pi_2)-\mathcal{Q}_2(\vartheta,\kappa)(\pi_1)\|_{\mathcal{B}_3}\\&\leqslant \frac{2-\delta}{\phi(\delta-1)}(\pi_2-\pi_1)\frac{k_2}{2}\nu+h_2+\frac{1-\delta}{\phi(\delta-1)\Gamma(\delta)}\frac{(\pi_2-\pi_1)^\delta}{\delta}\left(\frac{k_2}{2}\nu+h_2\right)\to 0 \text{ as } \pi_2\to\pi_1.\end{aligned}$$

From above inequalities, we know that $\mathcal{Q}$ is equicontinuous. So $\mathcal{Q}$ is relatively compact. Hence, by the Arzela-Ascoli theorem, then $\mathcal{Q}$ is compact. From Theorem 2.8, there is at least one solution $(\vartheta^*,\kappa^*)\in\mathcal{B}_\nu$. □

We are discussing the existence and uniqueness of solutions for the coupled $\mathcal{ABC}$ type fractional derivatives (1.2).

Lemma 3.3. *Let us assume $\delta\in(2,3]$, $\mathfrak{q}=\sigma(\aleph-\wp)-(\ell-\wp)\neq 0$, and $\mathfrak{g}_1,\mathfrak{g}_2\in\mathcal{C}(\mathcal{J}\times\mathbb{R}^3,\mathbb{R})$. If the coupled (ϑ,κ) satisfies the following fractional integral equations, then the functions ϑ and κ constitute a solution to the coupled $\mathcal{ABR}$ problem (1.1) as follows:*

$$\begin{aligned}\vartheta(\pi)&=\frac{(\pi-\wp)}{\mathfrak{q}}\left({}^{\mathcal{AB}}\mathcal{J}^{\delta}_{\wp^+}\mathfrak{g}_1(\ell,\vartheta(\ell),\vartheta(r\ell),\kappa(\ell))-\sigma{}^{\mathcal{AB}}\mathcal{J}^{\delta}_{\wp^+}\mathfrak{g}_1(\aleph,\vartheta(\aleph),\vartheta(r\aleph),\kappa(\aleph))\right)\\&\quad+{}^{\mathcal{AB}}\mathcal{J}^{\delta}_{\wp^+}\mathfrak{g}_1(\pi,\vartheta(\pi),\vartheta(r\pi),\kappa(\pi)),\\\kappa(\pi)&=\frac{(\pi-\wp)}{\mathfrak{q}}\left({}^{\mathcal{AB}}\mathcal{J}^{\delta}_{\wp^+}\mathfrak{g}_2(\ell,\vartheta(\ell),\kappa(\ell),\kappa(r\ell))-\lambda{}^{\mathcal{AB}}\mathcal{J}^{\delta}_{\wp^+}\mathfrak{g}_2(\aleph,\vartheta(\aleph),\kappa(\aleph),\kappa(r\aleph))\right)\\&\quad+{}^{\mathcal{AB}}\mathcal{J}^{\delta}_{\wp^+}\mathfrak{g}_2(\pi,\vartheta(\pi),\kappa(\pi),\kappa(r\pi)).\end{aligned}\tag{3.4}$$

or

$$\begin{aligned}\vartheta(\pi)&=\frac{(\pi-\wp)}{\mathfrak{q}}\left(\frac{2-\delta}{\phi(\delta-1)}\int_\wp^\ell\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\right.\\&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_\wp^\ell(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi-\sigma\left(\frac{2-\delta}{\phi(\delta-1)}\int_\wp^\aleph\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\right.\\&\quad\left.\left.+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_\wp^\aleph(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\right)\right)\\&\quad+\frac{2-\delta}{\phi(\delta-1)}\int_\wp^\pi\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_\wp^\pi(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi,\\\kappa(\pi)&=\frac{(\pi-\wp)}{\mathfrak{q}}\left(\frac{2-\delta}{\phi(\delta-1)}\int_\wp^\ell\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi\right.\\&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_\wp^\ell(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi-\lambda\left(\frac{2-\delta}{\phi(\delta-1)}\int_\wp^\aleph\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi\right.\\&\quad\left.\left.+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_\wp^\aleph(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi\right)\right)\\&\quad+\frac{2-\delta}{\phi(\delta-1)}\int_\wp^\pi\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_\wp^\pi(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi.\end{aligned}$$

Proof. Let ϑ and κ be the solution of (1.2). Lemmas 2.12 and 2.13 will be helpful to get the solution to (1.2) given as (3.4), where ${}^{\mathcal{AB}}\mathcal{J}^{\delta}_{\wp^+}\mathfrak{g}_1(\pi)$ is defined in (2.2). □

Theorem 3.4. *Suppose that $\mathfrak{g}_1,\mathfrak{g}_2$ are given nonlinear continuous functions such that*

(D1) *$\forall\, \vartheta, \bar{\vartheta}, \kappa, \bar{\kappa} \in \mathcal{C}(\mathcal{J}, \mathbb{R})$ and $\pi \in \mathcal{J}$, there exists $k_1, k_2 > 0$ such that*

$$|\mathfrak{g}_1(\pi, \vartheta(\pi), \vartheta(r\pi), \kappa(\pi)) - \mathfrak{g}_1(\pi, \bar{\vartheta}(\pi), \bar{\vartheta}(r\pi), \bar{\kappa}(\pi))| \leqslant \frac{k_1}{6}(2|\vartheta(\pi) - \bar{\vartheta}(\pi)| + |\kappa(\pi) - \bar{\kappa}(\pi)|)$$

and

$$|\mathfrak{g}_2(\pi, \vartheta(\pi), \kappa(\pi), \kappa(r\pi),) - \mathfrak{g}_2(\pi, \bar{\vartheta}(\pi), \bar{\kappa}(\pi), \bar{\kappa}(r\pi))| \leqslant \frac{k_2}{6}(|\vartheta(\pi) - \bar{\vartheta}(\pi)| + 2|\kappa(\pi) - \bar{\kappa}(\pi)|);$$

(D2) *if $U_{abc} + W_{abc} < 1$ and*

$$\left(\left(\frac{\mathfrak{q}_3(\pi - \wp)}{\mathfrak{q}}\right) + \mathfrak{q}_4\right)\left(\frac{k_1}{3} + \frac{k_2}{3}\right) < 1,$$

where

$$\begin{aligned} U_{abc} &= k_1\left(\frac{\mathfrak{q}_3(\pi - \wp)}{\mathfrak{q}} + \mathfrak{q}_4\right), \qquad W_{abc} = k_2\left(\frac{\mathfrak{q}_3(\pi - \wp)}{\mathfrak{q}} + \mathfrak{q}_4\right), \\ \mathfrak{q}_3 &= \frac{2-\delta}{\phi(\delta-1)}(\ell - \wp) + \frac{\delta - 1}{\phi(\delta-1)\Gamma(\delta+1)}\left((\pi - \ell)^{\delta} - (\pi - \wp)^{\delta}\right) \\ &\quad - \sigma\left(\frac{2-\delta}{\phi(\delta-1)}(\aleph - \wp) + \frac{\delta-1}{\phi(\delta-1)\Gamma(\delta+1)}\left((\pi - \aleph)^{\delta} - (\pi - \wp)^{\delta}\right)\right), \\ \mathfrak{q}_4 &= \frac{2-\delta}{\phi(\delta-1)}(\pi - \wp) + \frac{1-\delta}{\phi(\delta-1)\Gamma(\delta+1)}(\pi - \wp)^{\delta}. \end{aligned}$$

Then the coupled $\mathcal{ABC}$ type fractional derivatives (1.2) have a unique solution.

Proof. By the same method of Theorem 3.1, one can prove the above Theorem. □

4. Stability results

This section deals with the establishing results about $\mathcal{UH}$ and generalized $\mathcal{GUH}$ stabilities for considered coupled $\mathcal{ABR}$-type fractional differential equation 1.1 and $\mathcal{ABC}$-type fractional differential equation 1.2. To achieve our desired result, consider the following inequalities for $\epsilon > 0$:

$$|{}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\vartheta(\pi) - \mathfrak{g}_1(\pi, \vartheta(\pi), \vartheta(r\pi), \kappa(\pi))| \leqslant \epsilon, \quad |{}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\kappa(\pi) - \mathfrak{g}_2(\pi, \vartheta(\pi), \kappa(\pi), \kappa(r\pi))| \leqslant \epsilon. \tag{4.1}$$

Let us introduce the definition as follows.

Definition 4.1. The coupled $\mathcal{ABR}$-type fractional differential equations (1.1) are $\mathcal{UH}$ stable, if there exists a real number $S > 0$ so that for each $\epsilon > 0$ and for $\vartheta, \kappa \in \mathcal{C}(\mathcal{J}, \mathbb{R})$ of (4.1), there is a unique solution $\bar{\vartheta}, \bar{\kappa} \in \mathcal{C}(\mathcal{J}, \mathbb{R})$ of the suggested problem (1.1) so that $|\vartheta(\pi) - \bar{\vartheta}(\pi)| + |\kappa(\pi) - \bar{\kappa}(\pi)| \leqslant S\epsilon$. Also, the coupled $\mathcal{ABR}$-type fractional differential equations (1.1) are $\mathcal{GUH}$ stable, then the function $\Upsilon : \mathbb{R}^+ \to \mathbb{R}^+$ with $\Upsilon(0) = 0$ is such that $|\vartheta(\pi) - \bar{\vartheta}(\pi)| + |\kappa(\pi) - \bar{\kappa}(\pi)| \leqslant \Upsilon\epsilon$.

Remark 4.2. The functions $\bar{\vartheta}, \bar{\kappa} \in \mathcal{C}(\mathcal{J}, \mathbb{R})$ are a solution to equation (4.1) if continuous functions $U, V : \mathcal{J} \to \mathbb{R}$ can be identified depending on ϑ and κ, correspondingly, such that

(S1) $U(\pi) \leqslant \frac{\epsilon}{2}$ and $V(\pi) \leqslant \frac{\epsilon}{2}$, for all $\pi \in \mathcal{J}$;

(S2) ${}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\vartheta(\pi) = \mathfrak{g}_1(\pi, \vartheta(\pi), \vartheta(r\pi), \kappa(\pi)) + U(\pi), \pi \in \mathcal{J}$;

(S3) ${}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\kappa(\pi) = \mathfrak{g}_2(\pi, \vartheta(\pi), \kappa(\pi), \kappa(r\pi)) + V(\pi), \pi \in \mathcal{J}$.

Lemma 4.3. *If $\bar{\vartheta}$ and $\bar{\kappa}$ are a solution to equation (4.1) and $\bar{\vartheta}$ and $\bar{\kappa}$ satisfy the below conditions:*

$$\begin{aligned}&|\vartheta(\pi)-\Psi_1-\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\&\quad-\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi|\leqslant\frac{\epsilon}{2}\Big(\frac{(\pi-\wp)}{\mathfrak{q}}\mathfrak{q}_1+\mathfrak{q}_2\Big),\\&|\kappa(\pi)-\Psi_2+\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi\\&\quad+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi|\leqslant\frac{\epsilon}{2}\Big(\frac{(\pi-\wp)}{\mathfrak{q}}\mathfrak{q}_1+\mathfrak{q}_2\Big),\end{aligned}$$

where

$$\begin{aligned}\Psi_1=\frac{(\pi-\wp)}{\mathfrak{q}}\Big(&\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\ell}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\ell}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\\&-\sigma\Big(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\aleph}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\aleph}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi\Big)\Big)\end{aligned}$$

and

$$\begin{aligned}\Psi_2=\frac{(\pi-\wp)}{\mathfrak{q}}\Big(&\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\ell}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\ell}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi\\&-\lambda\Big(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\aleph}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\aleph}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi\Big)\Big).\end{aligned}$$

(???)

Proof. In the light of Theorem 3.1 and Remark 4.2, we have

$$\begin{aligned}\vartheta(\pi)=\frac{(\pi-\wp)}{\mathfrak{q}}\Big(&\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\ell}(\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))+\mathsf{U}(\chi))d\chi\\&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\ell}(\pi-\chi)^{\delta-1}(\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))+\mathsf{U}(\chi))d\chi\\&-\sigma\Big(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\aleph}(\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))+\mathsf{U}(\chi))d\chi\\&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\aleph}(\pi-\chi)^{\delta-1}(\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))+\mathsf{U}(\chi))d\chi\Big)\Big)\\&+\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}(\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))+\mathsf{U}(\chi))d\chi\\&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}(\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))+\mathsf{U}(\chi))d\chi,\end{aligned}$$

which implies that

$$\begin{aligned}&|\vartheta(\pi)-\Psi_1-\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi-\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi|\\&\leqslant\frac{(\pi-\wp)}{\mathfrak{q}}\Big(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\ell}|\mathsf{U}(\chi)|d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\ell}(\pi-\chi)^{\delta-1}|\mathsf{U}(\chi)|d\chi-\sigma\Big(\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\aleph}|\mathsf{U}(\chi)|d\chi\end{aligned}$$

$$
\begin{aligned}
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\aleph}(\pi-\chi)^{\delta-1}|U(\chi)|d\chi\Big)\Big)+\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}|U(\chi)|d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}|U(\chi)|d\chi\\
&\leqslant\frac{\epsilon(\pi-\wp)}{2q}\Big(\frac{2-\delta}{\phi(\delta-1)}(\ell-\wp)+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\frac{((\pi-\ell)^{\delta}-(\pi-\wp)^{\delta})}{\delta}-\sigma\Big(\frac{2-\delta}{\phi(\delta-1)}(\aleph-\wp)\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\frac{((\pi-\aleph)^{\delta}-(\pi-\wp)^{\delta})}{\delta}\Big)\Big)+\frac{\epsilon(2-\delta)}{2\phi(\delta-1)}(\pi-\wp)+\frac{\epsilon(1-\delta)}{2\phi(\delta-1)\Gamma(\delta)}\frac{(\pi-\wp)^{\delta}}{\delta}\\
&=\frac{\epsilon}{2}\Big(\frac{(\pi-\wp)}{q}\Big(\frac{2-\delta}{\phi(\delta-1)}(\ell-\wp)+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta+1)}((\pi-\ell)^{\delta}-(\pi-\wp)^{\delta})-\sigma\Big(\frac{2-\delta}{\phi(\delta-1)}(\aleph-\wp)\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta+1)}((\pi-\aleph)^{\delta}-(\pi-\wp)^{\delta})\Big)\Big)+\frac{\epsilon}{2}\Big(\frac{(2-\delta)}{\phi(\delta-1)}(\pi-\wp)+\frac{(1-\delta)}{\phi(\delta-1)\Gamma(\delta+1)}(\pi-\wp)^{\delta}\Big)\\
&=\frac{\epsilon}{2}\Big(\frac{(\pi-\wp)}{q}q_1+q_2\Big).
\end{aligned}
$$

Similarly, one can prove that

$$
\begin{aligned}
|\kappa(\pi)-\Psi_2+\frac{2-\delta}{\phi(\delta-1)}&\int_{\wp}^{\pi}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi\\
&+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi|\leqslant\frac{\epsilon}{2}\Big(\frac{(\pi-\wp)}{q}q_1+q_2\Big).
\end{aligned}
$$

□

Theorem 4.4. *Let condition (B1) holds. Then the coupled $\mathcal{ABR}$-type fractional differential equations (1.1) is $\mathcal{UH}$ stable if the following hypothesis holds:*

$$
\Big(\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp)+\frac{1-\delta}{\phi(\delta-1)\Gamma(\delta+1)}(\pi-\wp)^{\delta}\Big)\Big(\frac{k_1}{3}+\frac{k_2}{3}\Big)<1.
$$

Proof. Let us assume that $\epsilon>0$ and $\vartheta,\kappa\in\mathcal{C}(\mathcal{J},\mathbb{R})$ are functions satisfying (4.1). Let $\bar{\vartheta},\bar{\kappa}\in\mathcal{C}(\mathcal{J},\mathbb{R})$ be a unique solution to the following coupled system:

$$
\begin{cases}
{}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\vartheta(\pi)=\mathfrak{g}_1(\pi,\vartheta(\pi),\vartheta(r\pi),\kappa(\pi)), & \pi\in[\wp,\ell],\delta\in(2,3],\\
{}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\kappa(\pi)=\mathfrak{g}_2(\pi,\vartheta(\pi),\kappa(\pi),\kappa(r\pi)), & \pi\in[\wp,\ell],\delta\in(2,3],\\
\vartheta(\wp)=\bar{\vartheta}(\wp)=0,\vartheta(\ell)=\bar{\vartheta}(\ell)=\sigma\vartheta(\aleph), & \aleph\in(\wp,\ell),\\
\kappa(\wp)=\bar{\kappa}(\wp)=0,\kappa(\ell)=\bar{\kappa}(\wp)=\lambda\kappa(\aleph_1), & \aleph_1\in(\wp,\ell).
\end{cases}
\tag{4.2}
$$

From Lemma 2.13, we have

$$
\vartheta(\pi)=\Psi_1+\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))d\chi
$$

and

$$
\kappa(\pi)=\Psi_2+\frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi+\frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))d\chi.
$$

From our assumptions in (4.2), we have $\Psi_1=\overline{\Psi}_1$ and $\Psi_2=\overline{\Psi}_2$. Hence, by Theorem 3.1 and Lemma 4.3, we obtain

$|\vartheta(\pi)-\bar{\vartheta}(\pi)|$

$$
\begin{aligned}
&= |\vartheta(\pi) - \Psi_1 - \frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi} \mathfrak{g}_1(\chi, \bar{\vartheta}(\chi), \bar{\vartheta}(r\chi), \bar{\kappa}(\chi))d\chi - \frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_1(\chi, \bar{\vartheta}(\chi), \bar{\vartheta}(r\chi), \bar{\kappa}(\chi))d\chi| \\
&\leqslant \frac{\epsilon}{2}\left(\frac{(\pi-\wp)}{\mathfrak{q}}\mathfrak{q}_1 + \mathfrak{q}_2\right) + \frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi} |\mathfrak{g}_1(\chi, \vartheta(\chi), \vartheta(r\chi), \kappa(\chi)) - \mathfrak{g}_1(\chi, \bar{\vartheta}(\chi), \bar{\vartheta}(r\chi), \bar{\kappa}(\chi))|d\chi \\
&\quad + \frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}|\mathfrak{g}_1(\chi, \vartheta(\chi), \vartheta(r\chi), \kappa(\chi)) - \mathfrak{g}_1(\chi, \bar{\vartheta}(\chi), \bar{\vartheta}(r\chi), \bar{\kappa}(\chi))|d\chi \\
&\leqslant \frac{\epsilon}{2}\left(\frac{(\pi-\wp)}{\mathfrak{q}}\mathfrak{q}_1 + \mathfrak{q}_2\right) + \frac{2-\delta}{\phi(\delta-1)}(\pi-\wp)\frac{k_1}{6}(2|\vartheta(\pi) - \bar{\vartheta}(\pi)| + |\kappa(\pi) - \bar{\kappa}(\pi)|) \\
&\quad + \frac{1-\delta}{\phi(\delta-1)\Gamma(\delta+1)}(\pi-\wp)^{\delta}\frac{k_1}{6}(2|\vartheta(\pi) - \bar{\vartheta}(\pi)| + |\kappa(\pi) - \bar{\kappa}(\pi)|) \\
&\leqslant \frac{\epsilon}{2}\left(\frac{(\pi-\wp)}{\mathfrak{q}}\mathfrak{q}_1 + \mathfrak{q}_2\right) + \left(\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp) + \frac{1-\delta}{\phi(\delta-1)\Gamma(\delta+1)}(\pi-\wp)^{\delta}\right)\frac{k_1}{3}(|\vartheta(\pi) - \bar{\vartheta}(\pi)| + |\kappa(\pi) - \bar{\kappa}(\pi)|).
\end{aligned}
$$

Similarly, we can prove that

$$
\begin{aligned}
|\kappa(\pi) - \bar{\kappa}(\pi)| &= |\kappa(\pi) - \Psi_1 - \frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi} \mathfrak{g}_2(\chi, \bar{\vartheta}(\chi), \bar{\kappa}(\chi), \bar{\kappa}(r\chi))d\chi \\
&\quad - \frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}\mathfrak{g}_2(\chi, \bar{\vartheta}(\chi), \bar{\kappa}(\chi), \bar{\kappa}(r\chi))d\chi| \\
&\leqslant \frac{\epsilon}{2}\left(\frac{(\pi-\wp)}{\mathfrak{q}}\mathfrak{q}_1 + \mathfrak{q}_2\right) + \frac{2-\delta}{\phi(\delta-1)}\int_{\wp}^{\pi} |\mathfrak{g}_2(\chi, \vartheta(\chi), \kappa(\chi), \kappa(r\chi)) - \mathfrak{g}_2(\chi, \bar{\vartheta}(\chi), \bar{\kappa}(\chi), \bar{\kappa}(r\chi))|d\chi \\
&\quad + \frac{\delta-1}{\phi(\delta-1)\Gamma(\delta)}\int_{\wp}^{\pi}(\pi-\chi)^{\delta-1}|\mathfrak{g}_2(\chi, \vartheta(\chi), \kappa(\chi), \kappa(r\chi)) - \mathfrak{g}_2(\chi, \bar{\vartheta}(\chi), \bar{\kappa}(\chi), \bar{\kappa}(r\chi))|d\chi \\
&\leqslant \frac{\epsilon}{2}\left(\frac{(\pi-\wp)}{\mathfrak{q}}\mathfrak{q}_1 + \mathfrak{q}_2\right) + \frac{2-\delta}{\phi(\delta-1)}(\pi-\wp)\frac{k_2}{6}(|\vartheta(\pi) - \bar{\vartheta}(\pi)| + 2|\kappa(\pi) - \bar{\kappa}(\pi)|) \\
&\quad + \frac{1-\delta}{\phi(\delta-1)\Gamma(\delta+1)}(\pi-\wp)^{\delta}\frac{k_2}{6}(|\vartheta(\pi) - \bar{\vartheta}(\pi)| + 2|\kappa(\pi) - \bar{\kappa}(\pi)|) \\
&\leqslant \frac{\epsilon}{2}\left(\frac{(\pi-\wp)}{\mathfrak{q}}\mathfrak{q}_1 + \mathfrak{q}_2\right) + \left(\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp) + \frac{1-\delta}{\phi(\delta-1)\Gamma(\delta+1)}(\pi-\wp)^{\delta}\right)\frac{k_2}{3}(|\vartheta(\pi) - \bar{\vartheta}(\pi)| \\
&\quad + |\kappa(\pi) - \bar{\kappa}(\pi)|).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|\vartheta(\pi) - \bar{\vartheta}(\pi)\| + \|\kappa(\pi) - \bar{\kappa}(\pi)\| &\leqslant \epsilon\left(\frac{(\pi-\wp)}{\mathfrak{q}}\mathfrak{q}_1 + \mathfrak{q}_2\right) + \left(\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp)\right. \\
&\quad \left. + \frac{1-\delta}{\phi(\delta-1)\Gamma(\delta+1)}(\pi-\wp)^{\delta}\right)\left(\frac{k_1}{3} + \frac{k_2}{3}\right)(\|\vartheta(\pi) - \bar{\vartheta}(\pi)\| + \|\kappa(\pi) - \bar{\kappa}(\pi)\|).
\end{aligned}
$$

Consequently,

$$
\|\vartheta(\pi) - \bar{\vartheta}(\pi)\| + \|\kappa(\pi) - \bar{\kappa}(\pi)\| \leqslant \frac{\epsilon\left(\frac{(\pi-\wp)}{\mathfrak{q}}\mathfrak{q}_1 + \mathfrak{q}_2\right)}{1 - \left(\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp) + \frac{1-\delta}{\phi(\delta-1)\Gamma(\delta+1)}(\pi-\wp)^{\delta}\right)\left(\frac{k_1}{3} + \frac{k_2}{3}\right)} = S\epsilon,
$$

where

$$
S = \frac{\left(\frac{(\pi-\wp)}{\mathfrak{q}}\mathfrak{q}_1 + \mathfrak{q}_2\right)}{1 - \left(\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp) + \frac{1-\delta}{\phi(\delta-1)\Gamma(\delta+1)}(\pi-\wp)^{\delta}\right)\left(\frac{k_1}{3} + \frac{k_2}{3}\right)}.
$$

Let us consider $\Upsilon(\epsilon) = S\epsilon$. Thus $\Upsilon(0) = 0$, then the coupled $\mathcal{ABR}$-type fractional differential equations (1.1) are $\mathcal{GUH}$ stable. □

Theorem 4.5. *Let condition (B1) hold. Then the coupled $\mathcal{ABC}$-type fractional differential equations (1.2) are $\mathcal{UH}$ stable if the following hypothesis holds:*

$$\left(\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp)+\frac{1-\delta}{\phi(\delta-1)\Gamma(\delta+1)}(\pi-\wp)^{\delta}\right)\left(\frac{k_1}{3}+\frac{k_2}{3}\right)<1.$$

Proof. From Theorem 4.4 we can derive the proof of the theorem as

$$\|\vartheta(\pi)-\bar{\vartheta}(\pi)\|+\|\kappa(\pi)-\bar{\kappa}(\pi)\|\leqslant S^{*}\epsilon,$$

where

$$S^{*}=\frac{\left(\frac{(\pi-\wp)}{q}q_3+q_4\right)}{1-\left(\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp)+\frac{1-\delta}{\phi(\delta-1)\Gamma(\delta+1)}(\pi-\wp)^{\delta}\right)\left(\frac{k_1}{3}+\frac{k_2}{3}\right)}.$$

Let us consider $\Upsilon(\epsilon) = S^*\epsilon$. Thus $\Upsilon(0) = 0$, then the coupled $\mathcal{ABC}$-type fractional differential equation (1.2) are $\mathcal{GUH}$ stable. □

Example 4.1. For $\delta \in (2, 3]$, consider the following system:

$$\begin{cases} {}^{\mathcal{ABR}}\mathcal{D}^{2.6}_{0^+}\vartheta(\pi)=\frac{1}{\exp^{\pi}}+\frac{1}{6}\sin|\vartheta|-\frac{1}{12}\sin|\kappa|,\ \pi\in\mathcal{J},\\ {}^{\mathcal{ABR}}\mathcal{D}^{2.6}_{0^+}\kappa(\pi)=\frac{1}{\exp^{2\pi}}+\frac{1}{18}\cos|\vartheta|-\frac{1}{9}\cos|\kappa|,\ \pi\in\mathcal{J},\\ \vartheta(0)=0,\ \vartheta(1)=0.8\vartheta(0.8),\\ \kappa(0)=0,\ \kappa(1)=0.7\kappa(0.7). \end{cases} \tag{4.3}$$

Here, $\mathcal{J} = [0, 1]$, $\delta = 2.6 \in (2, 3]$, $\wp = 0$, $\ell = 1$, $\sigma = \aleph = 0.8 \in (0, 1), \lambda = \aleph_1 = 0.7 \in (0, 1)$,

$$\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))=\frac{1}{\exp^{\pi}}+\frac{1}{6}\sin|\vartheta|-\frac{1}{12}\sin|\kappa|$$

and

$$\mathfrak{g}_2(\chi,\vartheta(\chi),\kappa(\chi),\kappa(r\chi))=\frac{1}{\exp^{2\pi}}+\frac{1}{18}\cos|\vartheta|-\frac{1}{9}\cos|\kappa|.$$

If we consider $\pi \in \mathcal{J}$ and $\vartheta, \kappa, \bar{\vartheta}, \bar{\kappa} \in \mathbb{R}$, we can find that

$$|\mathfrak{g}_1(\chi,\vartheta(\chi),\vartheta(r\chi),\kappa(\chi))-\mathfrak{g}_1(\chi,\bar{\vartheta}(\chi),\bar{\vartheta}(r\chi),\bar{\kappa}(\chi))|=\frac{1}{12}(2|\vartheta-\bar{\vartheta}|+|\kappa-\bar{\kappa}|)$$

and

$$|\mathfrak{g}_2(\chi,\vartheta(\chi),,\kappa(\chi),\kappa(r\chi))-\mathfrak{g}_2(\chi,\bar{\vartheta}(\chi),\bar{\kappa}(\chi),\bar{\kappa}(r\chi))|=\frac{1}{18}(|\vartheta-\bar{\vartheta}|+2|\kappa-\bar{\kappa}|).$$

Therefore, condition (B1) is fulfilled with $k_1 = \frac{1}{2}$ and $k_2 = \frac{1}{3}$. Thus, all of Theorem 3.1's assumptions hold. Thus, on $[0, 1]$, there exists a unique solution to the coupled $\mathcal{ABR}$ fractional problem (4.3). Additionally, for every $\epsilon = \max\{\epsilon_1, \epsilon_2\}$ and each $\vartheta, \kappa \in \mathcal{C}(\mathcal{J}, \mathbb{R})$ satisfying

$$|{}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\vartheta(\pi) - \mathfrak{g}_1(\pi, \vartheta(\pi), \vartheta(r\pi), \kappa(\pi))| \leqslant \frac{\epsilon}{2} \quad \text{and} \quad |{}^{\mathcal{ABR}}\mathcal{D}^{\delta}_{\wp^+}\kappa(\pi) - \mathfrak{g}_2(\pi, \vartheta(\pi), \kappa(\pi), \kappa(r\pi))| \leqslant \frac{\epsilon}{2},$$

there are a solution $\bar{\vartheta}$, $\bar{\kappa} \in \mathcal{C}(\mathcal{J}, \mathbb{R})$ of the coupled $\mathcal{ABR}$ fractional problem (4.3) with

$$\|\vartheta(\pi) - \bar{\vartheta}(\pi)\| + \|\kappa(\pi) - \bar{\kappa}(\pi)\| \leqslant S\epsilon,$$

where S can be easily calculated from

$$S = \frac{\left(\frac{(\pi-\wp)}{q} q_1 + q_2\right)}{1 - \left(\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp) + \frac{1-\delta}{\phi(\delta-1)\Gamma(\delta+1)}(\pi-\wp)^{\delta}\right)\left(\frac{k_1}{3} + \frac{k_2}{3}\right)} > 0.$$

Thus, all the axioms of Theorem 4.4 are executed. Hence, (4.3) of the coupled $\mathcal{ABR}$ fractional problem is $\mathcal{UH}$ stable.

Example 4.2. For $\delta \in (2, 3]$, consider the following system:

$$\begin{cases} {}^{\mathcal{ABR}}\mathcal{D}^{2.8}_{0^+}\vartheta(\pi) = \frac{1}{\exp^{\pi}} + \frac{1}{6}\sin|\vartheta| - \frac{1}{12}\sin|\kappa|, \ \pi \in \mathcal{J}, \\ {}^{\mathcal{ABR}}\mathcal{D}^{2.8}_{0^+}\kappa(\pi) = \frac{1}{\exp^{2\pi}} + \frac{1}{18}\cos|\vartheta| - \frac{1}{9}\cos|\kappa|, \ \pi \in \mathcal{J}, \\ \vartheta(0) = 0, \ \vartheta(1) = 0.8\vartheta(0.8), \\ \kappa(0) = 0, \ \kappa(1) = 0.7\kappa(0.7). \end{cases} \tag{4.4}$$

Here, $\mathcal{J} = [0, 1]$, $\delta = 2.8 \in (2, 3]$, $\wp = 0$, $\ell = 1$, $\sigma = \aleph = 0.8 \in (0, 1)$, $\lambda = \aleph_1 = 0.7 \in (0, 1)$,

$$\mathfrak{g}_1(\chi, \vartheta(\chi), \vartheta(r\chi), \kappa(\chi)) = \frac{1}{\exp^{\pi}} + \frac{1}{6}\sin|\vartheta| - \frac{1}{12}\sin|\kappa|$$

and

$$\mathfrak{g}_2(\chi, \vartheta(\chi), \kappa(\chi), \kappa(r\chi)) = \frac{1}{\exp^{2\pi}} + \frac{1}{18}\cos|\vartheta| - \frac{1}{9}\cos|\kappa|.$$

If we consider $\pi \in \mathcal{J}$ and $\vartheta, \kappa, \bar{\vartheta}, \bar{\kappa} \in \mathbb{R}$, we can find that

$$|\mathfrak{g}_1(\chi, \vartheta(\chi), \vartheta(r\chi), \kappa(\chi)) - \mathfrak{g}_1(\chi, \bar{\vartheta}(\chi), \bar{\vartheta}(r\chi), \bar{\kappa}(\chi))| = \frac{1}{12}(2|\vartheta - \bar{\vartheta}| + |\kappa - \bar{\kappa}|)$$

and

$$|\mathfrak{g}_2(\chi, \vartheta(\chi), , \kappa(\chi), \kappa(r\chi)) - \mathfrak{g}_2(\chi, \bar{\vartheta}(\chi), \bar{\kappa}(\chi), \bar{\kappa}(r\chi))| = \frac{1}{18}(|\vartheta - \bar{\vartheta}| + 2|\kappa - \bar{\kappa}|).$$

Therefore, condition (B1) is fulfilled with $k_1 = \frac{1}{2}$ and $k_2 = \frac{1}{3}$. Hence, all assumptions of Theorem 3.4. Thus the coupled $\mathcal{ABC}$ fractional problem (4.4) has a unique solution on $[0, 1]$. Additionally, for every $\epsilon = \max\{\epsilon_1, \epsilon_2\}$ and each $\vartheta, \kappa \in \mathcal{C}(\mathcal{J}, \mathbb{R})$ satisfying

$$|{}^{\mathcal{ABC}}\mathcal{D}^{\delta}_{\wp^+}\vartheta(\pi) - \mathfrak{g}_1(\pi, \vartheta(\pi), \vartheta(r\pi), \kappa(\pi))| \leqslant \frac{\epsilon}{2} \quad \text{and} \quad |{}^{\mathcal{ABC}}\mathcal{D}^{\delta}_{\wp^+}\kappa(\pi) - \mathfrak{g}_2(\pi, \vartheta(\pi), \kappa(\pi), \kappa(r\pi))| \leqslant \frac{\epsilon}{2},$$

there are a solution $\bar{\vartheta}$, $\bar{\kappa} \in \mathcal{C}(\mathcal{J}, \mathbb{R})$ of the coupled $\mathcal{ABC}$ fractional problem (4.4) with

$$\|\vartheta(\pi) - \bar{\vartheta}(\pi)\| + \|\kappa(\pi) - \bar{\kappa}(\pi)\| \leqslant S^*\epsilon,$$

where S^* can be easily calculated from

$$S^* = \frac{\left(\frac{(\pi-\wp)}{q} q_1 + q_2\right)}{1 - \left(\frac{2-\delta}{\phi(\delta-1)}(\pi-\wp) + \frac{1-\delta}{\phi(\delta-1)\Gamma(\delta+1)}(\pi-\wp)^{\delta}\right)\left(\frac{k_1}{3} + \frac{k_2}{3}\right)} > 0.$$

As a result, Theorem 4.5 satisfies all of its conditions. Hence, (4.4) of the coupled $\mathcal{ABC}$ fractional problem is $\mathcal{UH}$ stable.

5. Conclusion

This work investigated the coupled BVP of $\mathcal{ABC}$ and $\mathcal{ABR}$ fractional differential equations, a topic that has not yet been studied by any scholars. We established existence and uniqueness of solutions for the given problem using the Banach fixed point theorem and Krasnoselskii's fixed point theorem. Moreover, $\mathcal{UH}$ and generalized $\mathcal{GUH}$ stability are both investigated.

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