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Oscillation of 3rd order non-canonical delay differential equations of limited neutral coefficient

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Abstract

The main aim of this paper is to obtain new criteria for oscillating all solutions of second-order differential equations with distributed deviating arguments and superlinear neutral terms. Using the comparative and integral averaging techniques, we find new conditions for oscillation that generalize and add to some of the already found results. There are examples to show how important the main results are.

Keywords: Neutral differential equation, noncanonical, third-order, oscillation.

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1. Introduction

In this paper, we are interested in the oscillatory features in this study of the 3rd-order noncanonical neutral delay differential equation

$$\left(\mu_2\left(\mu_1\phi'\right)'\right)'(\zeta)+f(\zeta)\theta(\delta(\zeta))=0, \quad \zeta\geqslant\zeta_0>0, \tag{E}$$

where $\phi(\zeta)=\theta(\zeta)+g(\zeta)\theta(\tau(\zeta))$. Throughout we assume that:

(H_1) $\delta,\tau\in C^1\left([\zeta_0,\infty),\mathbb{R}\right),\delta(\zeta)<\zeta,\tau(\zeta)<\zeta,\tau'(\zeta)\geqslant\tau_0$ and $\lim_{\zeta\to\infty}\tau(\zeta)=\lim_{\zeta\to\infty}\delta(\zeta)=\infty$;

(H_2) $g,f\in C\left([\zeta_0,\infty),\mathbb{R}^+\right),0\leqslant g(\zeta)\leqslant g_0<\infty$, and f does not vanish identically;

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(H_3) $\mu_1, \mu_2 \in C([\zeta_0, \infty), (0, \infty))$ satisfy

$$\int_{\zeta_0}^{\infty} \frac{1}{\mu_1(\zeta)} d\zeta < \infty \quad \text{and} \quad \int_{\zeta_0}^{\infty} \frac{1}{\mu_2(\zeta)} d\zeta < \infty,$$

that is, (E) is in noncanonical form;

(H_4) τ and δ commute.

Let's define the following operators for the purposes of simplicity:

$$L_0\phi = \phi, \quad L_1\phi = \mu_1\phi', \quad L_2\phi = \mu_2(L_1\phi)', \quad L_3\phi = (\mu_2 L_2\phi)'.$$

By a solution of (E), we mean a function $\theta \in C([T_\theta, \infty), \mathbb{R})$ with $T_\theta > \zeta_0$, which has the property $L_i \phi \in C^1([T_\theta, \infty), \mathbb{R}), i = 0, 1, 2,$ and satisfies (E) on $[T_\theta, \infty)$. We only consider those solutions of (E) which exist on some half-line $[T_\theta, \infty)$ and satisfy the condition $\sup\{|\theta(\zeta)| : T \leqslant \zeta < \infty\} > 0$ for any $T \geqslant T_\theta$. We assume that (E) possesses such a solution. A solution of (E) is said to be oscillatory if it is neither eventually negative nor eventually positive and it is called nonoscillatory otherwise. The equation itself is referred to as oscillatory if all of the solutions are oscillatory.

In dynamical models, delay and oscillation scenarios are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems, see, e.g. [7, 14, 21, 24, 28, 38]. A fundamental feature of the qualitative theory of functional differential equations is the existence of oscillation problems. Researchers have been paying close attention to the oscillation theory of third/fourth-order functional differential equations for the past few years due to the significance of these equations in applications and the volume of mathematical problems they involve (for more information, see the papers [1, 2, 4, 5, 8, 10, 11, 18, 25–27, 32–35].

Recent years have seen an increasing amount of attention paid, in particular, to the oscillation theory of third-order neutral type differential equations; see, [3, 13, 23, 30, 31, 36, 39, 43, 45] and the references listed therein. It is known from the review of the literature that the examples represent the majority of research efforts addressing the oscillation behavior of solutions of (E) with the following assumptions:

$$\int_{\zeta_0}^{\infty} \frac{1}{\mu_1(\zeta)} d\zeta = \int_{\zeta_0}^{\infty} \frac{1}{\mu_2(\zeta)} d\zeta = \infty,$$

or

$$\int_{\zeta_0}^{\infty} \frac{1}{\mu_1(\zeta)} d\zeta < \infty \quad \text{and} \quad \int_{\zeta_0}^{\infty} \frac{1}{\mu_1(\zeta)} d\zeta = \infty,$$

or

$$\int_{\zeta_0}^{\infty} \frac{1}{\mu_1(\zeta)} d\zeta = \infty \quad \text{and} \quad \int_{\zeta_0}^{\infty} \frac{1}{\mu_2(\zeta)} d\zeta < \infty.$$

Depending on various ranges of g, a variety of results for property of (E), its generalizations or particular cases, reported in the literature, see, for example [6, 9, 12, 15–17, 19, 20, 22, 29, 37, 40–42, 44] and the references cited therein. Among them, the work in [29] considered (E) in noncanonical form with $0 \leqslant g(\zeta) < 1$ and proved that every solution of (E) is either oscillatory or tends to zero as $\zeta \to \infty$.

Finding sufficient conditions which ensure that all solutions of (E) are oscillatory remained open till only lately. The first of these findings for (E) was reported in [4] in canonical form under the conditions $0 \leqslant g(\zeta) \leqslant g_0 < \infty$ and $\delta \circ \tau = \tau \circ \delta$. Very recently in [11, 37], the authors provided enough parameters for (E) to oscillate in the noncanonical or semi-canonical cases with unbounded neutral coefficient, that is, $g(\zeta) \geqslant g_0 > 1$ since in this case one can easily find the relation between $\theta(\zeta)$ and $\phi(\zeta)$. This is generally essential to obtain oscillation criteria for neutral type differential equations.

To the best of authors' knowledge, there is no result available on the oscillation of all solutions of (E) under the assumptions (H_1)-(H_4). In view of this fact, we intend to fill this gap by first transforming the equation (E) into semi-canonical form and this reduced the number of nonoscillatory solutions of (E) into three types instead of four. Then employing comparison and integral averaging method to get oscillation of all solutions of (E). An example is included to show the importance and novelty of the main results that already known [4, 11, 29, 37].

2. Main results

In view of (H_3), one can use the following notations:

$$\Omega_j(\zeta)=\int_\zeta^\infty \frac{ds}{\mu_j(s)},\ j=1,2,\qquad \beta_1(\zeta)=\mu_1(\zeta)\Omega_1^2(\zeta),\qquad \beta_2(\zeta)=\frac{\mu_2(\zeta)}{\Omega_1(\zeta)},$$
$$F(\zeta)=\min\{f(\zeta),f(\tau(\zeta))\},\qquad N(\zeta)=F(\zeta)\Omega_1(\delta(\zeta)),\qquad A(\zeta,u)=\int_u^\zeta \frac{1}{\beta_1(s)}\int_s^\zeta \frac{ds_1}{\beta_2(s_1)}ds,$$

for all $\zeta > u \geqslant \zeta_1 \geqslant \zeta_0$. From the form of (E), it is enough to consider positive solutions for nonoscillatory solutions of (E). The following is a standard one and can be found in [11].

Lemma 2.1. *Let* (H_1)-(H_3) *hold. If* θ *is an eventually positive solution of* (E)*, then there exists* $\zeta_1 \in [\zeta_0,\infty)$ *such that the corresponding function* ϕ *is also positive and satisfies one of the following four classes:*

$$S_1:\phi>0,\ L_1\phi<0,\ L_2\phi<0,\ L_3\phi\leqslant 0,\qquad S_2:\phi>0,\ L_1\phi<0,\ L_2\phi>0,\ L_3\phi\leqslant 0,$$
$$S_3:\phi>0,\ L_1\phi>0,\ L_2\phi>0,\ L_3\phi\leqslant 0,\qquad S_4:\phi>0,\ L_1\phi>0,\ L_2\phi<0,\ L_3\phi\leqslant 0,$$

eventually.

Hence, we want to derive oscillation criteria for the noncanonical equation (E), we have to eliminate the above said four classes. However if we transform (E) into semi-canonical form, then the number of classes reduced to three without making any additional condition. Thus, this greatly streamlines the analysis of oscillation of (E).

Theorem 2.2. *The noncanonical operator* $L_3\phi$ *has the semi-canonical representation*

$$L_3\phi(\zeta)=\left(\frac{\mu_2}{\Omega_1}\left(\mu_1\Omega_1^2\left(\frac{\phi}{\Omega_1}\right)'\right)'\right)'(\zeta).$$

Proof. Direct calculation shows that

$$\frac{\mu_2(\zeta)}{\Omega_1(\zeta)}\left(\mu_1(\zeta)\Omega_1^2(\zeta)\left(\frac{\phi(\zeta)}{\Omega_1(\zeta)}\right)'\right)'=\frac{\mu_2(\zeta)}{\Omega_1(\zeta)}\left(\Omega_1(t)\mu_1(\zeta)\phi'(\zeta)+\phi(\zeta)\right)'=\frac{\mu_2(\zeta)}{\Omega_1(\zeta)}\left(\Omega_1(\zeta)\left(\mu_1(\zeta)\phi'(\zeta)'\right)\right),$$

that is,

$$\left(\frac{\mu_2(\zeta)}{\Omega_1(\zeta)}\left(\mu_1(\zeta)\Omega_1^2(\zeta)\left(\frac{\phi(\zeta)}{\Omega_1(\zeta)}\right)'\right)'\right)'=\left(\mu_2(\zeta)\left(\mu_1(\zeta)\phi'(\zeta)\right)'\right)'.$$

Further note that

$$\int_{\zeta_0}^\infty \frac{d\zeta}{\mu_1(\zeta)\Omega_1^2(\zeta)}=\int_{\zeta_0}^\infty d\left(\frac{1}{\Omega_1(\zeta)}\right)=\lim_{\zeta\to\infty}\frac{1}{\Omega_1(\zeta)}-\frac{1}{\Omega_1(\zeta_0)}=\infty,$$

and

$$\int_{\zeta_0}^\infty \frac{\Omega_1(\zeta)}{\mu_2(\zeta)}d\zeta=\Omega_1(\zeta_0)\int_{\zeta_0}^\infty \frac{1}{\mu_2(\zeta)}d\zeta<\infty.$$

Hence $L_3\phi$ transformed into semi-canonical form. This ends the proof. □

Now it follows from Theorem 2.2 that (E) can be written in the equivalent semi-canonical form

$$\left(\beta_2(\zeta)\left(\beta_1(\zeta)\left(\frac{\phi(\zeta)}{\Omega_1(\zeta)}\right)'\right)'\right)' + f(\zeta)\theta(\delta(\zeta)) = 0.$$

By letting $\alpha(\zeta) = \frac{\phi(\zeta)}{\Omega_1(\zeta)}$, the following result is at once.

Theorem 2.3. *Noncanonical neutral delay differential equation* (E) *has a solution* $\theta(\zeta)$ *if and only if the semi-canonical equation*

$$\left(\beta_2(\nu)\left(\beta_1(\zeta)\alpha'(\zeta)\right)'\right)' + f(\zeta)\theta(\delta(\zeta)) = 0, \tag{E_s}$$

has the solution $\theta(\zeta)$.

Corollary 2.4. *Noncanonical neutral differential equation* (E) *has an eventually positive solution if and only if the semi-canonical equation* (E_s) *has an eventually positive solution.*

Now set $D_0\alpha = \alpha$, $D_1\alpha = \beta_1\alpha'$, $D_2\alpha = \beta_2\left(\beta_1\alpha'\right)'$, $D_3\alpha = \left(\beta_2\left(\beta_1\alpha'\right)'\right)'$. Corollary 2.4 clearly simplifies investigation of (E) since for (E_s) we deal with only three classes of an eventually positive solution, see, for example, [29, Theorem 2.2], namely

$$\begin{aligned}
&O_1 : \alpha(\zeta) > 0,\ D_1\alpha(\zeta) < 0,\ D_2\alpha(\zeta) > 0,\ D_3\alpha(\zeta) \leqslant 0,\\
&O_2 : \alpha(\zeta) > 0,\ D_1\alpha(\zeta) > 0,\ D_2\alpha(\zeta) > 0,\ D_3\alpha(\zeta) \leqslant 0,\\
&O_3 : \alpha(\zeta) > 0,\ D_1\alpha(\zeta) > 0,\ D_2\alpha(\zeta) < 0,\ D_3\alpha(\zeta) \leqslant 0,
\end{aligned}$$

eventually.

Lemma 2.5. *Let* (H_1)-(H_4) *hold. If* θ *is an eventually positive solution of* (E)*, then the corresponding function satisfies the inequality*

$$D_3\alpha(\zeta) + \frac{g_0}{\tau_0}D_3\alpha(\tau(\zeta)) + N(\zeta)\alpha(\delta(\zeta)) \leqslant 0 \tag{2.1}$$

for all $\zeta \geqslant \zeta_1 \geqslant \zeta_0$.

Proof. Let θ be an eventually positive solution of (E). Then there is a $\zeta_1 \geqslant \zeta_0$ such that $\theta(\zeta) > 0$, $\theta(\tau(\zeta)) > 0$, and $\theta(\delta(\zeta)) > 0$ for all $t \geqslant \zeta_1$. From Corollary 2.4, the corresponding function $\alpha(\zeta)$ is a positive solution of (E_s) for all $\zeta \geqslant \zeta_1$. Now, from (E_s), (H_1), and (H_4), we see that

$$\begin{aligned}
0 &= \frac{g_0}{\tau'(\zeta)}\left(D_2\alpha(\tau(\zeta))\right)' + g_0 f(\tau(\zeta))\theta(\delta(\tau(\zeta)))\\
&\geqslant \frac{g_0}{\tau_0}\left(D_2\alpha(\tau(\zeta))\right)' + g_0 f(\tau(\zeta))\theta(\delta(\tau(\zeta))) = \frac{g_0}{\tau_0}\left(D_2\alpha(\tau(\zeta))\right)' + g_0 f(\tau(\zeta))\theta(\tau(\delta(\zeta))).
\end{aligned} \tag{2.2}$$

Combining (E_s) along with the last inequality, we obtain

$$\begin{aligned}
0 &\geqslant D_3\alpha(\zeta) + \frac{g_0}{\tau_0}D_3\alpha(\tau(\zeta)) + f(\zeta)\theta(\delta(\zeta)) + g_0 f(\tau(\zeta))\theta(\tau(\delta(\zeta)))\\
&\geqslant D_3\alpha(\zeta) + \frac{g_0}{\tau_0}D_3\alpha(\tau(\zeta)) + F(\zeta)\left(\theta(\delta(\zeta)) + g_0\theta(\tau(\delta(\zeta)))\right).
\end{aligned}$$

Using (H_2) in the definition of $\phi(\zeta)$, we get

$$\Omega_1(\delta(\zeta))\alpha(\delta(\zeta)) = \phi(\delta(\zeta)) = \theta(\delta(\zeta)) + g(\delta(\zeta))\theta(\tau(\delta(\zeta))) \leqslant \theta(\delta(\zeta)) + g_0\theta(\tau(\delta(\zeta))).$$

In view of the latter inequality (2.2) becomes

$$D_3\alpha(\zeta)+\frac{g_0}{\tau_0}D_3\alpha(\tau(\zeta))+N(\zeta)\alpha(\delta(\zeta))\leqslant 0 \quad \text{or} \quad (D_2\alpha(\zeta)+\frac{g_0}{\tau_0}D_2\alpha(\tau(\zeta)))'+N(\zeta)\alpha(\delta(\zeta))\leqslant 0, \tag{2.3}$$

which proves (2.1). □

Before we state and prove our main results, let us define $B_1(\zeta)=\int_{\zeta_0}^{\zeta}\frac{1}{\beta_1(s)}ds$, $B(\zeta)=\int_{\zeta}^{\infty}\frac{1}{\beta_2(s)}ds$, for all $\zeta\geqslant\zeta_0$.

Theorem 2.6. *Let α be an eventually positive solution of (E_s). If*

$$\int_{\zeta_0}^{\infty}N(\zeta)B_1(\zeta)d\zeta=\infty, \tag{2.4}$$

then class O_2 is empty.

Proof. Assume the contrary that class O_2 is not empty. Then there exists a $\zeta_1\geqslant\zeta_0$ such that $\alpha(\zeta)>0$, $\alpha(\tau(\zeta))>0$, $\alpha(\delta(\zeta))>0$, for all $\zeta\geqslant\zeta_1$, such that the function $\alpha(\zeta)$ in class O_2 for all $\zeta\geqslant\zeta_1$. Since $\beta_1(\zeta)\alpha'(\zeta)>0$ is increasing, we have

$$\beta_1(\zeta)\alpha'(\zeta)\geqslant\beta_1(\zeta_1)\alpha'(\zeta_1)=M \text{ on } [\zeta_1,\infty).$$

Dividing the last inequality by $\beta_1(\zeta)$, and then integrating the resulting inequality, we obtain

$$\alpha(\delta(\zeta))\geqslant MB_1(\delta(\zeta)),\ \zeta\geqslant\zeta_2>\zeta_1. \tag{2.5}$$

Integrating (E_s) from ζ_2 to ζ and using (2.5) in the resulting inequality, we obtain

$$\begin{aligned}D_2\alpha(\zeta)+\frac{g_0}{\tau_0}D_2\alpha(\tau(\zeta))&=D_2\alpha(\zeta_2)+\frac{g_0}{\tau_0}D_2\alpha(\tau(\zeta_2))-\int_{\zeta_2}^{\zeta}N(s)\alpha(\delta(s))ds\\&\leqslant D_2\alpha(t_2)+\frac{g_0}{\tau_0}D_2\alpha(\tau(\zeta_0))-M\int_{\zeta_2}^{\zeta}N(s)B_1(\delta(s))ds,\end{aligned}$$

which tends to ∞ as $\zeta\to\infty$. This contradiction ends the proof. □

Lemma 2.7. *Let α be an eventually positive increasing solution of (E_s). If*

$$\int_{\zeta_0}^{\infty}\frac{1}{\beta_2(\zeta)}\left(\int_{\zeta_0}^{\zeta}N(s)B_1(\delta(s))ds\right)d\zeta=\infty, \tag{2.6}$$

then α satisfies the class O_3 for $\zeta\geqslant\zeta_1$ for some $\zeta_1\geqslant\zeta_0$ and further

$$\alpha(\zeta)\geqslant B_1(\zeta)\beta_1(\zeta)\alpha'(\zeta) \ \textit{for}\ \ \zeta\geqslant\zeta_1. \tag{2.7}$$

Proof. Since α is a positive increasing solution, so class O_1 is empty and hence by Theorem 2.6, $\alpha\in O_2\cup O_3$ for $\zeta\geqslant\zeta_1$, where $\zeta_1\geqslant\zeta_0$ is such that $\alpha(\delta(\zeta))>0$ and $\alpha(\tau(\zeta))>0$ for $\zeta\geqslant\zeta_1$. In view of (H_3), we see that (2.6) implies (2.4) and hence α satisfies class O_3 for $\zeta\geqslant\zeta_1$. Since $D_1\alpha$ is positive and decreasing, we see that

$$\alpha(\zeta)=\alpha(\zeta_1)+\int_{\zeta_1}^{\zeta}\frac{\beta_1(s)\alpha'(s)}{\beta_1(s)}ds\geqslant B_1(\zeta)\beta_1(\zeta)\alpha'(\zeta).$$

This ends the proof. □

Theorem 2.8. *Let α be an eventually positive solution of* (E_s). *If*

$$\liminf_{\zeta\to\infty}\int_{\delta(\zeta)}^{\zeta}\frac{1}{\beta_2(s)}\left(\int_{\zeta_0}^{s}N(s_1)B_1(\delta(s_1))ds_1\right)ds>\frac{\tau_0+g_0}{e\tau_0}, \tag{2.8}$$

then the classes O_2 and O_3 are empty.

Proof. Assume that (2.8) holds but α belongs to classes O_2 and O_3. Pick $\zeta_1\geqslant\zeta_0$ such that $\alpha(\tau(\zeta))>0$ and $\alpha(\delta(\zeta))>0$ for $\zeta\geqslant\zeta_1$. Clearly, it is necessary for the validity of (2.8) that (2.6) holds. Hence by Theorem 2.6 and Lemma 2.7, one can see that α satisfies class O_3. Proceeding as in the proof of Lemma 2.7, we see that (2.7) holds and so we obtain $\alpha(\delta(\zeta))\geqslant B_1(\delta(\zeta))\beta_1(\delta(\zeta))\alpha'(\delta(\zeta))$ for $\zeta\geqslant\zeta_2$ for some $\zeta_2\geqslant\zeta_1$. From the latter inequality and equation (E_s), we observe that

$$-\left(D_3\alpha(\zeta)+\frac{g_0}{\tau_0}D_3\alpha(\tau(\zeta))\right)=N(\zeta)\alpha(\delta(\nu))\geqslant N(\zeta)B_1\left(\delta(\zeta)\right)\beta_1\left(\delta(\zeta)\right)\alpha'(\delta(\zeta))\,.$$

Integrating the last inequality from ζ_2 to ζ, we have

$$\begin{aligned}-\left(D_2\alpha(\zeta)+\frac{g_0}{\tau_0}D_2\alpha(\tau(\zeta))\right)&\geqslant\int_{\zeta_2}^{\zeta}N(s)B_1\left(\delta(s)\right)\beta_1(\delta(s))\alpha'(\delta(s))ds\\&\geqslant\beta_1(\delta(\zeta))\alpha'(\delta(\zeta))\int_{\zeta_2}^{\zeta}N(s)B_1(\delta(s))ds.\end{aligned}\tag{2.9}$$

Since $D_2\alpha(\zeta)$ is decreasing and $\tau(\zeta)<\zeta$, we have $D_2\alpha(\zeta)\leqslant D_2\alpha(\tau(\zeta))$ and using this in (2.9), we get

$$\begin{aligned}-\left(1+\frac{g_0}{\tau_0}\right)D_2\alpha(\zeta)&\geqslant\beta_1(\delta(\zeta))\alpha'(\delta(\zeta))\int_{\zeta_2}^{\zeta}N(s)B_1(\delta(s))ds,\\-\left(\beta_1(\zeta)\alpha'(\zeta)\right)'&\geqslant\left(\frac{\tau_0}{\tau_0+g_0}\right)\frac{\beta_1(\delta(\zeta))}{\beta_2(\zeta)}\alpha'(\delta(\zeta))\int_{\zeta_2}^{\zeta}N(s)B_1(\delta(s))ds.\end{aligned}\tag{2.10}$$

Let $\omega(\zeta)=\beta_1(\zeta)\alpha'(\zeta)>0$ be a positive solution of the first-order delay differential inequality

$$\omega'(\zeta)+\left(\frac{\tau_0}{\tau_0+g_0}\right)\left(\frac{1}{\beta_2(\zeta)}\int_{\zeta_2}^{\zeta}N(s)B_1(\delta(s))ds\right)\omega(\delta(\zeta))\leqslant 0. \tag{2.11}$$

However by [22, Theorem 2.11], we see that $\omega(t)$ is not a positive solution of the inequality (2.11). This contradicts our initial assumption and the proof is complete. □

Theorem 2.9. *Assume that* (2.4) *holds. If*

$$\limsup_{\zeta\to\infty}B(\zeta)\int_{\zeta_0}^{\zeta}N(s)B_1(\delta(s))ds>\frac{g_0+\tau_0}{\tau_0}, \tag{2.12}$$

then the classes O_2 and O_3 are empty.

Proof. Assume to the contrary that α satisfies class O_2 or O_3 for $\zeta\geqslant\zeta_1$. First note that $\lim_{\zeta\to\infty}B(\zeta)=0$ holds which together with (2.12) implies (2.4). So by Lemma 2.7 we conclude that α satisfies O_3 and the asymptotic property (2.7) for all $\zeta\geqslant\zeta_1\geqslant\zeta_0$. Proceeding as in the proof of Theorem 2.8, we arrive at (2.10). Now from the monotonicity of $D_2\alpha(\zeta)$, we obtain

$$\beta_1(\zeta)\alpha'(\zeta)\geqslant-\int_{\zeta}^{\infty}\frac{1}{\beta_2(s)}\beta_2(s)\left(\beta_1(s)\alpha'(s)\right)'ds\geqslant-B(\zeta)\beta_2(\zeta)\left(\beta_1(\zeta)\alpha'(\zeta)\right)'$$

and using this in (2.10), we get

$$-\left(\beta_1(\zeta)\alpha'(\zeta)\right)' \geqslant -\left(\frac{\tau_0}{g_0+\tau_0}\right) B(\zeta)\left(\beta_1(\zeta)\alpha'(\zeta)\right)' \int_{\zeta_2}^{\zeta} N(s)B_1(\delta(s)ds, \tag{2.13}$$

where we have used $\beta_1(\delta(\zeta))\alpha'(\delta(\zeta)) \geqslant \beta_1(\zeta)\alpha'(\zeta)$. From (2.13) we obtain

$$\frac{g_0+\tau_0}{\tau_0} \geqslant B(\zeta)\int_{\zeta_2}^{\zeta} N(s)B_1(\delta(s))ds.$$

But the last inequality contradicts (2.12) and the proof is complete. □

Theorem 2.10. *Let α be an eventually positive solution of (E_s). If $\delta(\zeta) < \tau(\tau(\zeta))$ and*

$$\limsup_{\zeta\to\infty} \int_{\tau(\zeta)}^{\zeta} N(s)A(\delta(\zeta),\delta(s))ds > \frac{\tau_0+g_0}{\tau_0}, \tag{2.14}$$

then the class O_1 is empty.

Proof. Assume to the contrary that (2.14) holds but α belongs to class O_1. Choose $\zeta_1 \geqslant \zeta_0$ such that $\delta(\zeta) \geqslant \zeta_1$ for $\zeta \geqslant \zeta_1$. From the monotonicity of $D_2\alpha(\zeta)$ that for $\nu \geqslant u$,

$$-\beta_1(u)\alpha'(u) \geqslant \int_u^\nu \tfrac{\beta_2(s)}{\beta_2(s)}\left(\beta_1(s)\alpha'(s)\right)' ds \geqslant \beta_2(\nu)\left(\beta_1(\nu)\alpha'(\nu)\right)' \int_u^\nu \tfrac{ds}{\beta_2(s)}.$$

Dividing by $\beta_1(u)$ and then integrating the resulting inequality again from u to $\nu \geqslant u$ in u, we obtain

$$\alpha(u) \geqslant \beta_2(\nu)\left(\beta_1(\nu)\alpha'(\nu)\right)' \int_u^\nu \frac{1}{\beta_1(x)}\int_x^\nu \frac{ds}{\beta_2(s)}dx = \beta_2(\nu)\left(\beta_1(\nu)\alpha'(\nu)\right)' A(\nu,u). \tag{2.15}$$

Integrating (2.1) from $\tau(\zeta)$ to ζ and using (2.15) with $u=\delta(s)$ and $\nu=\delta(\zeta)$, we obtain

$$D_2\alpha(\tau(\zeta)) + \frac{g_0}{\tau_0}D_2\alpha(\tau(\tau(\zeta))) \geqslant \int_{\tau(\zeta)}^{\zeta} N(s)\alpha(\delta(s))ds \geqslant D_2\alpha(\delta(\zeta))\int_{\tau(\zeta)}^{\zeta} N(s)A(\delta(\zeta),\delta(s))ds. \tag{2.16}$$

From $\delta(\zeta) < \tau(\tau(\zeta))$ and $\tau(\tau(\zeta)) < \tau(\zeta)$, we see that

$$D_2\alpha(\delta(\zeta)) \geqslant D_2\alpha(\tau(\tau(\zeta))) \quad \text{and} \quad D_2\,\alpha(\tau(\tau(\zeta))) \geqslant D_2\alpha(\tau(\zeta))$$

and using these in (2.16), we obtain

$$\left(1+\frac{g_0}{\tau_0}\right) \geqslant \int_{\tau(\zeta)}^{\zeta} N(s)A(\delta(\zeta),\delta(s))ds,$$

which contradicts (2.14) and the proof is complete. □

Theorem 2.11. *Let α be an eventually positive solution of (E_s). If there exists a function $\xi(\zeta) \in C([\zeta_0,\infty),(0,\infty))$ satisfying $\delta(\zeta) < \xi(\zeta) < \tau(\zeta)$, such that*

$$\liminf_{\zeta\to\infty} \int_{\tau^{-1}(\xi(\zeta))}^{\zeta} N(s)A(\xi(s),\delta(s))ds > \frac{\tau_0+g_0}{\tau_0 e}, \tag{2.17}$$

then the class O_1 is empty.

Proof. Assume to the contrary that (2.17) holds but α belongs to class O_1. Proceeding as in the proof of

Theorem 2.10 we arrive at (2.9). Setting $u = \delta(\zeta)$ and $v = \xi(\zeta), \zeta \geqslant x \geqslant \zeta_1$, in (2.9), we get

$$\alpha(\delta(\zeta)) \geqslant D_2\alpha(\xi(\zeta))A(\xi(\zeta), \delta(\zeta)). \tag{2.18}$$

On the other hand, using (2.18) in (2.3) yields

$$(D_2\alpha(\zeta) + \frac{g_0}{\tau_0}D_2\alpha(\tau(\zeta)))' + N(\zeta)A(\xi(\zeta), \delta(\zeta))D_2\alpha(\xi(\zeta)) \leqslant 0. \tag{2.19}$$

Now, let

$$\omega(\zeta) = D_2\alpha(\zeta) + \frac{g_0}{\tau_0}D_2\alpha(\tau(\zeta)) > 0.$$

Using the fact that $\tau(\zeta) < \zeta$ and $D_2\alpha(\zeta)$ is nonincreasing, we have

$$\omega(\zeta) \leqslant \left(1 + \frac{g_0}{\tau_0}\right) D_2\alpha(\tau(\zeta))$$

or equivalently

$$D_2\alpha(\xi(\zeta)) \geqslant \frac{\tau_0}{g_0 + \tau_0}\omega(\tau^{-1}(\xi(\zeta))). \tag{2.20}$$

From (2.20) and (2.19), we see that $w(\zeta)$ is a positive solution of the first-order delay differential inequality

$$\omega'(\zeta) + \frac{\tau_0}{\tau_0 + g_0}N(\zeta)A(\xi(\zeta), \delta(\zeta))\omega\left(\tau^{-1}\xi(\zeta)\right) \leqslant 0. \tag{2.21}$$

In view of a well-known result [22, Theorem 2.11], one can deduce that $w(t)$ is not a positive solution of (2.21). This contradiction completes the proof. □

The primary outcome of the study is the following oscillation condition for (E).

Theorem 2.12. *Assume that* (H_1)-(H_4) *hold. If* (2.8) *(or* (2.12)*) and* (2.14) *(or* (2.17)*) are satisfied, then equation* (E) *is oscillatory.*

Proof. Let θ be a nonoscillatory solution of (E) and without loss of generality assume that there exists a $\zeta_1 \geqslant \zeta_0$ such that $\theta(\zeta) > 0$, $\theta(\tau(\zeta)) > 0$, and $\theta(\delta(\zeta)) > 0$ for all $\zeta \geqslant \zeta_1$. Then by Corollary 2.4, the function $\theta(\zeta)$ is also a positive solution of (E_s) as well as the related function $\alpha(\zeta)$ satisfies one of the three classes O_1 or O_2 or O_3 for $\zeta \geqslant \zeta_1$. In view of Theorem 2.8 (or Theorem 2.9), the classes O_2 and O_3 are empty. On the other hand from Theorem 2.10 (or Theorem 2.11) the class O_1 is empty. This contradiction implies that the equation (E) is oscillatory. This ends the proof. □

Example 2.13. Examine the third-order Euler type neutral differential equation

$$\left(\zeta^2\left(\zeta^2\left(\theta(\zeta) + g_0\theta\left(\lambda_1\eta\right)'\right)'\right)' + f_0\zeta\theta\left(\lambda_2\zeta\right) = 0, \quad \zeta \geqslant 1, \tag{2.22}$$

where $f_0 > 0$, $g_0 > 0$, $\lambda_1 \in (0, 1)$, and $\lambda_2 \in (0, 1)$. A simple calculation shows that

$$\Omega_1(\zeta) = \Omega_2(\zeta) = \frac{1}{\zeta}, \beta_1(\zeta) = 1, \quad \beta_2(\zeta) = \zeta^3, \quad \text{and} \quad \tau_0 = \lambda_1.$$

The transformed equation is $\left(\zeta^3\alpha''(\zeta)\right)' + f_0\zeta\theta\left(\lambda_2\zeta\right) = 0$, which is in semi-canonical form. Further calculation shows that

$$N(\zeta) = \frac{f_0\lambda_1}{\lambda_2}, \quad B_1(\zeta) \approx \zeta, \quad \text{and} \quad B(\zeta) = \frac{1}{2\zeta^2}.$$

The condition (2.8) becomes

$$\liminf_{\zeta\to\infty}\int_{\lambda_2\eta}^{\zeta}\left(\frac{1}{s^3}\int_1^s \frac{f_0\lambda_1}{\lambda_2}\lambda_2 s_1\,ds_1\right)ds=\frac{f_0\lambda_1}{2}\ln\frac{1}{\lambda_2}>\frac{\lambda_1+g_0}{\lambda_1 e},$$

that is, condition (2.8) is satisfied if

$$f_0>\frac{2\,(\lambda_1+g_0)}{\lambda_1^2 e\ln\frac{1}{\lambda_2}}.$$

Choosing λ_3 such that $\lambda_2<\lambda_3<\lambda_1$, then the condition (2.17) becomes

$$\liminf_{\zeta\to\infty}\int_{\frac{\lambda_3\zeta}{\lambda_1}}^{\zeta}\frac{f_0\lambda_1}{\lambda_2}\left(\frac{1}{2\lambda_2}-\frac{1}{\lambda_3}+\frac{\lambda_2}{2\lambda_3^2}\right)\frac{1}{s}ds=\frac{f_0\lambda_1}{\lambda_2}\left(\frac{1}{2\lambda_2}-\frac{1}{\lambda_3}+\frac{\lambda_2}{2\lambda_3^2}\right)\ln\frac{\lambda_1}{\lambda_3}>\frac{\lambda_1+g_0}{\lambda_1 e},$$

that is, condition (2.17) is satisfied if

$$f_0\left(\frac{1}{2\lambda_2}-\frac{1}{\lambda_3}+\frac{\lambda_2}{2\lambda_3^2}\right)>\frac{\lambda_2\,(\lambda_1+g_0)}{\lambda_1^2 e\ln\frac{\lambda_1}{\lambda_3}}.$$

Therefore equation (2.22) is oscillatory if

$$f_0>\frac{2\,(\lambda_1+g_0)}{\lambda_1^2 e\ln\frac{1}{\lambda_2}}\quad\text{and}\quad f_0\left(\frac{1}{2\lambda_2}-\frac{1}{\lambda_3}+\frac{\lambda_2}{2\lambda_3^2}\right)>\frac{\lambda_2\,(\lambda_1+g_0)}{\lambda_1^2 e\ln\left(\frac{\lambda_1}{\lambda_3}\right)},.$$

In particular, if we assume $\lambda_1=1/2$, $\lambda_2=1/4$, $\lambda_3=1/3$, $g_0=1/2$, then we get $f_0>29.033674$. So in this case the equation (2.22) is oscillatory if $f_0>29.033674$.

Take note that none of the outcomes listed in [4, 11, 29] can yield this conclusion since $g_0<1$ and the equation is noncanonical.

We provide an example at the end of this section to highlight the significance of our primary findings.

3. Conclusion

In this work, we converted the noncanonical equation (E) into a semi-canonical equation in order to construct oscillation criteria for it. Since there are now three rather than four classes of nonoscillatory solutions, the analysis of (E) is made simpler. Our key findings are obtained through the application of the integral averaging approach and comparison technique. An example is given to highlight the significance and originality of the primary findings. Extending the (E_s) results to nonlinear equations containing linear, sublinear, or superlinear neutral terms is an intriguing idea to get similar results for (E).

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