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New comprehensive two subclasses related to Gregory numbers of analytic bi-univalent functions

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Abstract

In this paper, using subordinations with the functions whose coefficients are Gregory numbers, we present two novel subclasses $\wp_{\Pi}(\vartheta, \gamma, \beta)$, and $\mathcal{F}_{\Pi}(\phi)$ within the bi-univalent function family. We study the estimates $|a_2|$ and $|a_3|$ of the Maclaurin coefficients and the Fekete-Szegö inequality regarding functions in every one of these two subclasses. Following the originality of the characterizations and the proofs may encourage additional research on these kinds of similarly defined analytic bi-univalent function subclasses.

Keywords: Gregory numbers, analytic, univalent, bi-univalent, Fekete-Szegö.

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1. Introduction and preliminaries

Let $\mathcal{H}$ be the class of all analytic functions h in the open unit disk $\Omega = \{\tau \in \mathbb{C} : |\tau| < 1\}$ normalized by $h(0) = h'(0) - 1 = 0$ and has the form:

$$h(\tau) = \tau + \sum_{k=2}^{\infty} a_k \tau^k, \quad (\tau \in \Omega). \tag{1.1}$$

Further, let $\mathcal{D}$ be the class of univalent functions in the class $\mathcal{H}$ (for details, see [9]). Now, $I \prec J$ (the subordination of analytic functions I and J) if for all $\tau \in \Omega$ there exists a function λ with $\lambda(0) = 0$ and $|\lambda(\tau)| < 1$, such that$I(\tau) = J(\lambda(\tau))$. Also, if $I(0) = J(0)$ and $I(\Omega) \subset J(\Omega)$ iff J is univalent in Ω, then $I(\tau) \prec J(\tau)$ (see [11, 15]).

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The inverse function is well-known $J(\tau) = I^{-1}(\tau)$ for analytic and univalent function $I(\tau) : \mathbb{D}_1 \to \mathbb{D}_2$ defined by

$$J(I(\tau)) = \tau, \quad (\tau \in \mathbb{D}_1),$$

is a univalent and analytic function. Furthermore (see [9]), there is an inverse map I^{-1} for any function $I \in \mathcal{D}$, such that

$$I^{-1}(I(\tau)) = \tau \quad (\tau \in \Omega),$$

and

$$I\left(I^{-1}(\lambda)\right) = \lambda \quad \left(|\lambda| < r_0(I); r_0(I) \geqslant \frac{1}{4}\right).$$

Actually, the inverse function is written as

$$I^{-1}(\lambda) = \lambda - a_2\lambda^2 + (2a_2^2 - a_3)\lambda^3 - (5a_2^3 - 5a_2a_3 + a_4)\lambda^4 + \cdots .$$

A function I given by (1.1) will be in the class Π (Π the class of bi-univalent functions in Ω) if $I \in \mathcal{H}$ and both $I(\tau)$ and $I^{-1}(\tau)$ are univalent in Ω. For details of the class Π, see [1, 7, 16, 19, 21, 24, 25].

Examining the class Π, Brannan and Clunie [6] conjectured that $a_2 < \sqrt{2}$, Lewin [13] found that $a_2 < 1.51$, and Netanyahu [18] showed that $\max a_2 = \frac{4}{3}$. Estimating the coefficient a_k of $k \geqslant 3, k \in \mathbb{N}$, is still an open problem.

The estimate for functions in the class Π was done by Tan [22] in 1984, and it is $|a_2| < 1.485$.

It is widely known that the Fekete-Szegöproblem relates to the coefficients of functions in $\mathcal{D}$. The first to address this was [10], if $I \in \Pi$,

$$|a_3 - \sigma a_2^2| \leqslant 1 + 2e^{-2\sigma/(1-\mu)}, \quad \sigma \in \mathbb{R}.$$

Lately, several scholars have begun to examining bi-univalent functions connected to orthogonal polynomials, like [2–5, 12, 23]. Gregory coefficients Θ_k, are the numbers $\frac{1}{2}, \frac{-1}{12}, \frac{1}{24}, \frac{-19}{720}, \frac{3}{160}, \frac{-863}{60.480}, \ldots$. They appear in the expansion of the reciprocal logarithm Maclaurin series

$$\frac{\tau}{\log(\tau+1)} = 1 + \frac{1}{2}\tau - \frac{1}{12}\tau^2 + \frac{1}{24}\tau^3 - \frac{19}{720}\tau^4 + \frac{3}{160}\tau^5 - \frac{863}{60.480}\tau^6 + \cdots .$$

James Gregory in 1670 introduced these numbers and subsequently brought back to life by numerous mathematicians and appear in the works of contemporary authors.

The generating function of the Gregory coefficients Θ_k (see [20]), are given by

$$\Upsilon(\tau) = \frac{\tau}{\log(\tau+1)} = \sum_{k=0}^{\infty} \Theta_k \tau^k = 1 + \frac{1}{2}\tau - \frac{1}{12}\tau^2 + \frac{1}{24}\tau^3 - \frac{19}{720}\tau^4 + \frac{3}{160}\tau^5 - \frac{863}{60.480}\tau^6 + \cdots, \quad \tau \in \Omega. \quad (1.2)$$

Clearly, Θ_k for some values of $k \in \mathbb{N}$ are

$$\Theta_0 = 1, \ \Theta_1 = \frac{1}{2}, \ \Theta_2 = \frac{-1}{12}, \ \Theta_3 = \frac{1}{24}, \ \Theta_4 = \frac{-19}{720}, \ \Theta_5 = \frac{3}{160}, \ \Theta_6 = \frac{-863}{60.480}.$$

The aim of this paper is to make an attempt to improve the initial coefficients $|a_2|$, $|a_3|$, and $\left|a_3 - \xi a_2^2\right|$ for certain subclasses of class Π.

2. The class $\wp_\Pi(\vartheta, \gamma, \beta)$ coefficient bounds

In this section, we examine a subclass $\wp_\Pi(\vartheta, \gamma, \beta)$ using the Gregory coefficients generating functions of analytic bi-univalent functions and obtain initial coefficients $|a_2|$, $|a_3|$, and Fekete-Szegö inequality. In our study, the next two lemmas are employed.

Lemma 2.1 ([8]). *If* $\varepsilon \in \mathcal{V}$, *then* $|s_k| \leqslant 2$, $k \geqslant 1$ *for each* k, *where* ε *analytic in* Ω *and has the form* $\varepsilon(\tau) = 1 + s_1\tau + s_2\tau^2 + \cdots$, *and sharp for all* $k \in \mathbb{N}$.

Lemma 2.2 ([26]). *Let* $j, g \in \mathbb{R}$ *and* $\tau_1, \tau_2 \in \mathbb{C}$. *If* $|\tau_1|, |\tau_2| < \hbar$, *then*

$$|(j+g)\tau_1 + (j-g)\tau_2| \leqslant \begin{cases} 2|j|\hbar, & \text{for } |j| \geqslant |g|, \\ 2|g|\hbar, & \text{for } |j| \leqslant |g|. \end{cases}$$

Definition 2.3. Let $\vartheta \geqslant 0$, $\beta \geqslant 1$, $\gamma \in \mathbb{C}$ and $\text{Re}(\gamma) \geqslant 0$. A function $h \in \Pi$ given by (1.1) is in the class $\wp_\Pi(\vartheta, \gamma, \beta)$ if satisfies the subordinations

$$(1-\vartheta)h'(z) + \vartheta\left(h'(z)\right)^\beta \left(\frac{h(z)}{z}\right)^{\gamma-1} \prec \Upsilon(z) \tag{2.1}$$

and

$$(1-\vartheta)p'(\varpi) + \vartheta\left(p'(\varpi)\right)^\beta \left(\frac{p(\varpi)}{\varpi}\right)^{\gamma-1} \prec \Upsilon(\varpi), \tag{2.2}$$

where $\Upsilon(z)$ is given by (1.2) and $p(\varpi) = h^{-1}(\varpi)$.

Remark 2.4.

1) By utilizing particular values for the parameters ϑ, β, and γ in Definition 2.3, we receive several popular subclasses in $\mathcal{H}$ studied by several authors.
2) If we consider $h_*(\tau) = \frac{\tau}{1-c\tau}$, $|c| \leqslant 1$, then we can check that $h_*(\tau) \in \wp_\Pi(\vartheta, \gamma, \beta)$, so the class $\wp_\Pi(\vartheta, \gamma, \beta)$ is not empty (see [26, Remark 1]).

Theorem 2.5. *If* $h \in \wp_\Pi(\vartheta, \gamma, \beta)$. *Then*

$$|a_2| \leqslant \min\left\{\frac{1}{2\vartheta(2\beta+\gamma-3)+4}, \frac{1}{\sqrt{\vartheta\left((\gamma+2)(\gamma-3)+4\beta(\gamma-1)+2\beta(2\beta+1)\right)+\frac{14}{3}\left(\vartheta(2\beta+\gamma-3)+2\right)^2+6}}\right\},$$

$$|a_3| \leqslant \min\left\{\frac{1}{4\left(\vartheta(2\beta+\gamma-3)+2\right)^2} + \frac{1}{2\vartheta(3\beta+\gamma-4)+6}, \frac{1}{\vartheta\left((\gamma+2)(\gamma-3)+4\beta(\gamma-1)+2\beta(2\beta+1)\right)+\frac{14}{3}\left(\vartheta(2\beta+\gamma-3)+2\right)^2+6} + \frac{1}{2\vartheta(3\beta+\gamma-4)+6}\right\},$$

and

$$\left|a_3 - \xi a_2^2\right| \leqslant \begin{cases} \frac{1}{2\vartheta(3\beta+\gamma-4)+6}, & 0 \leqslant |\Theta(\xi)| \leqslant \frac{1}{8\vartheta(3\beta+\gamma-4)+24}, \\ 4|\Theta(\xi)|, & |\Theta(\xi)| \geqslant \frac{1}{8\vartheta(3\beta+\gamma-4)+24}, \end{cases}$$

where

$$\Theta(\xi) = \frac{1-\xi}{4\vartheta\left((\gamma+2)(\gamma-3)+4\beta(\gamma-1)+2\beta(2\beta+1)\right)+\frac{56}{3}\left(\vartheta(2\beta+\gamma-3)+2\right)^2+24}.$$

Proof. Let $h \in \wp_\Pi(\vartheta, \gamma, \beta)$. From subordinations (2.1) and (2.2), there exists two analytic functions c and b such that $c(0) = 0 = b(0)$ and $|c(z)|, |b(\varpi)| < 1$, while

$$(1-\vartheta)h'(z) + \vartheta\left(h'(z)\right)^\beta \left(\frac{h(z)}{z}\right)^{\gamma-1} = \Upsilon(c(z)), \quad z \in \Omega,$$

and

$$(1-\vartheta)p'(\varpi)+\vartheta\left(p'(\varpi)\right)^{\beta}\left(\frac{p(\varpi)}{\varpi}\right)^{\gamma-1}=\Upsilon(b(\varpi)),\quad \varpi\in\Omega.$$

So, the function

$$t(z)=\frac{1+c(z)}{1-c(z)}=1+s_1z+s_2z^2+\cdots\in\mathcal{V},$$

hence

$$c(z)=\frac{s_1}{2}z+\frac{1}{2}\left(s_2-\frac{s_1^2}{2}\right)z^2+\frac{1}{2}\left(s_3-s_1s_2+\frac{s_1^3}{4}\right)z^3+\cdots$$

and

$$\Upsilon(c(z))=1+\frac{s_1}{4}z+\frac{1}{48}\left(12s_2-7s_1^2\right)z^2+\frac{1}{192}\left(17s_1^3-56s_1s_2+48s_3\right)z^3+\cdots,\ z\in\Omega.$$

Also, the function

$$l(\varpi)=\frac{1+b(\varpi)}{1-b(\varpi)}=1+e_1\varpi+e_2\varpi^2+\cdots\in\mathcal{V},$$

$$b(\varpi)=\frac{e_1}{2}\varpi+\frac{1}{2}\left(e_2-\frac{e_1^2}{2}\right)\varpi^2+\frac{1}{2}\left(e_3-e_1e_2+\frac{e_1^3}{4}\right)\varpi^3+\cdots,$$

and

$$\Upsilon(b(\varpi))=1+\frac{e_1}{4}\varpi+\frac{1}{48}\left(12e_2-7e_1^2\right)\varpi^2+\frac{1}{192}\left(17e_1^3-56e_1e_2+48e_3\right)\varpi^3+\cdots,\ \varpi\in\Omega.$$

Consequently, we get

$$\begin{aligned}&(1-\vartheta)h'(z)+\vartheta\left(h'(z)\right)^{\beta}\left(\frac{h(z)}{z}\right)^{\gamma-1}\\&\quad=1+\frac{s_1}{4}z+\frac{1}{48}\left(12s_2-7s_1^2\right)z^2+\frac{1}{192}\left(17s_1^3-56s_1s_2+48s_3\right)z^3+\cdots,\ z\in\Omega.\end{aligned}$$

and

$$\begin{aligned}&(1-\vartheta)p'(\varpi)+\vartheta\left(p'(\varpi)\right)^{\beta}\left(\frac{p(\varpi)}{\varpi}\right)^{\gamma-1}\\&\quad=1+\frac{e_1}{4}\varpi+\frac{1}{48}\left(12e_2-7e_1^2\right)\varpi^2+\frac{1}{192}\left(17e_1^3-56e_1e_2+48e_3\right)\varpi^3+\cdots,\ \varpi\in\Omega.\end{aligned}$$

Comparing the coefficients in the last two equations, we have

$$\left(\vartheta\left(2\beta+\gamma-3\right)+2\right)a_2=\frac{s_1}{4},\tag{2.3}$$

$$\left[\vartheta\left(\frac{(\gamma-1)(\gamma-2)}{2}+2\beta(\gamma-1)+2\beta(\beta-1)\right)\right]a_2^2+\left[\vartheta\left(3\beta+\gamma-4\right)+3\right]a_3=\frac{1}{48}\left(12s_2-7s_1^2\right),\tag{2.4}$$

$$-\left(\vartheta\left(2\beta+\gamma-3\right)+2\right)a_2=\frac{e_1}{4},\tag{2.5}$$

and

$$\left[\vartheta\left(\frac{(\gamma-2)(\gamma+3)}{2}+2\beta(\gamma-1)+2\beta(\beta+2)-4\right)+6\right]a_2^2-\left[\vartheta\left(3\beta+\gamma-4\right)+3\right]a_3=\frac{1}{48}\left(12e_2-7e_1^2\right).\tag{2.6}$$

From (2.3) and (2.5) it follows that

$$s_1=-e_1\tag{2.7}$$

and

$$s_1^2+e_1^2=32\left(\vartheta\left(2\beta+\gamma-3\right)+2\right)^2 a_2^2. \tag{2.8}$$

If we add (2.4) to (2.6), we get

$$\left[\vartheta\left(\left(\gamma+2\right)\left(\gamma-3\right)+4\beta\left(\gamma-1\right)+2\beta(2\beta+1)\right)+6\right] a_2^2=\frac{1}{4}\left(s_2+e_2\right)-\frac{7}{48}\left(s_1^2+e_1^2\right). \tag{2.9}$$

Substituting the value of $s_1^2+e_1^2$ from (2.8) in (2.9), we have

$$\left\{\left[\vartheta\left(\left(\gamma+2\right)\left(\gamma-3\right)+4\beta\left(\gamma-1\right)+2\beta(2\beta+1)\right)+6\right]+\frac{14}{3}\left(\vartheta\left(2\beta+\gamma-3\right)+2\right)^2\right\} a_2^2=\frac{1}{4}\left(s_2+e_2\right). \tag{2.10}$$

By applying Lemma 2.1 and the triangle inequality for the relations (2.3) and (2.10), we get, respectively,

$$|a_2|\leqslant\frac{1}{2\vartheta\left(2\beta+\gamma-3\right)+4}$$

and

$$|a_2|\leqslant\frac{1}{\sqrt{\vartheta\left(\left(\gamma+2\right)\left(\gamma-3\right)+4\beta\left(\gamma-1\right)+2\beta(2\beta+1)\right)+\frac{14}{3}\left(\vartheta\left(2\beta+\gamma-3\right)+2\right)^2+6}}.$$

Moreover, if we subtract (2.6) from (2.4), we have

$$2\left[\vartheta\left(3\beta+\gamma-4\right)+3\right]\left(a_3-a_2^2\right)=\frac{1}{4}\left(s_2-e_2\right)-\frac{7}{48}\left(s_1^2-e_1^2\right).$$

Then, in view of (2.7), last equation becomes

$$a_3=a_2^2+\frac{s_2-e_2}{8\vartheta\left(3\beta+\gamma-4\right)+24}. \tag{2.11}$$

The above equation with (2.3) becomes

$$a_3=\frac{s_1^2}{16\left(\vartheta\left(2\beta+\gamma-3\right)+2\right)^2}+\frac{s_2-e_2}{8\vartheta\left(3\beta+\gamma-4\right)+24}.$$

By applying Lemma 2.1 and the triangle inequality for the last relation, we get

$$|a_3|\leqslant\frac{1}{4\left(\vartheta\left(2\beta+\gamma-3\right)+2\right)^2}+\frac{1}{2\vartheta\left(3\beta+\gamma-4\right)+6}$$

and using our first assertion with (2.11), it follows that

$$|a_3|\leqslant\frac{1}{\vartheta\left(\left(\gamma+2\right)\left(\gamma-3\right)+4\beta\left(\gamma-1\right)+2\beta(2\beta+1)\right)+\frac{14}{3}\left(\vartheta\left(2\beta+\gamma-3\right)+2\right)^2+6}+\frac{1}{2\vartheta\left(3\beta+\gamma-4\right)+6}.$$

Also, from (2.11) we have

$$\begin{aligned}a_3-\xi a_2^2&=\frac{s_2-e_2}{8\vartheta\left(3\beta+\gamma-4\right)+24}+(1-\xi)a_2^2\\&=\frac{s_2-e_2}{8\vartheta\left(3\beta+\gamma-4\right)+24}\\&\quad+\frac{(1-\xi)\left(s_2+e_2\right)}{4\vartheta\left(\left(\gamma+2\right)\left(\gamma-3\right)+4\beta\left(\gamma-1\right)+2\beta(2\beta+1)\right)+\frac{56}{3}\left(\vartheta\left(2\beta+\gamma-3\right)+2\right)^2+24}\end{aligned}$$

$$= \left(\Theta(\xi) + \frac{1}{8\vartheta\,(3\beta+\gamma-4)+24}\right) s_2 + \left(\Theta(\xi) - \frac{1}{8\vartheta\,(3\beta+\gamma-4)+24}\right) e_2,$$

where

$$\Theta(\xi) = \frac{1-\xi}{4\vartheta\left((\gamma+2)(\gamma-3)+4\beta(\gamma-1)+2\beta(2\beta+1)\right)+\frac{56}{3}\left(\vartheta(2\beta+\gamma-3)+2\right)^2+24}.$$

Then, in view of Lemma 2.1 for s_2 and e_2, and Lemma 2.2, we obtain

$$\left|a_3-\xi a_2^2\right| \leqslant \begin{cases} \frac{1}{2\vartheta(3\beta+\gamma-4)+6}, & 0 \leqslant |\Theta(\xi)| \leqslant \frac{1}{8\vartheta(3\beta+\gamma-4)+24}, \\ 4\,|\Theta(\xi)|, & |\Theta(\xi)| \geqslant \frac{1}{8\vartheta(3\beta+\gamma-4)+24}, \end{cases}$$

which completes the proof. □

Fixing $\vartheta = 0$ or $\vartheta = \beta = \gamma = 1$ in Theorem 2.5, we get the following coefficient estimates, which is due to [17] in Theorems 1 and 2.

Corollary 2.6. *If* $h \in \wp_\Pi(\vartheta,\gamma,\beta)$, *then* $|a_2| \leqslant \sqrt{\frac{3}{74}}$, $|a_3| \leqslant \frac{23}{111}$, *and*

$$\left|a_3-\xi a_2^2\right| \leqslant \begin{cases} \frac{1}{6}, & |K(\xi)| \leqslant \frac{1}{24}, \\ 4\,|K(\xi)|, & |K(\xi)| \geqslant \frac{1}{24}, \end{cases}$$

where $K(\xi) = \frac{3(1-\xi)}{296}$.

3. The class $\mathcal{F}_\Pi(\phi)$ coefficient bounds

In this part, we examine a subclass $\mathcal{F}_\Pi(\phi)$ of analytic bi-univalent functions and obtain initial coefficients $|a_2|$, $|a_3|$, and Fekete-Szegö inequality. We utilize the subsequent lemma to establish our results for the class $\mathcal{F}_\Pi(\phi)$.

Lemma 3.1 ([14]). *If* $\varepsilon(\tau) = 1 + s_1\tau + s_2\tau^2 + \cdots \in \mathcal{V}$, $\tau \in \Omega$, *then there exist some* α, δ *with* $|\alpha|, |\delta| \leqslant 1$, *such that*

$$2s_2 = s_1^2 + \alpha(4-s_1^2) \quad \text{and} \quad 4s_3 = s_1^3 + 2s_1\alpha(4-s_1^2) - (4-s_1^2)s_1\alpha^2 + 2(4-s_1^2)(1-|\alpha|^2)\delta.$$

Definition 3.2. A function $h \in \Pi$ given by (1.1) is in the class $\mathcal{F}_\Pi(\phi)$, where $\phi \in (-\pi,\pi]$, iff

$$h'(z) + \left(\frac{e^{i\phi}+1}{2}\right) zh''(z) \prec \Upsilon(z) \quad \text{and} \quad p'(\varpi) + \left(\frac{e^{i\phi}+1}{2}\right) \varpi p''(\varpi) \prec \Upsilon(\varpi), \tag{3.1}$$

where $\Upsilon(z)$ is given by (1.2) and $p(\varpi) = h^{-1}(\varpi)$.

Remark 3.3.

1) By utilizing particular values for $\phi \in (-\pi,\pi]$ in Definition 3.2, we obtain several popular subclasses of $\mathcal{H}$.
2) If we consider $h_*(\tau) = \frac{\tau}{1-c\tau}$, $|c| \leqslant 1$, then we can check that $h_*(\tau) \in \mathcal{F}_\Pi(\phi)$, so the class $\mathcal{F}_\Pi(\phi)$ is not empty (see [26, Remark 5]).

Theorem 3.4. *If* $h \in \mathcal{F}_\Pi(\phi)$. *Then*

$$|a_2| \leqslant \min\left\{\frac{1}{\sqrt{2}\,|e^{i\phi}+3|}, \frac{1}{\sqrt{6\,|e^{i\phi}+2| + \frac{7}{3}\,|e^{i\phi}+3|^2}}\right\},$$

$$|a_3| \leqslant \left\{ \frac{1}{2\left|e^{i\phi}+3\right|^2} + \frac{1}{6\left|e^{i\phi}+2\right|}, \; \frac{1}{\left|33+20e^{i\phi}+\frac{7}{3}e^{i2\phi}\right|} + \frac{1}{6\left|e^{i\phi}+2\right|} \right\},$$

and

$$\left|a_3-\rho a_2^2\right| \leqslant \begin{cases} \frac{1}{6|e^{i\phi}+2|}, & |1-\rho| \leqslant \frac{|e^{i\phi}+3|^2}{3|e^{i\phi}+2|}, \\ \frac{|1-\rho|}{2|e^{i\phi}+3|^2}, & |1-\rho| \geqslant \frac{|e^{i\phi}+3|^2}{3|e^{i\phi}+2|}. \end{cases} \tag{3.2}$$

Proof. Let $h \in \mathcal{F}_{\Pi}(\phi)$. From subordinations (3.1), there exists two analytic functions c and b such that $c(0) = 0 = b(0)$ and $|c(z)|, |b(\varpi)| < 1$, such that

$$h'(z) + \left(\frac{e^{i\phi}+1}{2}\right) zh''(z) = \Upsilon(c(z)), \quad z \in \Omega,$$

and

$$p'(\varpi) + \left(\frac{e^{i\phi}+1}{2}\right) \varpi p''(\varpi) = \Upsilon(e(\varpi)), \quad \varpi \in \Omega.$$

Thus we have

$$h'(z) + \left(\frac{e^{i\phi}+1}{2}\right) zh''(z) = 1 + \frac{s_1}{4}z + \frac{1}{48}\left(12s_2-7s_1^2\right) z^2 + \frac{1}{192}\left(17s_1^3-56s_1s_2+48s_3\right) z^3 + \cdots, \; z \in \Omega,$$

and

$$\begin{aligned} p'(\varpi) + \left(\frac{e^{i\phi}+1}{2}\right) \varpi p''(\varpi) = 1 &+ \frac{e_1}{4}\varpi + \frac{1}{48}\left(12e_2-7e_1^2\right) \varpi^2 \\ &+ \frac{1}{192}\left(17e_1^3-56e_1e_2+48e_3\right) \varpi^3 + \cdots, \; \varpi \in \Omega. \end{aligned}$$

Comparing the coefficients in last two equations, we have

$$\left(e^{i\phi}+3\right) a_2 = \frac{s_1}{4},$$

$$3\left(e^{i\phi}+2\right) a_3 = \frac{1}{48}\left(12s_2-7s_1^2\right), \tag{3.3}$$

$$-\left(e^{i\phi}+3\right) a_2 = \frac{e_1}{4},$$

$$3\left(e^{i\phi}+2\right)\left(2a_2^2-a_3\right) = \frac{1}{48}\left(12e_2-7e_1^2\right). \tag{3.4}$$

From (2.3) and (2.5) it follows that

$$s_1 = -e_1 \tag{3.5}$$

and

$$16\left(e^{i\phi}+3\right)^2 a_2^2 = s_1^2+e_1^2. \tag{3.6}$$

If we add (3.3) to (3.4), we get

$$6\left(e^{i\phi}+2\right) a_2^2 = \frac{1}{4}\left(s_2+e_2\right) - \frac{7}{48}\left(s_1^2+e_1^2\right). \tag{3.7}$$

Substituting the value of $s_1^2+e_1^2$ from (3.6) in (3.7), we have

$$\left\{6\left(e^{i\phi}+2\right) + \frac{7}{3}\left(e^{i\phi}+3\right)^2\right\} a_2^2 = \frac{1}{4}\left(s_2+e_2\right). \tag{3.8}$$

By applying Lemma 2.1 and the triangle inequality for the relations (3.6) and (3.8), we get, respectively,

$$|a_2| \leqslant \frac{1}{\sqrt{2}\,|e^{i\phi}+3|} \quad \text{and} \quad |a_2| \leqslant \frac{1}{\sqrt{6\,|e^{i\phi}+2| + \frac{7}{3}\,|e^{i\phi}+3|^2}}.$$

Moreover, if we subtract (3.4) from (3.3), we have

$$6\left(e^{i\phi}+2\right)\left(a_3 - a_2^2\right) = \frac{1}{4}\left(s_2 - e_2\right) - \frac{7}{48}\left(s_1^2 - e_1^2\right).$$

Then, in view of (3.5), last equation becomes

$$a_3 = a_2^2 + \frac{s_2 - e_2}{24\left(e^{i\phi}+2\right)}. \tag{3.9}$$

The above equation with (3.6) becomes

$$a_3 = \frac{s_1^2 + e_1^2}{16\left(e^{i\phi}+3\right)^2} + \frac{s_2 - e_2}{24\left(e^{i\phi}+2\right)}.$$

By applying Lemma 2.1 and the triangle inequality for the last relation, we get

$$|a_3| \leqslant \frac{1}{2\,|e^{i\phi}+3|^2} + \frac{1}{6\,|e^{i\phi}+2|}.$$

Similarly, using of (3.8) in relation (3.9) follows that

$$|a_3| \leqslant \frac{1}{\left|33 + 20e^{i\phi} + \frac{7}{3}e^{i2\phi}\right|} + \frac{1}{6\,|e^{i\phi}+2|}.$$

Also, using (3.5) and (3.6), we get $a_2^2 = \frac{s_1^2}{8(e^{i\phi}+3)^2}$. Thus, from (3.9), we have

$$a_3 - \rho a_2^2 = \frac{s_2 - e_2}{24\left(e^{i\phi}+2\right)} + (1-\rho)a_2^2 = \frac{s_2 - e_2}{24\left(e^{i\phi}+2\right)} + (1-\rho)\frac{s_1^2}{8\left(e^{i\phi}+3\right)^2}.$$

From Lemma 3.1, we have $2s_2 = s_1^2 + \alpha(4 - s_1^2)$ and $2e_2 = e_1^2 + \delta(4 - e_1^2)$, $|\alpha|, |\delta| \leqslant 1$, and using (3.5), we obtain

$$s_2 - e_2 = \frac{4 - s_1^2}{2}(\alpha - \delta),$$

and thus

$$a_3 - \rho a_2^2 = \frac{(4 - s_1^2)(\alpha - \delta)}{48\left(e^{i\phi}+2\right)} + \frac{(1-\rho)s_1^2}{8\left(e^{i\phi}+3\right)^2}.$$

Using the triangle inequality, taking $|\alpha| = \mu$, $|\delta| = \eta$, $\mu, \eta \in [0,1]$, and assuming that $s_1 \in \mathbb{R}$, $s_1 = d \in [0,2]$, thus, we get

$$\left|a_3 - \rho a_2^2\right| \leqslant \frac{(4 - d^2)(\mu + \eta)}{48\,|e^{i\phi}+2|} + \frac{|1-\rho|\,d^2}{8\,|e^{i\phi}+3|^2}. \tag{3.10}$$

Assume that $\Psi(d) = \frac{|1-\rho|\,d^2}{8|e^{i\phi}+3|^2} \geqslant 0$ and $\Phi(d) = \frac{4-d^2}{48|e^{i\phi}+2|} \geqslant 0$, the relation (3.10) can be rewritten as

$$\left|a_3 - \rho a_2^2\right| \leqslant \Psi(d) + \Phi(d)(\mu + \eta) =: \mathcal{L}(\mu, \eta), \;\; \mu, \eta \in [0,1].$$

Therefore

$$\max\{\mathcal{L}(\mu,\eta): \ \mu,\eta \in [0,1]\} = \mathcal{L}(1,1) = \Psi(d) + 2\Phi(d) =: K(d), \ d \in [0,2],$$

where

$$K(d) = \frac{1}{8\left|e^{i\phi}+3\right|^2}\left(|1-\rho| - \frac{\left|e^{i\phi}+3\right|^2}{3\left|e^{i\phi}+2\right|}\right)d^2 + \frac{1}{6\left|e^{i\phi}+2\right|}.$$

Since

$$K'(d) = \frac{1}{4\left|e^{i\phi}+3\right|^2}\left(|1-\rho| - \frac{\left|e^{i\phi}+3\right|^2}{3\left|e^{i\phi}+2\right|}\right)d,$$

it is clear that $K'(d) \leqslant 0$ iff $|1-\rho| \leqslant \frac{|e^{i\phi}+3|^2}{3|e^{i\phi}+2|}$, hence, the function K is decreasing on $[0,2]$, therefore,

$$\max\{K(d) : d \in [0,2]\} = K(0) = \frac{1}{6\left|e^{i\phi}+2\right|}.$$

Also, $K'(d) \geqslant 0$ iff $|1-\rho| \geqslant \frac{|e^{i\phi}+3|^2}{3|e^{i\phi}+2|}$, so, K is an increasing function over $[0,2]$, so

$$\max\{K(d) : d \in [0,2]\} = K(2) = \frac{|1-\rho|}{2\left|e^{i\phi}+3\right|^2}$$

and the estimation (3.2) has been validated. □

Remark 3.5. By taking specific values for ϑ, γ, and β in Theorem 2.5, and for ϕ in Theorem 3.4 we get various well-known subclasses of Π. Also, there are several functions $\Upsilon(z)$ that may produce interesting subclasses of function class Π. As an example, if

$$\Upsilon(z) = \frac{1+(1-2\hbar)z}{1-z} = 1 + 2(1-\hbar)z + 2(1-\hbar)z^2 + \cdots \ (0 \leqslant \hbar < 1)$$

or

$$\Upsilon(z) = \left(\frac{1+z}{1-z}\right)^{\hbar} = 1 + 2\hbar z + 2\hbar^2 z^2 + \cdots \ (0 < \hbar \leqslant 1),$$

which gives special cases for our previous results.

4. Conclusions

In this study, utilizing the functions whose coefficients are Gregory numbers, for the class of bi-univalent functions we present two new subclasses $\wp_{\Pi}(\vartheta, \gamma, \beta)$ and $\mathcal{F}_{\Pi}(\phi)$. We study the estimates $|a_2|$ and $|a_3|$ and the Fekete-Szegö inequality regarding the functions in these classes. The originality of the characterizations and the proofs in this research may encourage additional research on these kinds of similarly defined analytic bi-univalent function subclasses.

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