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New oscillation criteria of fourth-order neutral noncanonical differential equations

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Abstract

The purpose of this paper is deal with the oscillatory behavior of solutions of neutral delay differential equations of fourth-order in noncanonical form. We use a different techniques which significantly reduce the number of conditions assuring that all the solutions are oscillates. We provided two examples to demonstrate the power and relevance of our findings.

Keywords: Noncanonical operator, oscillation, neutral differential equation, fourth-order.

2020 MSC: 34C10, 34K11.

1. Introduction

This paper is concerned with the fourth-order nonlinear neutral delay differential equation

$$\mathcal{D}_4 z(\kappa) + q(\kappa) y^{\alpha}(\tau(\kappa)) = 0, \ \kappa \geqslant \kappa_0 > 0, \tag{1.1}$$

where $z(\kappa) = y(\kappa) + \mathcal{P}(\kappa) y(\sigma(\kappa))$, and

$$\mathcal{D}_0 z = z, \ \mathcal{D}_j z = b_j(\kappa)(\mathcal{D}_{j-1} z)', \ \ j = 1, 2, 3, \ \ \mathcal{D}_4 z = (\mathcal{D}_3 z)'.$$

Let us make the following assumptions.

(L_1) $b_j \in \mathcal{C}([\kappa_0, \infty), \mathbb{R})$, $b_j > 0$ for $j = 1, 2, 3$ and holds $\Omega_j(\kappa_0) = \int_{\kappa_0}^{\infty} \frac{d\kappa}{b_j(\kappa)} < \infty$;

(L_2) $\mathcal{P} \in \mathcal{C}([\kappa_0, \infty), \mathbb{R})$ with $0 \leqslant \mathcal{P}(\kappa) \leqslant \mathcal{P} < 1$;

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(L_3) $q \in \mathcal{C}([\kappa_0, \infty), \mathbb{R})$ is non-negative and does not vanish eventually;
(L_4) $\sigma, \tau \in \mathcal{C}^1([\kappa_0, \infty), \mathbb{R})$, $\sigma(\kappa) \leqslant \kappa$, $\tau(\kappa) \leqslant \kappa$, and $\lim_{\kappa\to\infty} \sigma(\kappa) = \lim_{\kappa\to\infty} \tau(\kappa) = \infty$;
(L_5) α is a ratio of odd positive integers.

Let $\kappa_* = \min\{\min_{\kappa \geqslant \kappa_0} \sigma(\kappa), \min_{\kappa\geqslant\kappa_0} \tau(\kappa)\}$. Under a solution of (1.1), we mean a function $y \in \mathcal{C}([\kappa_*, \infty), \mathbb{R})$ such that $\mathcal{D}_j z \in \mathcal{C}^1([\kappa_*, \infty), \mathbb{R})$ for $j = 1, 2, 3$ and satisfies (1.1) on $[\kappa_0, \infty)$. Only we consider the solutions of (1.1) which satisfy $\sup\{|y(\kappa)| : T \leqslant \kappa < \infty\} > 0$ for any $T \geqslant \kappa_0$, and tacitly assuming that (1.1) possesses such solutions. "A solution (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is said to nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory."

Fourth-order functional differential equations have been arrived in many modeling of various physical, biological, engineering and chemical phenomena, see [2, 19, 21, 23] for more details. Further several applications of fourth-order functional differential equations are described in the recent papers [11, 13]. During the past several years, the researchers studied oscillatory behavior of solutions of various classes of functional differential equations. See, for example, the monographs [12, 15], the papers [1, 3–10, 12, 14–18, 20, 22], and the reference therein. From the literature survey, we see that there are numerous outcomes available in the literature for all the solutions of (1.1) oscillate, when $\mathcal{P}(\kappa) \equiv 0$, and

$$\Omega_j(\kappa_0) = \infty, \quad j = 1, 2, 3,$$

or

$$\Omega_3(\kappa_0) < \infty, \quad \Omega_2(\kappa_0) = \Omega_1(\kappa_0) = \infty,$$

or

$$\Omega_3(\kappa_0) = \infty, \quad \Omega_2(\kappa_0) < \infty, \quad \Omega_1(\kappa_0) = \infty,$$

or

$$\Omega_3(\kappa_0) < \infty, \quad \Omega_2(\kappa_0) < \infty, \quad \Omega_1(\kappa_0) = \infty.$$

Very recently in [11], the authors studied the oscillatory properties of (1.1) when $\mathcal{P}(\kappa) \equiv 0$, under condition ($L_1$).

To the greatest of our knowledge, oscillation of (1.1) is unresolved according to the assumption (L_1). This is due to the fact that to get relation between $y(\kappa)$ and the corresponding function $z(\kappa)$ is very difficult but this is needed to obtain for the oscillation criteria of the neutral type equation (1.1). Therefore, in order to cover this gap, we obtained some new criteria for the oscillation of all solutions of (1.1). Also we proposed an innovative approach that will serve as an information source for the less discussed theory for neutral type non-canonical fourth-order differential equations. Finally, we provide two examples that shows the significance of the established results via Euler-type neutral differential equations.

2. Oscillation results

For our convenience, we provide the following list of functions to be used in this paper. Let

$$\Omega_{12}(\kappa) = \int_\kappa^\infty \frac{\Omega_2(s)}{b_1(s)} ds, \quad \Omega_{23}(\kappa) = \int_\kappa^\infty \frac{\Omega_3(s)}{b_2(s)} ds, \quad \Omega_{123}(\kappa) = \int_\kappa^\infty \frac{\Omega_{23}(s)}{b_1(s)} ds,$$

for all $\kappa \geqslant \kappa_0$. We consider an eventually positive solutions of (1.1), since if y satisfies (1.1), then so does $-y$.

Lemma 2.1. *Suppose that* (L_1)-(L_5) *remains true and* y *is an eventually positive solution of* (1.1). *Then* $\exists t_1 \geqslant t_0$, $\ni z > 0$ *and satisfies one of the following eight cases:*

(I) $z > 0$, $\mathcal{D}_1 z > 0$, $\mathcal{D}_2 z > 0$, $\mathcal{D}_3 z > 0$, $\mathcal{D}_4 z \leqslant 0$;
(II) $z > 0$, $\mathcal{D}_1 z > 0$, $\mathcal{D}_2 z > 0$, $\mathcal{D}_3 z < 0$, $\mathcal{D}_4 z \leqslant 0$;

(III) $z > 0$, $\mathcal{D}_1 z > 0$, $\mathcal{D}_2 z < 0$, $\mathcal{D}_3 z > 0$, $\mathcal{D}_4 z \leqslant 0$;
(IV) $z > 0$, $\mathcal{D}_1 z > 0$, $\mathcal{D}_2 z < 0$, $\mathcal{D}_3 z < 0$, $\mathcal{D}_4 z \leqslant 0$;
(V) $z > 0$, $\mathcal{D}_1 z < 0$, $\mathcal{D}_2 z > 0$, $\mathcal{D}_3 z > 0$, $\mathcal{D}_4 z \leqslant 0$;
(VI) $z > 0$, $\mathcal{D}_1 z < 0$, $\mathcal{D}_2 z > 0$, $\mathcal{D}_3 z < 0$, $\mathcal{D}_4 z \leqslant 0$;
(VII) $z > 0$, $\mathcal{D}_1 z < 0$, $\mathcal{D}_2 z < 0$, $\mathcal{D}_3 z > 0$, $\mathcal{D}_4 z \leqslant 0$;
(VIII) $z > 0$, $\mathcal{D}_1 z < 0$, $\mathcal{D}_2 z < 0$, $\mathcal{D}_3 z < 0$, $\mathcal{D}_4 z \leqslant 0$, $\forall\, t \geqslant t_1$.

The proof of the lemma is quite obvious and so we omit it.

First, we find the relation between the function $y(\kappa)$ and its relevant function $z(\kappa)$ when it satisfies any of the eight possible cases as in Lemma 2.1.

Lemma 2.2. *Let $z(\kappa)$ satisfies cases* (I)-(IV) *of Lemma* 2.1, $\forall \kappa \geqslant \kappa_1 \geqslant \kappa_0$. *Then*

$$y(\kappa) \geqslant (1 - \mathcal{P}(\kappa))\, z(\kappa), \ \forall \ \ \kappa \geqslant \kappa_1. \tag{2.1}$$

Proof. By the definition of $z(\kappa)$, we get

$$y(\kappa) = z(\kappa) - \mathcal{P}(\kappa) y(\sigma(\kappa)) \geqslant z(\kappa) - \mathcal{P}(\kappa) z(\sigma(\kappa)). \tag{2.2}$$

Since $z(\kappa)$ is increasing in all cases (I)-(IV) and $\sigma(\kappa) \leqslant \kappa$, we attain from (2.2) that

$$y(\kappa) \geqslant (1 - \mathcal{P}(\kappa))\, z(\kappa), \ \kappa \geqslant \kappa_1.$$

This completes the proof. □

Lemma 2.3. *Assuming $z(\kappa)$ holds in case* (V) *of Lemma* 2.1. *Then*

$$y(\kappa) \geqslant \left(1 - \frac{\mathcal{P}(\kappa)\Omega_{12}(\sigma(\kappa))}{\Omega_{12}(\kappa)}\right) z(\kappa), \ \forall \ \kappa \geqslant \kappa_1 \geqslant \kappa_0. \tag{2.3}$$

Proof. From the monotonicity of $\mathcal{D}_2 z$, we see that

$$-\mathcal{D}_1 z(\kappa) \geqslant \mathcal{D}_1 z(\infty) - \mathcal{D}_1 z(\kappa) = \int_\kappa^\infty \frac{1}{b_2(s)} \mathcal{D}_2 z(s) ds \geqslant \Omega_2(\kappa) \mathcal{D}_2 z(\kappa), \tag{2.4}$$

so,

$$\left(\frac{-\mathcal{D}_1 z(\kappa)}{\Omega_2(\kappa)}\right)' = -\left(\frac{\Omega_2(\kappa)\mathcal{D}_2 z(\kappa) + \mathcal{D}_1 z(\kappa)}{\Omega_2^2(\kappa) b_2(\kappa)}\right) \geqslant 0$$

by (2.4). Therefore

$$\frac{-\mathcal{D}_1 z(\kappa)}{\Omega_2(\kappa)} \text{ is nondecreasing.} \tag{2.5}$$

Now, using (2.5) we obtain

$$z(\kappa) \geqslant -\int_\kappa^\infty \frac{\Omega_2(s)\mathcal{D}_1 z(s)}{b_1(s)\Omega_2(s)} ds \geqslant \frac{-\mathcal{D}_1 z(\kappa)}{\Omega_2(\kappa)} \Omega_{12}(\kappa)$$

and so

$$\left(\frac{z(\kappa)}{\Omega_{12}(\kappa)}\right)' = \frac{\Omega_{12}(\kappa)\mathcal{D}_1 z(\kappa) + \Omega_2(\kappa) z(\kappa)}{b_1(\kappa)\Omega_{12}^2(\kappa)} \geqslant 0.$$

Hence

$$\frac{z(\kappa)}{\Omega_{12}(\kappa)} \text{ is nondecreasing.} \tag{2.6}$$

Using the definition of $z(\kappa)$ and (2.6),

$$y(\kappa) = z(\kappa) - \mathcal{P}(\kappa)y(\sigma(\kappa)) \geqslant z(\kappa) - \mathcal{P}(\kappa)z(\sigma(\kappa)) \geqslant \left(1 - \frac{\mathcal{P}(\kappa)\Omega_{12}(\sigma(\kappa))}{\Omega_{12}(\kappa)}\right) z(\kappa), \text{ for all } \kappa \geqslant \kappa_1.$$

This completes the proof. □

Lemma 2.4. *Let* $z(\kappa)$ *satisfies case* (VI) *of Lemma* 2.1. *Then*

$$y(\kappa) \geqslant \left(1 - \frac{\mathcal{P}(\kappa)\Omega_{123}(\sigma(\kappa))}{\Omega_{123}(\kappa)}\right) z(\kappa), \textit{ for all } \kappa \geqslant \kappa_1 \geqslant \kappa_0. \tag{2.7}$$

Proof. From the monotonicity of $\mathcal{D}_3 z$, we see that

$$\mathcal{D}_2 z(\kappa) = \mathcal{D}_2 z(\infty) - \int_\kappa^\infty \frac{1}{b_3(s)} \mathcal{D}_3 z(s) ds \geqslant -\Omega_3(\kappa)\mathcal{D}_3 z(\kappa). \tag{2.8}$$

Hence

$$\left(\frac{\mathcal{D}_2 z(\kappa)}{\Omega_3(\kappa)}\right)' = \frac{\Omega_3(\kappa)\mathcal{D}_3 z(\kappa) + \mathcal{D}_2 z(\kappa)}{b_3(\kappa)\Omega_3^2(\kappa)} \geqslant 0,$$

which shows that $\frac{\mathcal{D}_2 z(\kappa)}{\Omega_3(\kappa)}$ in nondecreasing. Using this property, we see that

$$-\mathcal{D}_1 z(\kappa) \geqslant \int_\kappa^\infty \frac{1}{b_2(s)} \mathcal{D}_2 z(s) ds \geqslant \frac{\mathcal{D}_2 z(\kappa)}{\Omega_3(\kappa)} \int_\kappa^\infty \frac{\Omega_3(s)}{b_2(s)} ds = \frac{\Omega_{23}(\kappa)}{\Omega_3(\kappa)} \mathcal{D}_2 z(\kappa).$$

Hence

$$\left(-\frac{\mathcal{D}_1 z(\kappa)}{\Omega_{23}(\kappa)}\right)' = \frac{-\Omega_{23}(\kappa)\mathcal{D}_2 z(\kappa) - \Omega_3(\kappa)\mathcal{D}_1 z(\kappa)}{b_2(\kappa)\Omega_{23}^2(\kappa)} \geqslant 0$$

and so $\frac{-\mathcal{D}_1 z(\kappa)}{\Omega_{23}(\kappa)}$ is nondecreasing. Finally, one obtains

$$z(\kappa) \geqslant -\int_\kappa^\infty \frac{1}{b_1(s)} \mathcal{D}_1 z(s) ds \geqslant \frac{-\mathcal{D}_1 z(\kappa)}{\Omega_{23}(\kappa)} \int_\kappa^\infty \frac{\Omega_{23}(s)}{b_1(s)} ds = \frac{-\Omega_{123}(\kappa)}{\Omega_{23}(\kappa)} \mathcal{D}_1 z(\kappa).$$

Hence

$$\left(\frac{z(\kappa)}{\Omega_{123}(\kappa)}\right)' = \frac{\Omega_{123}(\kappa)\mathcal{D}_1 z(\kappa) + \Omega_{23}(\kappa) z(\kappa)}{b_1(\kappa)\Omega_{123}^2(\kappa)} \geqslant 0$$

and so

$$\frac{z(\kappa)}{\Omega_{123}(\kappa)} \text{ is nondecreasing.} \tag{2.9}$$

Now using the definition of $z(\kappa)$ and (2.9), we get

$$y(\kappa) = z(\kappa) - \mathcal{P}(\kappa)y(\sigma(\kappa)) \geqslant \left(1 - \frac{\mathcal{P}(\kappa)\Omega_{123}(\sigma(\kappa))}{\Omega_{123}(\kappa)}\right) z(\kappa), \ \kappa \geqslant \kappa_1.$$

Hence, the proof is complete. □

Lemma 2.5. *Let* $z(\kappa)$ *satisfies either case* (VII) *or* (VIII) *of Lemma* 2.1. *Then*

$$y(\kappa) \geqslant \left(1 - \frac{\mathcal{P}(\kappa)\Omega_1(\sigma(\kappa))}{\Omega_1(\kappa)}\right) z(\kappa), \textit{ for all } \kappa \geqslant \kappa_1 \geqslant \kappa_0. \tag{2.10}$$

Proof. From the monotonic property of $\mathcal{D}_1 z$, we have

$$z(\kappa) \geqslant z(\kappa) - z(\infty) = -\int_t^\infty \frac{\mathcal{D}_1 z(s)}{b_1(s)} ds \geqslant -\Omega_1(\kappa)\mathcal{D}_1 z(\kappa).$$

Now

$$\left(\frac{z(\kappa)}{\Omega_1(\kappa)}\right)' = \frac{\Omega_1(\kappa)\mathcal{D}_1 z(\kappa) + z(\kappa)}{b_1(\kappa)\Omega_1^2(\kappa)} \geqslant 0,$$

which gives $\frac{z(\kappa)}{\Omega_1(\kappa)}$ in nondecreasing. Using this in the definition of $z(\kappa)$, we find that

$$y(\kappa) \geqslant z(\kappa) - \mathcal{P}(\kappa) z(\sigma(\kappa)) \geqslant \left(1 - \frac{\mathcal{P}(\kappa)\Omega_1(\sigma(\kappa))}{\Omega_1(\kappa)}\right) z(\kappa)$$

for all $\kappa \geqslant \kappa_1 \geqslant \kappa_0$. This completes the proof. □

Let us define

$$d(\kappa) = \max\left\{\mathcal{P}(\kappa), \frac{\mathcal{P}(\kappa)\Omega_1(\sigma(\kappa))}{\Omega_1(\kappa)}, \frac{\mathcal{P}(\kappa)\Omega_{12}(\sigma(\kappa))}{\Omega_{12}(\kappa)}, \frac{\mathcal{P}(\kappa)\Omega_{123}(\sigma(\kappa))}{\Omega_{123}(\kappa)}\right\}$$

and assume that $1 - d(\kappa) > 0$, $\kappa \geqslant \kappa_1 \geqslant \kappa_0$. From (2.1), (2.3), (2.7), and (2.10), we obtain

$$y(\kappa) \geqslant (1 - d(\kappa))\, z(\kappa), \tag{2.11}$$

for all $\kappa \geqslant \kappa_1 \geqslant \kappa_0$.

Lemma 2.6. *Let $y(\kappa)$ be an eventually positive solution of* (1.1) *and the corresponding function $z(\kappa)$ satisfies cases* (I)-(VIII) *of Lemma* 2.1*. Then*

$$\mathcal{D}_4 z(\kappa) + q(\kappa)(1 - d(\tau(\kappa)))^\alpha z^\alpha(\tau(\kappa)) \leqslant 0, \ \forall\ \kappa \geqslant \kappa_1 \geqslant \kappa_0. \tag{2.12}$$

Proof. The proof follows from (2.11) and (1.1). To prove our results, let us denote

$$Q(\kappa, \kappa_*) = \int_{\kappa_*}^{\kappa} \frac{1}{b_2(s)} \int_{\kappa_*}^{s} \frac{1}{b_3(u)} \int_{\kappa_*}^{u} q(\nu)(1 - \mathcal{P}(\tau(\nu)))^\alpha d\nu du ds,$$

and

$$Q^*(\kappa, \kappa_*) = \int_{\kappa_*}^{\kappa} \frac{q(s)(1 - d(\tau(s)))^\alpha \Omega_{123}(\tau(s))}{\Omega_3(\tau(s))} ds, \text{ for all } \kappa \geqslant \kappa_* \geqslant \kappa_0.$$

□

Lemma 2.7. *Suppose that* (L_1)-(L_5) *remains true. Let y be an eventually positive solution of* (1.1)*. If*

$$Q(\infty, \kappa_0) = \infty, \tag{2.13}$$

then (I)-(IV) *of Lemma* 2.1 *do not hold.*

Proof. First note that from (L_1) and (2.13), we must have

$$\int_{\kappa_0}^{\infty} \frac{1}{b_3(s)} \int_{\kappa_0}^{s} q(u)(1 - d(\tau(u)))^\alpha du ds = \int_{\kappa_0}^{\infty} q(s)(1 - d(\tau(s)))^\alpha ds = \infty. \tag{2.14}$$

Now assume that $y(\kappa)$ be an eventually positive solution of (1.1) with the corresponding function $z(\kappa)$ is positive which satisfies one of the cases (I)-(IV) from Lemma 2.1. Since $z(\kappa)$ is increasing ∃ a constant

$c > 0$ and a $\kappa_2 \geqslant \kappa_1 \geqslant \kappa_0 \ni z(\tau(\kappa)) \geqslant c$, $\forall \kappa \geqslant \kappa_2$. Using this inequality in (2.12), we obtain

$$-\mathcal{D}_4 z(\kappa) \geqslant c^\alpha q(\kappa)(1-d(\tau(\kappa)))^\alpha, \ \kappa \geqslant \kappa_2. \tag{2.15}$$

Integrating (2.15) form κ_2 to κ, we get

$$-\mathcal{D}_3 z(\kappa) + \mathcal{D}_3 z(\kappa_2) \geqslant c^\alpha \int_{\kappa_2}^{\kappa} q(s)(1-d(\tau(s)))^\alpha ds. \tag{2.16}$$

Considering z is a part of either case (I) or (III), then from (2.14) and (2.16), we find that

$$\mathcal{D}_3 z(\kappa_2) \geqslant c^\alpha \int_{\kappa_2}^{\kappa} q(s)(1-d(\tau(s)))^\alpha ds \to \infty \text{ as } \kappa \to \infty, \tag{2.17}$$

which is a contradiction. For case (II), (2.16) becomes

$$-\mathcal{D}_3 z(\kappa) \geqslant c^\alpha \int_{\kappa_2}^{\kappa} q(s)(1-d(\tau(s)))^\alpha ds,$$

that is

$$-(\mathcal{D}_2 z(\kappa))' \geqslant \frac{c^\alpha}{b_3(\kappa)} \int_{\kappa_2}^{\kappa} q(s)(1-d(\tau(s)))^\alpha ds. \tag{2.18}$$

Taking integration (2.18) form κ_2 to κ, we have

$$\mathcal{D}_2 z(\kappa_2) - \mathcal{D}_2 z(\kappa) \geqslant c^\alpha \int_{\kappa_2}^{\kappa} \frac{1}{b_3(s)} \int_{\kappa_2}^{s} q(u)(1-d(\tau(u)))^\alpha du ds, \tag{2.19}$$

which in view of (2.14) yields

$$\mathcal{D}_2 z(\kappa_2) \geqslant c^\alpha \int_{\kappa_2}^{\kappa} \frac{1}{b_3(s)} \int_{\kappa_2}^{s} q(u)(1-d(\tau(u)))^\alpha du ds \to \infty \text{ as } \kappa \to \infty, \tag{2.20}$$

which is a contradiction. Finally assume that case (IV) holds. As in the last case we arrive at (2.19), that is,

$$-(\mathcal{D}_1 z(\kappa))' \geqslant \frac{c^\alpha}{b_2(\kappa)} \int_{\kappa_2}^{\kappa} \frac{1}{b_3(s)} \int_{\kappa_2}^{s} q(u)(1-d(\tau(u)))^\alpha du ds.$$

Applying integration from t_2 to t in the above inequality, we obtain

$$\mathcal{D}_1 z(\kappa_2) - \mathcal{D}_1 z(\kappa) \geqslant c^\alpha \int_{\kappa_2}^{\kappa} \frac{1}{b_2(s)} \int_{\kappa_2}^{s} \frac{1}{b_3(u)} \int_{\kappa_2}^{u} q(v)(1-d(\tau(v)))^\alpha dv du ds = c^\alpha Q(\kappa, \kappa_2), \tag{2.21}$$

which by virtue of (2.13) yields $\mathcal{D}_1 z(\kappa_2) \geqslant c^\alpha Q(\kappa, \kappa_2) \to \infty$ as $\kappa \to \infty$, which is again a contradiction. This completes the proof. □

In the next theorem, we present a condition which shows that every nonoscillatory solution of (1.1) converges to zero whenever $\kappa \to \infty$.

Theorem 2.8. *Assume that* (L_1)-(L_5) *hold. If*

$$\int_{\kappa_0}^{\infty} \frac{Q(\kappa, \kappa_0)}{b_1(\kappa)} d\kappa = \infty, \tag{2.22}$$

then any solution of y *of* (1.1) *is either oscillatory or* $\lim_{\kappa \to \infty} y(\kappa) = 0$.

Proof. Let $y(\kappa)$ be nonoscillatory solution of (1.1). Then $\exists \kappa_1 \geqslant \kappa_0$, $\ni y(\tau(\kappa)) > 0$ and $y(\sigma(\kappa)) > 0$, $\forall \kappa \geqslant \kappa_1$.

Then $z(\kappa) > 0$ and by Lemma 2.1, eight possible cases may occur for $\kappa \geqslant \kappa_1$. Combining (2.22) with (L_1) gives $\int_{\kappa_0}^{\infty} Q(\kappa,\kappa_0)dt$ cannot be bounded, by Lemma 2.7, (I)-(IV) are not possible.

Suppose that one of the cases (V)-(VIII) hold. Given z is decreasing there exists a finite nonnegative limit $\lim_{\kappa\to\infty} z(\kappa) = c$. Let $c > 0$, $\exists$ a $\kappa_2 \geqslant \kappa_1$, $\ni z(\kappa) \geqslant c$, $\forall \kappa \geqslant \kappa_2$ and inequality (2.12) holds, which contradicts (2.17) in cases (V) and (VII), and (2.20) in case (VI). Hence we conclude that, $c = 0$. Suppose that case (VIII) remains true, then we get (2.21), i.e., $-\mathcal{D}_1 z(\kappa) \geqslant c^{\alpha} Q(\kappa,\kappa_2)$ or $-z'(\kappa) \geqslant \frac{c^{\alpha}}{b_1(\kappa)} Q(\kappa,\kappa_2)$. Integrating the last inequality κ_2 to κ, we get

$$z(\kappa_2) \geqslant c^{\alpha} \int_{\kappa_2}^{\kappa} \frac{Q(s,\kappa_2)}{b_1(s)} ds,$$

which contradicts (2.22) as $\kappa \to \infty$. Thus $c = 0$, that is, $\lim_{\kappa\to\infty} z(\kappa) = 0$. However $y(\kappa) \leqslant z(\kappa)$ implies that $\lim_{\kappa\to\infty} y(\kappa) = 0$. This completes the proof of the theorem. □

In the sequel, we present a condition for the oscillation of all solutions of (1.1).

Theorem 2.9. *Suppose* (L_1)-(L_4) *hold and* $\alpha = 1$ *and* τ *is nondecreasing. If*

$$\lim_{\kappa\to\infty} \sup R(\kappa,\kappa_1) > 1 \tag{2.23}$$

for any $\kappa_1 \geqslant \kappa_0$, *where* $R(\kappa,\kappa_1) = \min\{\Omega_1(\kappa)Q(\kappa,\kappa_1),\ \Omega_3(\kappa)Q^*(\kappa,\kappa_1)\}$, *then* (1.1) *is oscillatory.*

Proof. Let $y(\kappa)$ be an eventually positive solution of (1.1). Then $\exists \kappa_1 \geqslant \kappa_0$, $\ni y(\tau(\kappa)) > 0$ and $y(\sigma(\kappa)) > 0$, $\forall \kappa \geqslant \kappa_1$. Then $z(\kappa) > 0$ and by Lemma 2.1, there are eight possible cases may arise for $\kappa \geqslant \kappa_1$. Initially we note that, by virtue of (L_1), for the validity of (2.23),

$$Q(\infty,\kappa_0) = Q^*(\infty,\kappa_0) = \infty \tag{2.24}$$

is necessary and from Lemma 2.7, the condition (2.24) ensures that cases (I)-(IV) from Lemma 2.1 are impossible.

Assume that case (V) holds. From the proof of Lemma 2.3, we arrive at (2.6). Since $\frac{z(\kappa)}{\Omega_{12}(\kappa)}$ is nondecreasing, there exist constant $c > 0$ and a $\kappa_2 \geqslant \kappa_1$ such that $z(\kappa) \geqslant c\ \Omega_{12}(\kappa)$, $\kappa \geqslant \kappa_2$. Using this property in (2.12), we see that

$$-\mathcal{D}_4 z(\kappa) \geqslant c\ q(\kappa)(1 - d(\tau(\kappa)))\Omega_{12}(\tau(\kappa)),\ \kappa \geqslant \kappa_2.$$

Taking integration from κ_2 to κ, in the last inequality, we get

$$\mathcal{D}_3 z(\kappa_2) \geqslant \mathcal{D}_3 z(\kappa) + c \int_{\kappa_2}^{\kappa} q(s)(1 - d(\tau(s)))\Omega_{12}(\tau(s))ds. \tag{2.25}$$

Taking (L_1) and (2.24) into account, easily we get

$$\infty = Q^*(\infty,\kappa_0) = \int_{\kappa_0}^{\infty} \frac{q(s)(1 - d(\tau(s)))\Omega_{123}(\tau(s))}{\Omega_3(\tau(s))} ds \leqslant \int_{\kappa_0}^{\infty} q(s)(1 - d(\tau(s)))\Omega_{12}(\tau(s))ds. \tag{2.26}$$

Substituting (2.26) in (2.25), we obtain a contradiction as $\kappa \to \infty$.

Suppose case (VI) holds. From the proof of the Lemma 2.4, we have

$$z(\kappa) \geqslant \frac{\Omega_{123}(\kappa)}{\Omega_3(\kappa)} \mathcal{D}_2 z(\kappa),\ \kappa \geqslant \kappa_1. \tag{2.27}$$

Using (2.27) in (2.12) we obtain

$$-\mathcal{D}_4 z(\kappa) \geqslant \frac{q(\kappa)(1 - d(\tau(\kappa)))\Omega_{123}(\kappa)}{\Omega_3(\kappa)} \mathcal{D}_2 z(\tau(\kappa)).$$

Applying integration from κ_1 to κ in the above inequality and using the decreasing property of $\mathcal{D}_2 z(\kappa)$, we have

$$\begin{aligned} -\mathcal{D}_3 z(\kappa) &\geqslant \int_{\kappa_1}^{\kappa} \frac{q(s)(1-d(\tau(s)))\Omega_{123}(\tau(s))}{\Omega_3(\tau(s))} \mathcal{D}_2 z(\tau(s)) ds \\ &\geqslant \mathcal{D}_2 z(\tau(\kappa)) \int_{\kappa_1}^{\kappa} \frac{q(s)(1-d(\tau(s)))\Omega_{123}(\tau(s))}{\Omega_3(\tau(s))} ds \geqslant \mathcal{D}_2 z(\kappa) Q^*(\kappa,\kappa_1). \end{aligned} \tag{2.28}$$

Using (2.8) in (2.28), we obtain

$$-\mathcal{D}_3 z(\kappa) \geqslant -\Omega_3(\kappa) Q^*(\kappa,\kappa_1) \mathcal{D}_3 z(\kappa).$$

First divide by $-D_3 z$ in the last inequality, then taking lim sup of the resulting inequality on both sides, we obtain a contradiction with (2.23).

Once again assume case (VII) holds. Integrating (2.12) from t_1 to t and using the property that $\frac{z(\kappa)}{\Omega_1(\kappa)}$ is nondecreasing, we have

$$\mathcal{D}_3 z(\kappa_1) \geqslant \mathcal{D}_3 z(\kappa) + \int_{\kappa_1}^{\kappa} q(s)(1-d(\tau(s)))z(\tau(s)) ds \geqslant \frac{z(\kappa_1)}{\Omega_1(\kappa_1)} \int_{\kappa_1}^{\kappa} q(s)(1-d(\tau(s)))\Omega_1(s) ds. \tag{2.29}$$

On the other side, using (L_1) and (2.26), it is easy to see that for any constant $K > 0$,

$$\infty = \int_{\kappa_1}^{\infty} q(s)(1-d(\tau(s)))\Omega_{12}(s) ds \leqslant K \int_{\kappa_1}^{\infty} q(s)(1-d(\tau(s)))\Omega_1(s) ds.$$

From this in view of (2.29), we get a contradiction.

Finally assume case (VIII) holds. Integrating (2.12) from κ_1 to κ, we get

$$-\mathcal{D}_3 z(\kappa) \geqslant \int_{\kappa_1}^{\kappa} q(s)(1-d(\tau(s)))z(\tau(s)) ds \geqslant z(\tau(\kappa)) \int_{\kappa_1}^{\kappa} q(s)(1-d(\tau(s))) ds.$$

Dividing the last inequality by $b_3(\kappa)$ and then taking integration from κ_1 to κ, one obtains

$$-\mathcal{D}_2 z(\kappa) \geqslant \int_{\kappa_1}^{\kappa} \frac{z(\tau(s))}{b_3(s)} \int_{\kappa_1}^{s} q(u)(1-d(\tau(u))) du \geqslant z(\tau(\kappa)) \int_{\kappa_1}^{\kappa} \frac{1}{b_3(s)} \int_{\kappa_1}^{s} q(u)(1-d(\tau(u))) du ds.$$

Similarly, we have

$$\begin{aligned} -\mathcal{D}_1 z(\kappa) &\geqslant z(\tau(\kappa)) \int_{\kappa_1}^{s} \frac{1}{b_2(s)} \int_{\kappa_1}^{s} \frac{1}{b_3(u)} \int_{\kappa_1}^{u} q(\nu)(1-d(\tau(\nu))) d\nu du ds \\ &= z(\tau(\kappa)) Q(\kappa,\kappa_1) \geqslant z(\kappa) Q(\kappa,\kappa_1) \geqslant -\Omega_1(\kappa) Q(\kappa,\kappa_1) \mathcal{D}_1 z(\kappa), \end{aligned}$$

that is

$$1 \geqslant \Omega_1(\kappa) Q(\kappa,\kappa_1),$$

which contradicts (2.23). This completes the proof. □

Theorem 2.10. *Suppose* (L_1)-(L_4) *holds and* $\alpha = 1$ *with* τ *is nondecreasing. If*

$$\lim_{\kappa\to\infty} \inf \int_{z(\kappa)}^{\kappa} M(s,t_1) ds > \frac{1}{e} \tag{2.30}$$

for any $\kappa_1 \geqslant \kappa_0$*, where*

$$M(\kappa,\kappa_1) = \min\left\{\frac{Q(\kappa,\kappa_1)}{b_1(\kappa)}, \frac{Q^*(\kappa,\kappa_1)}{b_3(\kappa)}\right\}$$

then (1.1) *is oscillatory.*

Proof. Let $y(\kappa)$ be an eventually positive solution of (1.1). Then $\exists \kappa_1 \geqslant \kappa_0$, $\ni y(\sigma(\kappa)) > 0$ and $y(\tau(\kappa)) > 0$, $\forall \kappa \geqslant \kappa_1$. Then $z(\kappa) > 0$ and eight possible cases may occur for $\kappa \geqslant \kappa_1$ in Lemma 2.1. Initially we note that, for the validity of (2.30),

$$\int_{\kappa_0}^{\infty} M(\kappa,\kappa_1)dt = \infty$$

is necessary and which in view of (L_1) implies (2.24) satisfies. Form Lemma 2.7, it is evident that (I)-(IV) of Lemma 2.1 are not attainable. Now let us think about the possible cases (V)-(VIII) separately.

Since the proof of the cases (V) and (VII) are same in Theorem 2.9 and so omitted. Next, assume case (VI) holds. By Theorem 2.9 (case (VI)), we arrive at (2.28), that is

$$-\mathcal{D}_3 z(\kappa) \geqslant \mathcal{D}_2 z(\tau(\kappa)) \int_{\kappa_1}^{\kappa} \frac{q(s)(1-d(\tau(s)))\Omega_{123}(\tau(s))}{\Omega_3(\tau(s))} ds,$$

that is,

$$x'(\kappa) + \frac{Q^*(\kappa,\kappa_1)}{b_3 z(\kappa)} z(\tau(\kappa)) \leqslant 0, \tag{2.31}$$

where we let $x(\kappa) = \mathcal{D}_2 z(\kappa) > 0$. In view of (2.30),

$$\liminf_{\kappa\to\infty} \int_{\tau(\kappa)}^{\kappa} \frac{Q^*(\kappa,\kappa_1)}{b_3 z(\kappa)} ds > \frac{1}{e},$$

however, by [15, Theorem 2.1.1], the above inequality guarantees that (2.31) does not possess a positive solution, which contradicts our assumption.

Finally, we assume case (VIII) holds. Proceeding as in the proof of Theorem 2.9 case (VIII), we arrive at $-\mathcal{D}_1 z(\kappa) \geqslant z(\tau(\kappa))Q(\kappa,\kappa_1)$ or $z'(\kappa) + \frac{Q(\kappa,\kappa_1)}{b_1(\kappa)} z(\tau(\kappa)) \leqslant 0$, same as case (VII), which is a contradiction. This completes the proof. □

3. Examples

Two examples are presented in this section to highlight the significance of our findings.

Example 3.1. Consider the equation

$$\left(\kappa^2\left(\kappa^2\left(\kappa^2 z'(\kappa)\right)'\right)'\right)' + q_0\kappa^2 y^3(\lambda\kappa) = 0, \ \kappa \geqslant 1, \tag{3.1}$$

where $z(\kappa) = y(\kappa) + \mathcal{P}y(\mu\kappa)$, $q_0 > 0$, $\mu \in (0,1)$, $\lambda(0,1)$, and $\mathcal{P} < \lambda^3$. A simple calculation shows that $\Omega_1(\kappa) = \Omega_2(\kappa) = \Omega_3(\kappa) = \frac{1}{\kappa}$, $\Omega_{12}(\kappa) = \frac{1}{2\kappa^2}$, $\Omega_{23}(\kappa) = \frac{1}{2\kappa^2}$, and $\Omega_{123}(\kappa) = \frac{1}{6\kappa^3}$. With $d(\kappa) = \frac{\mathcal{P}}{\lambda^3}$, we see that condition $Q(\kappa,1) = \frac{q_0}{6}\left(1-\frac{\mathcal{P}}{\lambda^3}\right)^3\kappa$. By Theorem 2.8, the condition (2.22) is satisfied. Thus we conclude that any nonoscillatory solution of (3.1) converges to zero whenever $\kappa \to \infty$.

Example 3.2. Consider the equation

$$\left(\kappa^2\left(\kappa^2\left(\kappa^2\left(y(\kappa) + \frac{1}{16}y\left(\frac{\kappa}{3}\right)\right)'\right)'\right)'\right)' + q_0\kappa^2 y\left(\frac{\kappa}{2}\right) = 0, \ \kappa \geqslant 1, \tag{3.2}$$

where $q_0 > 0$. Here $b_1(\kappa) = b_2(\kappa) = b_3(\kappa) = \kappa^2$, $\mathcal{P}(\kappa) = \frac{1}{16}$, $q(s) = q_0\kappa^2$, $\tau(\kappa) = \frac{\kappa}{2}$, and $\sigma(\kappa) = \frac{\kappa}{3}$. A simple calculation shows that $\Omega_1(\kappa) = \Omega_2(\kappa) = \Omega_3(\kappa) = \frac{1}{\kappa}$, $\Omega_{12}(\kappa) = \Omega_{23}(\kappa) = \frac{1}{2\kappa^2}$, and $\Omega_{123}(\kappa) = \frac{1}{6\kappa^3}$. Further $d(\kappa) = \frac{1}{2}$ and $1 - d(\tau(\kappa)) = \frac{1}{2} > 0$. The condition (2.23) is clearly satisfied if $q_0 > 12$. So Theorem 2.9 implies that (3.2) is oscillatory if $q_0 > 12$. The same conclusion follows from Theorem 2.10 since the condition (2.30) is satisfied for $q_0 > 6.36886$. Hence Theorem 2.10 provides a stronger result than Theorem 2.10. In fact Theorem 2.10 is more efficient and depends on the delay arguments.

4. Conclusion

In this paper, we provide two new criteria for the oscillation of all solutions of (1.1). Also we presented two examples to demonstrate the significance of our findings and none of the results reported in the literature yield this conclusion.

A further extension of this article could be to use this results to study a class of systems of higher-order neutral differential equations as well as fractional-order equations.

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