J. Math. Computer Sci., 38 (2025), 1–15

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Numerical solution of mixed Volterra-Fredholm integral equations using different methods

Sahar M. Abusalim

Department of Mathematics, College of Science, Jouf University, Sakaka, Saudi Arabia.

Abstract

This study discusses the existence and unique solution of the second-kind mixed Volterra-Fredholm integral equation (MV-FIE). Using the projection approach, the best approximate solution can be found by providing two consecutive algorithms and primarily relying on the iterative projection process (P-IM). For every algorithm, we obtain the relative error and the approximate solution. Furthermore, we proved that the first algorithm's estimated P-IM error is better than the successive approximation method's (SAM) estimate. The error estimate was determined, and numerical results were calculated for each example. Some numerical experiments are performed to show the simplicity and efficiency of the presented method, and all results are performed by using the program Wolfram Mathematica 10.

Keywords: Mixed Volterra-Fredholm integral equation, projection-iteration method, Banach fixed point theorem.

2020 MSC: 45B05, 45G10, 65R20.

1. Introduction

In this study, a new numerical method is employed to solve the following mixed Volterra-Fredholm integral equation (MV-FIE):

$$\begin{aligned}\Xi(u,t) = g(u,t) + \gamma_1 \int_0^t \int_{-a}^{a} k(u,\nu)F(t,\tau)\Xi(\nu,\tau)d\nu d\tau \\ + \gamma_2 \int_{-a}^{a} k(u,\nu)\Xi(\nu,t)d\nu + \gamma_3 \int_0^t G(t,\tau)\Xi(u,\tau)d\tau,\end{aligned} \tag{1.1}$$

shere $\gamma_i, i = 1,2,3$ are constant parameters and the unknown function $\Xi(u,t)$ is called the potential function in the Banach space $L_2[-a,a] \times C[0,T]$, $0 \leqslant T < 1$. The kernels $F(t,\tau)$, $G(t,\tau)$ are continuous in $C[0,T]$ and the given function $g(u,t)$ is continuous in the space $L_2[-a,a] \times C[0,T]$. Also the kernel of position $k(u,\nu)$ belongs to $L_2([-a,a] \times [-a,a])$ and a continuous function.

This kind of MV-FIEs arises in a wide variety of applications in numerous areas including mathematical economics [12], theory of elasticity [35], generalized potential theory [6], quantum mechanics [20], population genetics [10, 41], fluid mechanics [38], radiation [21], nonlinear problems theory of boundary value [7, 10], contact problems in two layers of elastic materials [5], spectral relationships in laser theory [13], and electromagnetic and electrodynamics [11, 40].

Email address: sabosalem@ju.edu.sa (Sahar M. Abusalim)

doi: 10.22436/jmcs.038.01.01

Received: 2024-07-05 *Revised:* 2024-08-19 *Accepted:* 2024-09-23

It can be exceedingly challenging to discover exact solutions to these problems. It is therefore preferable to create a precise and efficient numerical approach to solve these kinds of problems. Many computing methods have been developed to solve the MV-FIE given by equation (1.1), including Block pulse functions [26], Chebyshev wavelets polynomials [37], Separation of variables technique [22], Resolvent method [3], Lagrange-collocation method [33], Lagrange polynomials [36], Lucas polynomial [23], Tau-collocation method [14], Legendre polynomials [34], Separation of variables method [32], Picard iteration method [24], Hat functions [17, 18], Legendre-Chebyshev collocation method [16], collocation methods [10, 19, 30], hybrid functions method [4], modified iterated projection method [15], Taylor polynomial method [39], operational matrices [28], modification of hat functions [27], Fibonacci collocation method [29], degenerate kernel method [9, 31], and Bell polynomials [25]. Our main goal for the study is to determine the numerical solution to the problem given by the equation (1.1), for which we have devised a new and accurate method.

The article aims to study the integral equation (1.1) analytically by discussing the existence and uniqueness of the solution to the equation. It also seeks to examine the equation numerically by finding applicable algorithms to obtain the approximate solution in case it is difficult to obtain the exact solution.

2. uniqueness and existence solution of MV-FIE (1.1)

We make the following assumptions in order to talk about the existence and uniqueness solution of MV-FIE (1.1).

(i) The kernel of position $k(u,v)$ satisfies $|k(u,v)| \leqslant A$, $\forall u,v \in [-a,a]$, where A is a constant.

(ii) Functions $F(t,\tau)$ and $G(t,\tau)$ satisfy $|F(t,\tau)|_{C[0,T]} \leqslant B_1$, $|G(t,\tau)|_{C[0,T]} \leqslant B_2$, $\forall t,\tau \in [0,T]$, where B_1, B_2 are constants.

(iii) Function $g(u,t)$ is defined as

$$\|g(u,t)\|_{L_2[-a,a]\times C[0,T]} = \max_{0\leqslant t\leqslant T} \left| \int_0^t \left[\int_{-a}^{a} g^2(u,\tau)du \right]^{\frac{1}{2}} d\tau \right| = D,$$

where D is a constant.

Theorem 2.1. *If the conditions* (i)-(iii) *are satisfied, and*

$$1 > (2a(|\gamma_1|B_1T + |\gamma_2|)A + |\gamma_3|B_2T),$$

then MV-FIE (1.1) *has a unique solution* $\Xi(u,t)$ *in the Banach space* $L_2[-a,a] \times C[0,T]$.

Proof. We employ the successive approximation approach (*Picard's method*) to demonstrate this theorem. As $\{i\}$ goes to ∞, a sequence of functions $\{\Xi_i(u,t)\}$ can be formed as a solution for MV-FIE (1.1); hence,

$$\Xi(u,t) = \lim_{m\to\infty} \Xi_i(u,t),$$

where

$$\Xi_i(u,t) = \sum_{j=0}^{i} \Psi_j(u,t),\ t \in [0,T],\ i = 0,1,2,\ldots,$$

where the functions $\Psi_j(u,t)$, $j = 0,1,\ldots,i$ are continuous and take the form:

$$\begin{cases} \Psi_i(u,t) = \Xi_i(u,t) - \Xi_{i-1}(u,t), \\ \Psi_0(u,t) = g(u,t). \end{cases} \tag{2.1}$$

□

To establish the previous theorem, we need to take into consideration the following lemmas.

Lemma 2.2. *If the series* $\sum_{j=0}^{i} \Psi_j(u,t)$ *is uniformly convergent, then* $\Xi(u,t)$ *represents a solution of MV-FIE* (1.1).

Proof. We design a series described by $\Xi_i(u,t)$,

$$\Xi_i(u,t) = g(u,t) + \gamma_1 \int_0^t \int_{-a}^{a} k(u,v)F(t,\tau)\Xi_{i-1}(v,\tau)dvd\tau + \gamma_2 \int_{-a}^{a} k(u,v)\Xi_{i-1}(v,t)dv + \gamma_3 \int_0^t G(t,\tau)\Xi_{i-1}(u,\tau)d\tau.$$

Next, we obtain

$$\Xi_i(u,t) - \Xi_{i-1}(u,t) = \gamma_1 \int_0^t \int_{-a}^{a} k(u,v)F(t,\tau)[\Xi_{i-1}(v,\tau) - \Xi_{i-2}(v,\tau)]dvd\tau + \gamma_2 \int_{-a}^{a} k(u,v)[\Xi_{i-1}(v,t) - \Xi_{i-2}(v,t)]dv + \gamma_3 \int_0^t G(t,\tau)[\Xi_{i-1}(u,\tau) - \Xi_{i-2}(u,\tau)]d\tau.$$

By applying the norm's characteristics to equation (2.1), we get

$$\|\Psi_i(u,t)\| \leqslant |\gamma_1| \left\| \int_0^t \int_{-a}^{a} k(u,v)F(t,\tau)\Psi_{i-1}(v,\tau)dvd\tau \right\| + |\gamma_2| \left\| \int_{-a}^{a} k(u,v)\Psi_{i-1}(v,t)dv \right\| + |\gamma_3| \left\| \int_0^t G(t,\tau)\Psi_{i-1}(u,\tau)d\tau \right\|.$$

By conditions (i) and (ii), we obtain

$$\|\Psi_i(u,t)\| \leqslant [2a(|\gamma_1|B_1T + |\gamma_2|)A + |\gamma_3|B_2T]\|\Psi_{i-1}(u,t)\|, \tag{2.2}$$

where $T = \max_{0<t\leqslant T} |t|$. Using condition (iii) and $i = 1$, we obtain from formula (2.2),

$$\|\Psi_1(u,t)\| \leqslant [2a(|\gamma_1|B_1T + |\gamma_2|)A + |\gamma_3|B_2T]\|\Psi_0(u,t)\| \leqslant [2a(|\gamma_1|B_1T + |\gamma_2|)A + |\gamma_3|B_2T]D,$$

by induction, we have

$$\|\Psi_i(u,t)\| \leqslant \vartheta^i D, \quad \vartheta = [2a(|\gamma_1|B_1T + |\gamma_2|)A + |\gamma_3|B_2T] < 1, \quad i = 0, 1, 2, \ldots.$$

This leads us to conclude that there is a convergent solution for the sequence $\Xi_i(u,t)$. In this way, we have for $i \to \infty$,

$$\begin{aligned}\Xi(u,t) &= \lim_{i\to\infty} \left(g(u,t) + \gamma_1 \int_0^t \int_{-a}^{a} k(u,v)F(t,\tau)\Xi_i(v,\tau)dvd\tau + \gamma_2 \int_{-a}^{a} k(u,v)\Xi_i(v,t)dv + \gamma_3 \int_0^t G(t,\tau)\Xi_i(u,\tau)d\tau \right) \\ &= g(u,t) + \gamma_1 \int_0^t \int_{-a}^{a} k(u,v)F(t,\tau)\Xi(v,\tau)dvd\tau + \gamma_2 \int_{-a}^{a} k(u,v)\Xi(v,t)dv + \gamma_3 \int_0^t G(t,\tau)\Xi(u,\tau)d\tau.\end{aligned}$$

Consequently, it is proven that there is an MV-FIE solution (1.1). □

Lemma 2.3. *The function $\Xi(u,t)$ represents a unique solution of MV-FIE (1.1).*

Proof. Assuming that there is another continuous solution $\Xi^*(u,t)$ of MV-FIE (1.1), we may demonstrate that $\Xi(u,t)$ is the only solution, then we obtain

$$\Xi^*(u,t) = g(u,t) + \gamma_1 \int_0^t \int_{-a}^{a} k(u,v)F(t,\tau)\Xi^*(v,\tau)dvd\tau$$

$$+\gamma_2\int_{-a}^{a}k(u,v)\Xi^*(v,t)dv+\gamma_3\int_0^t G(t,\tau)\Xi^*(u,\tau)d\tau,$$

and

$$\Xi(u,t)-\Xi^*(u,t)=\gamma_1\int_0^t\int_{-a}^{a}k(u,v)F(t,\tau)[\Xi(v,\tau)-\Xi^*(v,\tau)]dvd\tau$$
$$+\gamma_2\int_{-a}^{a}k(u,v)[\Xi(v,t)-\Xi^*(v,t)]dv+\gamma_3\int_0^t G(t,\tau)[\Xi(u,\tau)-\Xi^*(u,\tau)]d\tau.$$

By using the norm's properties, we have

$$\|\Xi(u,t)-\Xi^*(u,t)\|\leqslant|\gamma_1|\left\|\int_0^t\int_{-a}^{a}k(u,v)F(t,\tau)[\Xi(v,\tau)-\Xi^*(v,\tau)]dvd\tau\right\|$$
$$+|\gamma_2|\left\|\int_{-a}^{a}k(u,v)[\Xi(v,t)-\Xi^*(v,t)]dv\right\|+|\gamma_3|\left\|\int_0^t G(t,\tau)[\Xi(u,\tau)-\Xi^*(u,\tau)]d\tau\right\|.$$

Using conditions (i) and (ii), we have

$$\|\Xi(u,t)-\Xi^*(u,t)\|\leqslant[2a(|\gamma_1|B_1T+|\gamma_2|)A+|\gamma_3|B_2T]\|\Xi(u,t)-\Xi^*(u,t)\|\leqslant\vartheta\|\Xi(u,t)-\Xi^*(u,t)\|,\quad\vartheta<1.$$

If $\|\Xi(u,t)-\Xi^*(u,t)\|\neq 0$, then the last formula yields $\vartheta\geqslant 1$, which is a contradiction. Therefore, $\|\Xi(u,t)-\Xi^*(u,t)\|=0$ and it is implied that $\Xi(u,t)=\Xi^*(u,t)$, which means the solution is unique. □

3. Projection-iteration method (PIM)

The projection-iteration method is a technique often used in various fields such as optimization, numerical analysis, and machine learning, particularly for solving convex optimization problems or finding fixed-point solutions in non-linear equations. Here's a brief overview.

1. Iterative approach: The method is based on generating a sequence of iterates that converge to a desired solution.
2. Projection: At each iteration, the method typically involves projecting the current iterate onto a feasible set or a solution space. This projection is often done to ensure that constraints or certain properties of the solution are respected.
3. Iteration: The iterate is updated based on some rule, which may involve using information from previous iterates, gradients, or other iterative schemes.

Rewrite (1.1) in the operator effect that follows in order to apply the projection-iteration approach to the discussion of the numerical solution of the MV-FIE (1.1):

$$\begin{aligned}\bar{\Xi}&=g+\gamma_1KF(\Xi)+\gamma_2K(\Xi)+\gamma_3G(\Xi),\quad & KF(\Xi)&=\int_0^t\int_{-a}^{a}k(u,v)F(t,\tau)\Xi(v,\tau)dvd\tau,\\ K(\Xi)&=\int_{-a}^{a}k(u,v)\Xi(v,t)dv, & G(\Xi)&=\int_0^t G(t,\tau)\Xi(u,\tau)d\tau.\end{aligned}\tag{3.1}$$

Next, based on the following lemma, we provide different versions of the projection-iteration approach of MV-FIE (1.1) in the space $L_2[-a,a]\times C[0,T]$.

Lemma 3.1. *There is an orthogonal projection operator* Q *for the projection operator* P *that satisfies the relation*

$$\|Pu+Qv\|^2=\|Pu\|^2+\|Qv\|^2\quad(Q=I-P).\tag{3.2}$$

To obtain the approximate solution of Eq. (3.1), we will now build two distinct logarithm types using the projection-iteration method.

3.1. The first algorithm

Create the numerical solution $\{\Xi_k\}$ series in the following format:

$$\Xi_k = g + \gamma_1 KF[\Xi_{k-1} + \Theta_k] + \gamma_2 K[\Xi_{k-1} + \Theta_k] + \gamma_3 G[\Xi_{k-1} + \Theta_k] \quad (k \geqslant 1),$$

where $\Theta_k = P[\Xi_k - \Xi_{k-1}]$, $k = 1, 2, 3, \ldots$, and $\Xi_0 \in L_2[-a, a] \times C[0, T]$. Then, we have

$$\Xi_k = g + \gamma_1 KF[P(\Xi_k) + Q(\Xi_{k-1})] + \gamma_2 K[P(\Xi_k) + Q(\Xi_{k-1})] + \gamma_3 G[P(\Xi_k) + Q(\Xi_{k-1})] \quad (k \geqslant 1), \tag{3.3}$$

the final equation yields to

$$\Xi_k = g + \gamma_1 KFP(\Xi_k) + \gamma_2 KP(\Xi_k) + \gamma_3 GP(\Xi_k) + \gamma_1 KFQ(\Xi_{k-1}) + \gamma_2 KQ(\Xi_{k-1}) + \gamma_3 GQ(\Xi_{k-1}). \tag{3.4}$$

Theorem 3.2. *Assume that in the space $L_2[-a, a] \times C[0, T]$, the two operators P and Q are constrained by the constants M_1 and M_2, respectively, and the condition*

$$1 > [|\gamma_1|\|KF\| + |\gamma_2|\|K\| + |\gamma_3|\|G\|](M_1^2 + M_2^2)^{\frac{1}{2}} \tag{3.5}$$

is satisfied, then, in the space $L_2[-a, a] \times C[0, T]$, the sequence $\{\Xi_k\}$ of a unique solution of Eq. (3.4) converges to the unique solution $\bar{\Xi}$ of Eq. (3.1) and the estimating error is given by the relation,

$$\|\bar{\Xi} - \Xi_k\| \leqslant \frac{\delta_1^k}{1 - \delta_1} \|\Xi_1 - \Xi_0\|, \quad \delta_1 < 1,$$

where

$$\delta_1 = \frac{[\|\gamma_1 KF\| + \|\gamma_2 K\| + \|\gamma_3 G\|] M_2}{1 - [\|\gamma_1 KF\| + \|\gamma_2 K\| + \|\gamma_3 G\|] M_1}.$$

Proof. We will use the Banach fixed point theorem to demonstrate that there is only one solution to Eq. (3.4) in the space $L_2[-a, a] \times C[0, T]$. For this, we follow to

$$\begin{aligned} \Xi_k - \Xi_{k-1} = {} & \gamma_1 KFP(\Xi_k - \Xi_{k-1}) + \gamma_2 KP(\Xi_k - \Xi_{k-1}) + \gamma_3 GP(\Xi_k - \Xi_{k-1}) \\ & + \gamma_1 KFQ(\Xi_{k-1} - \Xi_{k-2}) + \gamma_2 KQ(\Xi_{k-1} - \Xi_{k-2}) + \gamma_3 GQ(\Xi_{k-1} - \Xi_{k-2}). \end{aligned}$$

Hence

$$\begin{aligned} \|\Xi_k - \Xi_{k-1}\| \leqslant {} & |\gamma_1|\|KF\|\|P(\Xi_k - \Xi_{k-1})\| + |\gamma_2|\|K\|\|P(\Xi_k - \Xi_{k-1})\| + |\gamma_3|\|\|G\|\|P(\Xi_k - \Xi_{k-1})\| \\ & + |\gamma_1|\|KF\|\|Q(\Xi_{k-1} - \Xi_{k-2})\| + |\gamma_2|\|K\|\|Q(\Xi_{k-1} - \Xi_{k-2})\| + |\gamma_3|\|G\|\|Q(\Xi_{k-1} - \Xi_{k-2}). \end{aligned}$$

Then

$$\begin{aligned} \|\Xi_k - \Xi_{k-1}\| \leqslant {} & |\gamma_1|\|KF\|M_1\|\Xi_k - \Xi_{k-1}\| + |\gamma_2|\|K\|M_1\|\Xi_k - \Xi_{k-1}\| + |\gamma_3|\|G\|M_1\|\Xi_k - \Xi_{k-1}\| \\ & + |\gamma_1|\|KF\|M_2\|\Xi_{k-1} - \Xi_{k-2}\| + |\gamma_2|\|K\|M_2\|\Xi_{k-1} - \Xi_{k-2}\| + |\gamma_3|\|G\|M_2\|\Xi_{k-1} - \Xi_{k-2}\|, \end{aligned}$$

it is adaptable to form $\|\Xi_k - \Xi_{k-1}\| \leqslant \delta_1\|\Xi_{k-1} - \Xi_{k-2}\|$. The same methods allow us to demonstrate that $\|\Xi_{k-1} - \Xi_{k-2}\| \leqslant \delta_1\|\Xi_{k-2} - \Xi_{k-3}\|$, then $\|\Xi_k - \Xi_{k-1}\| \leqslant \delta_1^2\|\Xi_{k-2} - \Xi_{k-3}\|$. Applying this successively,

$$\|\Xi_k - \Xi_{k-1}\| \leqslant \delta_1^2\|\Xi_{k-2} - \Xi_{k-3}\| \leqslant \delta_1^3\|\Xi_{k-3} - \Xi_{k-4}\| \leqslant \cdots.$$

So, by induction we obtain

$$\|\Xi_k - \Xi_{k-1}\| \leqslant \delta_1^{k-1}\|\Xi_1 - \Xi_0\|. \tag{3.6}$$

Now, we get

$$\|\Xi_{k+q} - \Xi_k\| = \|\Xi_{k+q} - \Xi_{k+q-1} + \Xi_{k+q-1} + \Xi_k\|.$$

Utilizing the norm's properties we have

$$\|\Xi_{k+q} - \Xi_k\| \leqslant \|\Xi_{k+q} - \Xi_{k+q-1}\| + \|\Xi_{k+q-1} + \Xi_k\|,$$

using Eq. (3.6) we obtain

$$\|\Xi_{k+q} - \Xi_k\| \leqslant \delta_1^{k+q-1}\|\Xi_1 - \Xi_0\| + \|\Xi_{k+q-1} - \Xi_{k+q-2}\| + \|\Xi_{k+q-2} + \Xi_k\|.$$

Then

$$\|\Xi_{k+q} - \Xi_k\| \leqslant (\delta_1^{k+q-1} + \delta_1^{k+q-2})\|\Xi_1 - \Xi_0\| + \|\Xi_{k+q-2} - \Xi_{k+q-3}\| + \|\Xi_{k+q-3} + \Xi_k\|.$$

By carrying out this process again, we get

$$\|\Xi_{k+q} - \Xi_k\| \leqslant (\delta_1^{k+q-1} + \delta_1^{k+q-2} + \delta_1^{k+q-3} + \cdots + \delta_1^{k+1})\|\Xi_1 - \Xi_0\| + \|\Xi_{k+1} - \Xi_k\|,$$

so, we have

$$\|\Xi_{k+q} - \Xi_k\| \leqslant (\delta_1^{k+q-1} + \delta_1^{k+q-2} + \delta_1^{k+q-3} + \cdots + \delta_1^{k+1} + \delta_1^{k})\|\Xi_1 - \Xi_0\|.$$

The formula mentioned above can be modified to form

$$\|\Xi_{k+q} - \Xi_k\| \leqslant \delta_1^{k}(1 + \delta_1 + \delta_1^2 + \cdots)\|\Xi_1 - \Xi_0\|,$$

the final inequality yields to

$$\|\Xi_{k+q} - \Xi_k\| \leqslant \frac{\delta_1^k}{1-\delta_1}\|\Xi_1 - \Xi_0\|.$$

The final inequality displays that $\{\Xi_k\}$ is the Cauchy sequence. Since the space $L_2[-a, a] \times C[0, T]$ is a complete space, then there exists $\bar{\Xi}$ such that $\|\bar{\Xi} - \Xi_k\| \to 0$. Then $\bar{\Xi}$ is the unique solution of Eq. (3.1) in the space $L_2[-a, a] \times C[0, T]$. Also, if $q \to \infty$, the estimate error can be obtained. Now, if the bounded operator is not confirmed in the complete space then we verify that $\|\bar{\Xi} - \Xi_k\| \to 0$ as $k \to \infty$ in a certain ball of Banach space. □

Theorem 3.3. *Suppose* $\|P(\Xi)\| \leqslant N_1(\wp)$ *and* $\|Q(\Xi)\| \leqslant N_2(\wp)$ *are satisfied in a certain ball* $V(\|\Xi\| \leqslant \wp)$*, where* N_1 *and* N_2 *are functions of positive values, and a radius* $\wp > 0$*. In the event that the condition* (3.5) *holds true, the inequality*

$$\wp \geqslant [|\gamma_1|\|KF\| + |\gamma_2|\|K\| + |\gamma_3|\|G\|](N_1^2(\wp) + N_2^2(\wp))^{\frac{1}{2}}. \tag{3.7}$$

Then, we can determine the sequence $\{\Xi_k\}$ *in the ball* $V(\|\Xi\| \leqslant \wp)$*, where the unique solution of Eq.* (3.4) *converges to the unique solution* $\bar{\Xi}$ *of Eq.* (3.1) *in that ball for every* $\Xi_0 \in V(\|\Xi\| \leqslant \wp)$ *and the estimating error holds.*

Proof. Taking Eq. (3.3) as a norm gives us,

$$\|\Xi_k\| \leqslant \|g\| + |\gamma_1|\|KF[P(\Xi_k) + Q(\Xi_{k-1})]\| + |\gamma_2|\|K[P(\Xi_k) + Q(\Xi_{k-1})]\| + |\gamma_3|\|G[P(\Xi_k) + Q(\Xi_{k-1})]\|.$$

This could be expressed as

$$\begin{aligned}\|\Xi_k\| \leqslant \|g\| &+ |\gamma_1|\|KF\|\sqrt{\|P(\Xi_k) + Q(\Xi_{k-1})\|^2} \\ &+ |\gamma_2|\|K\|\sqrt{\|P(\Xi_k) + Q(\Xi_{k-1})\|^2} + |\gamma_3|\|G\|\sqrt{\|P(\Xi_k) + Q(\Xi_{k-1})\|^2},\end{aligned}$$

using relation (3.2), we obtain

$$\begin{aligned}\|\Xi_k\| \leqslant \|g\| &+ |\gamma_1|\|KF\|\sqrt{\|P(\Xi_k)\|^2 + \|Q(\Xi_{k-1})\|^2} \\ &+ |\gamma_2|\|K\|\sqrt{\|P(\Xi_k)\|^2 + \|Q(\Xi_{k-1})\|^2} + |\gamma_3|\|G\|\sqrt{\|P(\Xi_k)\|^2 + \|Q(\Xi_{k-1})\|^2}.\end{aligned}$$

Then

$$\|\Xi_k\| \leqslant \|g\| + |\gamma_1|\|KF\|\sqrt{N_1^2(\wp)+N_2^2(\wp)} + |\gamma_2|\|K\|\sqrt{N_1^2(\wp)+N_2^2(\wp)} + |\gamma_3|\|G\|\sqrt{N_1^2(\wp)+N_2^2(\wp)}.$$

Then formula (3.7) gives $\|\Xi_k\| \leqslant \wp$. For any $\Xi \in V(\|\Xi\| \leqslant \wp)$, there exists $\Xi_k \in V(\|\Xi\| \leqslant \wp)$, as the aforementioned inequality holds true for $\wp$. In that case, the condition (3.5) is true and there is a unique solution for Eq. (3.4) in that ball. □

3.2. *The second algorithm*

One way to develop the second algorithm is as follows:

$$\Xi_k = g + \gamma_1[KF\Xi_{k-1} + A_k] + \gamma_2[K\Xi_{k-1} + B_k] + \gamma_3[G\Xi_{k-1} + C_k] \quad (k \geqslant 1), \tag{3.8}$$

where

$$A_k = P[KF\Xi_k - KF\Xi_{k-1}], \quad B_k = P[K\Xi_k - K\Xi_{k-1}], \quad C_k = P[G\Xi_k - G\Xi_{k-1}].$$

Utilizing Eq. (3.2), we can rewrite Eq. (3.8) in the form as

$$\Xi_k = g + \gamma_1[PKF(\Xi_k) + QKF(\Xi_{k-1})] + \gamma_2[PK(\Xi_k) + QK(\Xi_{k-1})] + \gamma_3[PG(\Xi_k) + QG(\Xi_{k-1})] \quad (k \geqslant 1). \tag{3.9}$$

Theorem 3.4. *If the condition $1 > [|\gamma_1|\|KF\| + |\gamma_2|\|K\| + |\gamma_3|\|G\|]$ is satisfied, then, in the space $L_2[-a,a] \times C[0,T]$, the sequence $\{\Xi_k\}$ of a unique solution of Eq. (3.9) converges to the unique solution $\bar{\Xi}$ of Eq. (3.1), and the estimating error is given by the relation $\|\bar{\Xi} - \Xi_k\| \leqslant \frac{\delta_2^k}{1-\delta_2}\|\Xi_1 - \Xi_0\|$, $\delta_2 < 1$, where*

$$\delta_2 = \frac{[\|\gamma_1 QKF\| + \|\gamma_2 QK\| + \|\gamma_3 QG\|]}{1 - [\|\gamma_1 PKF\| + \|\gamma_2 PK\| + \|\gamma_3 PG\|]}.$$

Proof. In the space $L_2[-a,a] \times C[0,T]$, we demonstrate that Eq. (3.9) has a unique solution by doing the following

$$\Xi_k - \Xi_{k-1} = \gamma_1 PKF(\Xi_k - \Xi_{k-1}) + \gamma_2 PK(\Xi_k - \Xi_{k-1}) + \gamma_3 PG(\Xi_k - \Xi_{k-1}) + \gamma_1 QKF(\Xi_{k-1} - \Xi_{k-2}) + \gamma_2 QK(\Xi_{k-1} - \Xi_{k-2}) + \gamma_3 QG(\Xi_{k-1} - \Xi_{k-2}),$$

Hence,

$$\|\Xi_k - \Xi_{k-1}\| \leqslant |\gamma_1|\|PKF(\Xi_k - \Xi_{k-1})\| + |\gamma_2|\|PK(\Xi_k - \Xi_{k-1})\| + |\gamma_3|\|\|PG(\Xi_k - \Xi_{k-1})\| \; |\gamma_1|\|QKF(\Xi_{k-1} - \Xi_{k-2})\| + |\gamma_2|\|QK(\Xi_{k-1} - \Xi_{k-2})\| + |\gamma_3|\|QG(\Xi_{k-1} - \Xi_{k-2})\|.$$

This formula can be expressed as $\|\Xi_k - \Xi_{k-1}\| \leqslant \delta_2\|\Xi_{k-1} - \Xi_{k-2}\|$. Next, via induction, we obtain $\|\Xi_k - \Xi_{k-1}\| \leqslant \delta_2^{k-1}\|\Xi_1 - \Xi_0\|$. Thus, the final inequality becomes to $\|\Xi_{k+q} - \Xi_k\| \leqslant \frac{\delta_2^k}{1-\delta_1}\|\Xi_1 - \Xi_0\|$. We finish the proof as stated in Theorem 3.2. □

Theorem 3.5. *If the two conditions $1 > [|\gamma_1|\|KF\| + |\gamma_2|\|K\| + |\gamma_3|\|G\|]$ and $\|\Xi\| \leqslant N(\wp)$ are satisfied in a certain ball $V(\|\Xi\| \leqslant \wp)$, where N is a positive value function, assuming that the inequality is $\wp > 0$,*

$$\wp \geqslant [|\gamma_1|\|KF\| + |\gamma_2|\|K\| + |\gamma_3|\|G\|]N(\wp).$$

Then, the sequence $\{\Xi_k\}$ in the ball $V(\|\Xi\| \leqslant \wp)$ of unique solutions of Eq. (3.9) converges to the unique solution of Eq. (3.1) in this ball for all $\Xi_0 \in V(\|\Xi\| \leqslant \wp)$ and the estimating error holds.

Proof. We can observe from Eq. (3.9) that

$$\|\Xi_k\| \leqslant \|g\| + |\gamma_1|\sqrt{\|PKF(\Xi_k) + QKF(\Xi_{k-1})\|^2} + |\gamma_2|\sqrt{\|PK(\Xi_k) + QK(\Xi_{k-1})\|^2} + |\gamma_3|\sqrt{\|PG(\Xi_k) + QG(\Xi_{k-1})\|^2},$$

using relation (3.2), we have

$$\|\Xi_k\| \leqslant \|g\| + |\gamma_1|\sqrt{\|PKF(\Xi_k)\|^2 + \|QKF(\Xi_{k-1})\|^2} + |\gamma_2|\sqrt{\|PK(\Xi_k)\|^2 + \|QK(\Xi_{k-1})\|^2} + |\gamma_3|\sqrt{\|PG(\Xi_k)\|^2 + \|QG(\Xi_{k-1})\|^2}.$$

Utilizing the last theorem's second condition, we obtain

$$\|\Xi_k\| \leqslant \|g\| + |\gamma_1|\sqrt{\|PKF\|^2 + \|QKF\|^2}N(\wp) + |\gamma_2|\sqrt{\|PK\|^2 + \|QK\|^2}N(\wp) + |\gamma_3|\sqrt{\|PG\|^2 + \|QG\|^2}N(\wp).$$

It is adaptable as

$$\|\Xi_k\| \leqslant \|g\| + [|\gamma_1|\|KF\| + |\gamma_2|\|K\| + |\gamma_3|\|G\|]N(\wp) \leqslant \wp.$$

If we discover $\wp > 0$ for which the above inequality is satisfied for all $\Xi_k \in V(\|\Xi\| \leqslant \wp)$, then, the believed ball of Eq. (3.9) has a unique solution. One can finish the proof by the exact method of Theorem 3.3. □

4. Examples

In this section, we examine a few issues that various researchers have resolved through the use of various numerical techniques. Next, we assess the relative inaccuracy between the approach employed in this study and the approach that was previously employed. In addition, we obtain the estimating errors δ_1, δ_2 at various times t, as well as the numerical solutions to a some problems.

Example 4.1. Consider the following MV-FIE:

$$\Xi(u,t) = g(u,t) + \int_0^t \int_{-1}^1 e^{uv} t^2 \tau^2 \Xi(v,\tau) dv d\tau + \int_{-1}^1 e^{uv} \Xi(v,t) dv + \int_0^t t\tau \Xi(u,\tau) d\tau, \tag{4.1}$$

where the function $g(u,t)$ is specified by laying $\Xi(u,t) = u^2 t^2$ as an exact solution. We apply P-IM for first and second algorithms with $k = 10$ on the integral equation (4.1). In Tables 1-4, for $u \in [-1,1]$, $t \in [0, 0.6]$, the errors of P-IM for first and second algorithms of (4.1) are calculated for $k = 10$.

Table 1: Error for Example 4.1 at $k = 10$ and $t = 0$.

u	Error-PIM for first algorithm	Error-PIM for second algorithm
-1.0	5.28471×10^{-10}	5.54223×10^{-10}
-0.8	4.12768×10^{-10}	3.41552×10^{-10}
-0.6	3.24856×10^{-10}	2.74264×10^{-10}
-0.4	5.74235×10^{-11}	4.97235×10^{-11}
-0.2	3.54328×10^{-11}	4.57236×10^{-11}
0.0	2.34268×10^{-11}	3.58214×10^{-11}
0.2	3.54785×10^{-11}	4.90574×10^{-11}
0.4	5.62741×10^{-11}	5.11752×10^{-11}
0.6	3.54782×10^{-10}	2.53147×10^{-10}
0.8	4.28304×10^{-10}	3.95201×10^{-10}
1.0	5.12749×10^{-10}	5.34721×10^{-10}

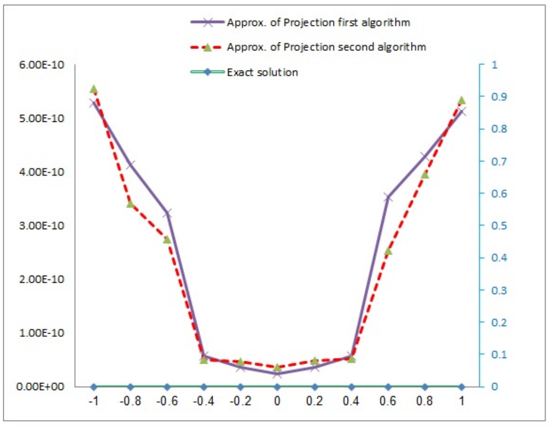

Figure 1: Exact solution, approximate solution of P-IM for first algorithm and approximate solution of P-IM for second algorithm at $t = 0$.

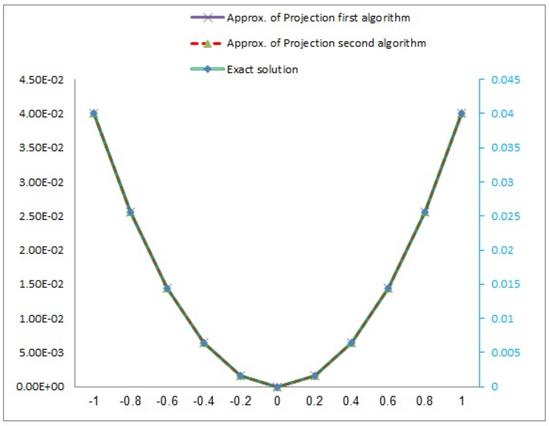

Figure 2: Exact solution, approximate solution of P-IM for first algorithm, and approximate solution of P-IM for second algorithm at $t = 0.2$.

Table 2: Error for Example 4.1 at $k = 10$ and $t = 0.2$.

u	Error-PIM for first algorithm	Error-PIM for second algorithm
-1.0	4.17528×10^{-9}	3.99832×10^{-9}
-0.8	3.87124×10^{-9}	3.49236×10^{-9}
-0.6	7.41253×10^{-10}	7.11546×10^{-10}
-0.4	6.87235×10^{-10}	6.52874×10^{-10}
-0.2	4.11752×10^{-10}	4.00713×10^{-10}
0.0	3.58219×10^{-10}	2.93756×10^{-10}
0.2	4.00785×10^{-10}	3.89475×10^{-10}
0.4	6.28165×10^{-10}	6.05732×10^{-10}
0.6	6.85274×10^{-10}	6.56874×10^{-10}
0.8	3.51276×10^{-9}	3.47251×10^{-9}
1.0	4.57268×10^{-9}	4.42789×10^{-9}

Table 3: Error for Example 4.1 at $k = 10$ and $t = 0.4$.

u	Error-PIM for first algorithm	Error-PIM for second algorithm
-1.0	8.52741×10^{-9}	8.27365×10^{-9}
-0.8	7.41856×10^{-9}	7.00475×10^{-9}
-0.6	5.97250×10^{-9}	5.52764×10^{-9}
-0.4	5.23147×10^{-9}	4.72985×10^{-9}
-0.2	3.84275×10^{-9}	2.94023×10^{-9}
0.0	2.85074×10^{-9}	1.58123×10^{-9}
0.2	3.57210×10^{-9}	3.11584×10^{-9}
0.4	4.98032×10^{-9}	4.86147×10^{-9}
0.6	6.12745×10^{-9}	5.87231×10^{-9}
0.8	7.59247×10^{-9}	7.48203×10^{-9}
1.0	1.74865×10^{-9}	1.68426×10^{-8}

In Figure 1, we computed exact solution, approximate solution of P-IM for first algorithm, and approximate solution of P-IM for second algorithm with different values of u and $k = 10$ at $t = 0$.

In Figure 2, we computed exact solution, approximate solution of P-IM for first algorithm, and approximate solution of P-IM for second algorithm with different values of u and $k = 10$ at $t = 0.2$.

In Figure 3, we computed exact solution, approximate solution of P-IM for first algorithm, and approximate solution of P-IM for second algorithm with different values of u and $k = 10$ at $t = 0.4$. In Figure 4, we computed exact solution, approximate solution of P-IM for first algorithm, and approximate solution of P-IM for second algorithm with different values of u and $k = 10$ at $t = 0.6$.

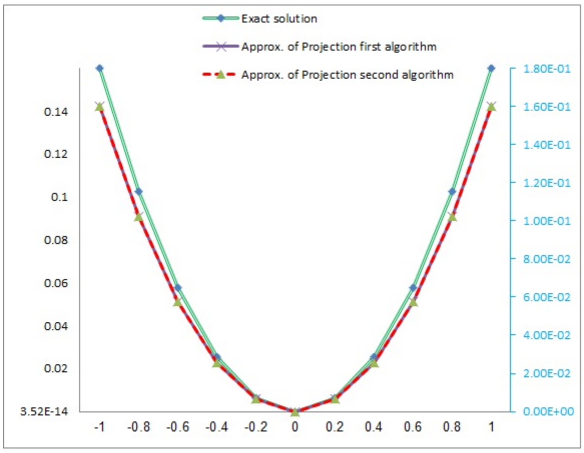

Figure 3: Exact solution, approximate solution of P-IM for first algorithm, and approximate solution of P-IM for second algorithm at $t = 0.4$.

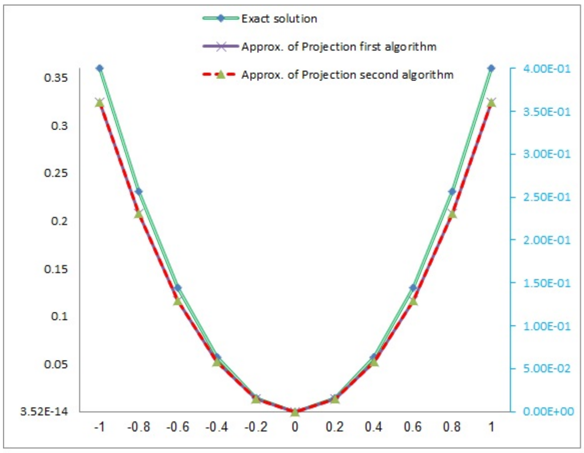

Figure 4: Exact solution, approximate solution of P-IM for first algorithm, and approximate solution of P-IM for second algorithm at $t = 0.6$.

Table 4: Error for Example 4.1 at $k = 10$ and $t = 0.6$.

u	Error-PIM for first algorithm	Error-PIM for second algorithm
-1.0	4.71856×10^{-8}	4.76230×10^{-8}
-0.8	3.85426×10^{-8}	3.41520×10^{-8}
-0.6	9.67416×10^{-9}	9.67834×10^{-9}
-0.4	8.52741×10^{-9}	8.26471×10^{-9}
-0.2	6.87163×10^{-9}	6.85234×10^{-9}
0.0	5.87204×10^{-9}	5.67287×10^{-9}
0.2	6.74268×10^{-9}	6.47205×10^{-9}
0.4	8.34521×10^{-9}	8.31276×10^{-9}
0.6	9.57269×10^{-9}	9.36824×10^{-9}
0.8	4.17201×10^{-8}	3.98247×10^{-8}
1.0	4.96852×10^{-8}	4.57128×10^{-8}

Example 4.2. Consider the following MV-FIE:

$$\begin{aligned}\Xi(u,t) = g(u,t) + 0.5\int_0^t\int_{-\pi}^{\pi}\cos(v)\sin(u)t\sin(\tau)\Xi(v,\tau)dvd\tau \\ + 2\int_{-\pi}^{\pi}\cos(v)\sin(u)\Xi(v,t)dv + \int_0^t te^{\tau}\Xi(u,\tau)d\tau,\end{aligned} \tag{4.2}$$

where the function $g(u,t)$ is specified by laying $\Xi(u,t) = (0.25+u)\sin(t)$ as an exact solution. We apply P-IM for first and second algorithms with $k = 20$ on the integral equation (4.2). In Tables 5-8, for $u \in [-\pi, \pi]$, $t = \{0.001, 0.03, 0.5, 0.9\}$, the errors of P-IM for first and second algorithms of (4.2) are calculated for $k = 20$.

In Figure 5, we computed exact solution, approximate solution of P-IM for first algorithm, and approximate solution of P-IM for second algorithm with different values of u and $k = 20$ at $t = 0.001$.

In Figure 6, we computed exact solution, approximate solution of P-IM for first algorithm, and approximate solution of P-IM for second algorithm with different values of u and $k = 20$ at $t = 0.03$.

In Figure 7, we computed exact solution, approximate solution of P-IM for first algorithm, and approximate solution of P-IM for second algorithm with different values of u and $k = 20$ at $t = 0.5$.

In Figure 8, we computed exact solution, approximate solution of P-IM for first algorithm, and approximate solution of P-IM for second algorithm with different values of u and $k = 20$ at $t = 0.9$.

Table 5: Error for Example 4.2 at $k = 20$ and $t = 0.001$.

u	Error-PIM for first algorithm	Error-PIM for second algorithm
$-\pi$	3.51472×10^{-15}	1.24138×10^{-15}
$-\frac{9\pi}{10}$	3.23854×10^{-15}	2.99453×10^{-16}
$-\frac{7\pi}{10}$	4.77201×10^{-16}	4.39245×10^{-16}
$-\frac{\pi}{2}$	4.71027×10^{-16}	4.23147×10^{-16}
$-\frac{3\pi}{10}$	3.94578×10^{-16}	3.59287×10^{-16}
$-\frac{\pi}{10}$	3.50745×10^{-16}	3.37056×10^{-16}
$\frac{\pi}{10}$	3.72864×10^{-16}	3.54238×10^{-16}
$\frac{3\pi}{10}$	3.62745×10^{-16}	3.36217×10^{-16}
$\frac{\pi}{5}$	4.35721×10^{-16}	4.17268×10^{-16}
$\frac{7\pi}{10}$	1.00712×10^{-15}	6.50764×10^{-16}
$\frac{9\pi}{10}$	3.17258×10^{-15}	2.57234×10^{-15}
π	3.36245×10^{-15}	3.12680×10^{-15}

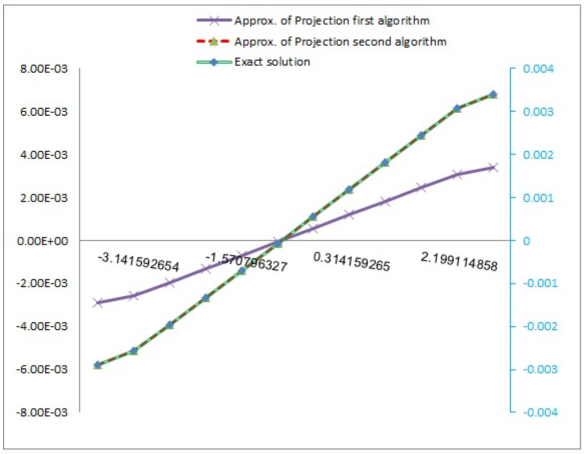

Figure 5: Exact solution, approximate solution of P-IM for first algorithm, and approximate solution of P-IM for second algorithm at $t = 0.001$.

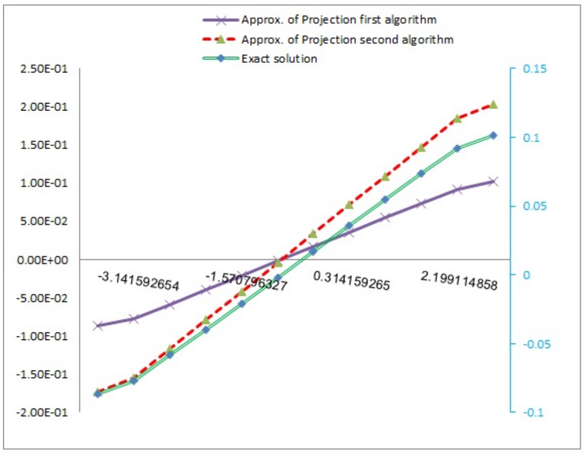

Figure 6: Exact solution, approximate solution of P-IM for first algorithm, and approximate solution of P-IM for second algorithm at $t = 0.03$.

Table 6: Error for Example 4.2 at $k = 20$ and $t = 0.03$.

u	Error-PIM for first algorithm	Error-PIM for second algorithm
$-\pi$	3.47236×10^{-14}	2.99354×10^{-14}
$-\frac{9\pi}{10}$	1.62745×10^{-14}	1.39524×10^{-14}
$-\frac{7\pi}{10}$	6.23741×10^{-15}	6.12864×10^{-15}
$-\frac{\pi}{2}$	4.12806×10^{-15}	4.02741×10^{-15}
$-\frac{3\pi}{10}$	2.47210×10^{-15}	1.98234×10^{-15}
$-\frac{\pi}{10}$	1.87264×10^{-15}	1.32647×10^{-15}
$\frac{\pi}{10}$	1.38247×10^{-15}	1.20875×10^{-15}
$\frac{3\pi}{10}$	2.11453×10^{-15}	1.98642×10^{-15}
$\frac{\pi}{5}$	4.78264×10^{-15}	4.47632×10^{-15}
$\frac{7\pi}{10}$	1.35248×10^{-14}	1.10876×10^{-14}
$\frac{9\pi}{10}$	1.88647×10^{-14}	1.27605×10^{-14}
π	2.74613×10^{-13}	2.35741×10^{-13}

Table 7: Error for Example 4.2 at $k = 20$ and $t = 0.5$.

u	Error-PIM for first algorithm	Error-PIM for second algorithm
$-\pi$	4.17526×10^{-12}	3.98526×10^{-12}
$-\frac{9\pi}{10}$	2.57324×10^{-13}	2.01856×10^{-13}
$-\frac{7\pi}{10}$	7.41586×10^{-14}	7.12864×10^{-14}
$-\frac{\pi}{2}$	5.74682×10^{-14}	5.28631×10^{-14}
$-\frac{3\pi}{10}$	3.57231×10^{-14}	3.37412×10^{-14}
$-\frac{\pi}{10}$	2.87521×10^{-14}	2.27496×10^{-14}
$\frac{\pi}{10}$	2.54231×10^{-14}	2.11864×10^{-14}
$\frac{3\pi}{10}$	3.74102×10^{-14}	3.57264×10^{-14}
$\frac{\pi}{5}$	5.45826×10^{-14}	5.15374×10^{-14}
$\frac{7\pi}{10}$	2.00749×10^{-13}	1.68347×10^{-13}
$\frac{9\pi}{10}$	2.64641×10^{-13}	2.24961×10^{-13}
π	4.25314×10^{-12}	4.00867×10^{-12}

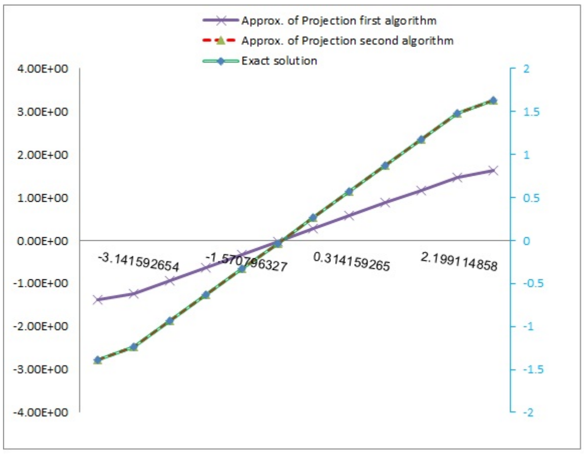

Figure 7: Exact solution, approximate solution of P-IM for first algorithm, and approximate solution of P-IM for second algorithm at $t = 0.5$.

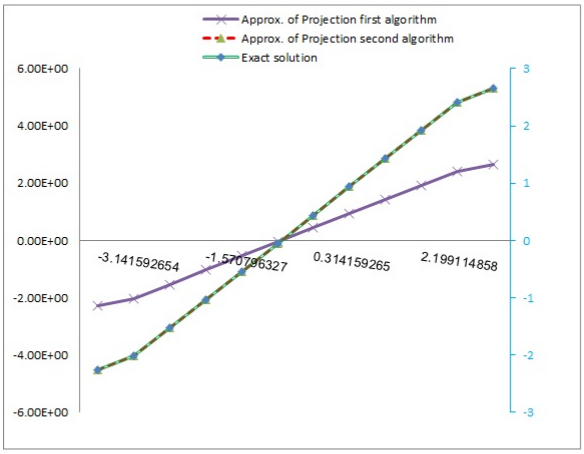

Figure 8: Exact solution, approximate solution of P-IM for first algorithm, and approximate solution of P-IM for second algorithm at $t = 0.9$.

Table 8: Error for Example 4.2 at $k = 20$ and $t = 0.9$.

u	Error-PIM for first algorithm	Error-PIM for second algorithm
$-\pi$	7.41567×10^{-10}	7.30854×10^{-10}
$-\frac{9\pi}{10}$	3.47621×10^{-10}	3.34287×10^{-10}
$-\frac{7\pi}{10}$	3.57283×10^{-11}	3.21765×10^{-11}
$-\frac{\pi}{2}$	7.34520×10^{-12}	6.98237×10^{-12}
$-\frac{3\pi}{10}$	5.87564×10^{-12}	5.67234×10^{-12}
$-\frac{\pi}{10}$	4.28634×10^{-12}	3.57621×10^{-12}
$\frac{\pi}{10}$	4.35742×10^{-12}	3.38264×10^{-12}
$\frac{3\pi}{10}$	5.74235×10^{-12}	5.56387×10^{-12}
$\frac{\pi}{5}$	7.58234×10^{-12}	7.28416×10^{-12}
$\frac{7\pi}{10}$	3.24705×10^{-11}	3.03874×10^{-11}
$\frac{9\pi}{10}$	3.25749×10^{-10}	3.11864×10^{-10}
π	7.21967×10^{-10}	6.87642×10^{-10}

5. General conclusion

From the work mentioned above, we can infer the following.

1. Equation (1.1) allows us to determine the following well-known special cases.
   (i) Suppose in (1.1), $F(t,\tau) = 0$, to get
   $$\Xi(u,t) = g(u,t) + \gamma_2 \int_{-a}^{a} k(u,v)\Xi(v,t)dv + \gamma_3 \int_0^t G(t,\tau)\Xi(u,\tau)d\tau. \tag{5.1}$$
   The Fredholm-Volterra integral equation of the second kind with discontinuous kernels is represented by the formula (5.1) above. Many researchers used various numerical techniques to determine the solution of (5.1), see [1].
   (ii) Fredholm-Volterra integral equations of the first kind, with various singular kernels, are acquired directly from (5.1).
   (iii) Suppose in (1.1), $G(t,\tau) = 0$, to get
   $$\Xi(u,t) = g(u,t) + \gamma_1 \int_0^t \int_{-a}^{a} k(u,v)F(t,\tau)\Xi(v,\tau)dvd\tau + \gamma_2 \int_{-a}^{a} k(u,v)\Xi(v,t)dv. \tag{5.2}$$
   The Volterra-Fredholm integral equation of the second kind is represented by formula (5.2). Many researchers have addressed the solution of (5.2) by various numerical techniques, for example, refer to [2, 3].
2. MV-FIE (1.1) has a unique solution $\Xi(u,t)$ in the Banach space $L_2[-a,a] \times C[0,T]$, under some conditions.
3. Since solving MV-FIEs with varied kernels is typically impossible analytically, approximate solutions must be obtained.
4. The paper's findings demonstrate the effectiveness and simplicity of this approach.
5. The solution sequence is used to hold the P-IM to assume two algorithms.
6. In Example 4.1, we considered the MV-FIE of the second kind when the exact solution is $\Xi(u,t) = u^2t^2$ and time $t \in [0, 0.6]$. As the value of u increases the error increases. Also, the error of the first algorithm is bigger than the error of the second algorithm.
7. In Example 4.2, the maximum error is at $u = \pm\pi$, and when the values of T are increasing, the error values increase slowly.
8. Wolfram Mathematica 10 was used to perform all of the calculations.

Future work

We will discuss the solution of the following equation in the case of a singular kernel:

$$\Xi(u,t) = g(u,t) + \gamma_1 \int_0^t \int_{-a}^{a} k(|u-v|)F(t,\tau)\Xi(v,\tau)dvd\tau + \gamma_2 \int_{-a}^{a} k(|u-v|)\Xi(v,t)dv + \gamma_3 \int_0^t G(t,\tau)\Xi(u,\tau)d\tau,$$

Acknowledgment

The authors express their gratitude to the Editorial Board and the reviewers for their insightful recommendations and remarks, which significantly enhanced the paper's final version. This research is funded by the Deanship of Graduate Studies and Scientific Research at Jouf University through the Fast-Track Research Funding Program.

References

[1] M. Abdel-Aty, M. Abdou, *Analytical and Numerical Discussion for the Phase-Lag Volterra-Fredholm Integral Equation with SingularKernel*, J. Appl. Anal. Comput., **13** (2023), 3203–3220. 1i
[2] M. A. Abdou, M. E. Nasr, M. A. Abdel-Aty, *Study of the normality and continuity for the mixed integral equations with phase-lag term*, Int. J. Math. Anal., **11** (2017), 787–799. 1iii
[3] M. A. Abdou, M. E. Nasr, M. A. Abdel-Aty, *A study of normality and continuity for mixed integral equations*, J. Fixed Point Theory Appl., **20** (2018), 19 pages. 1, 1iii

[4] S. A. Abusalim, M. A. Abdou, M. A. Abdel-Aty, M. E. Nasr, *Hybrid Functions Approach via Nonlinear Integral Equations with Symmetric and Nonsymmetrical Kernel in Two Dimensions*, Symmetry, **15** (2023), 19 pages. 1
[5] A. M. Al-Bugami, *Numerical treating of mixed integral equation two-dimensional in surface cracks in finite layers of materials*, Adv. Math. Phys., **2022** (2022), 12 pages. 1
[6] S. E. A. Alhazmi, *New model for solving mixed integral equation of the first kind with generalized potential kernel*, J. Math. Res., **9** (2017), 18–29. 1
[7] V. M. Aleksandov, E. V. Kovalenko, *Problems in continuum mechanics with mixed boundary conditions*, Nauk, Moscow, (1986). 1
[8] K. E. Atkinson, *The numerical solution of integral equations of the second kind*, Cambridge University Press, Cambridge, (1997).
[9] M. Basseem, *Degenerate method in mixed nonlinear three dimensions integral equation*, Alex. Eng. J., **58** (2019), 387–392. 1
[10] H. Brunner, *Collocation methods for Volterra integral and related functional differential equations*, Cambridge University Press, Cambridge, (2004). 1
[11] M. V. C. Chari, S. J. Salon, *Numerical methods in electromagnetism*, Academic Press, San Diego, (2000). 1
[12] L. M. Delves, J. L. Mohamed, *Computational methods for integral equations*, Cambridge University Press, Cambridge, (1985). 1
[13] J. Gao, M. Condon, A. Iserles, *Spectral computation of highly oscillatory integral equations in laser theory*, J. Compute. Phys., **395** (2019), 351–381. 1
[14] Z. Gouyandeh, T. Allahviranloo, A. Armand, *Numerical solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via tau-collocation method with convergence analysis*, J. Comput. Appl. Math., **308** (2016), 435–446. 1
[15] L. Grammont, P. B. Vasconcelos, M. Ahues, *A modified iterated projection method adapted to a nonlinear integral equations*, Appl. Math. Comput., **276** (2016), 432–441. 1
[16] R. M. Hafez, Y. H. Youssri, *Spectral Legendre-Chebyshev treatment of 2D linear and nonlinear mixed Volterra-Fredholm integral equation*, Math. Sci. Lett., **9** (2020), 37–47. 1
[17] B. H. Hashemi, M. Khodabin, K. Maleknejad, *Numerical method for solving linear stochastic Itô-Volterra integral equations driven by fractional Brownian motion using hat functions*, Turkish J. Math., **41** (2017), 611–624. 1
[18] M. H. Heydari, M. R. Hooshmandasl, F. M. M. Ghaini, C. Cattani, *A computational method for solving stochastic itô-volterra integral equations based on stochastic operational matrix for generalized hat basis functions*, J. Comput. Phys., **270** (2014), 402–415. 1
[19] A. R. Jan, *Solution of nonlinear mixed integral equation via collocation method basing on orthogonal polynomials*, Heliyen, **8** (2022), 9 pages. 1
[20] M. Lienert, R. Tumulka, *A new class of Volterra-type integral equations from relativistic quantum physics*, J. Integral Equ. Appl., **31** (2019), 535–569. 1
[21] N. Madbouly, *Solutions of Hammerstein integral equations arising from chemical reactor theory*, University of Strathclyde, PhD. thesis, (1996). 1
[22] A. M. S. Mahdy, M. A. Abdou, D. Sh. Mohamed, *A computational technique for computing second-type mixed integral equations with singular kernels*, J. Math. Comput. Sci., **32** (2024), 137–151. 1
[23] A. M. S. Mahdy, A. S. Nagdy, K. H. Hashem, D. S. Mohamed, *A computational technique for solving three-dimensional mixed volterra-fredholm integral equations*, Fractal Fract., **7** (2023), 14 pages. 1
[24] S. Micula, *An iterative numerical method for Fredholm-Volterra integral equations of the second kind*, Appl. Math. Comput., **270** (2015), 935–942. 1
[25] F. Mirzaee, *Numerical solution of nonlinear Fredholm-Volterra integral equations via Bell polynomials*, Comput. Methods Differ. Equ., **5** (2017), 88–102. 1
[26] F. Mirzaee, E. Hadadiyan, *Applying the modified block-pulse functions to solve the three-dimensional Volterra-Fredholm integral equations*, Appl. Math. Comput., **265** (2015), 759–767. 1
[27] F. Mirzaee, E. Hadadiyan, *Numerical solution of Volterra-Fredholm integral equations via modification of hat functions*, Appl. Math. Comput., **280** (2016), 110–123. 1
[28] F. Mirzaee, E. Hadadiyan, *Using operational matrix for solving nonlinear class of mixed volterra-fredholm integral equations*, Math. Methods Appl. Sci., **40** (2017), 3433–3444. 1
[29] F. Mirzaee, S. F. Hoseini, *Application of Fibonacci collocation method for solving Volterra-Fredholm integral equations*, Appl. Math. Comput., **273** (2016), 637–644. 1
[30] F. Mirzaee, N. Samadyar, *Convergence of 2D-orthonormal Bernstein collocation method for solving 2D-mixed Volterra-Fredholm integral equations*, Trans. A. Razmadze Math. Inst., **172** (2018), 631–641. 1
[31] M. E. Nasr, M. A. Abdel-Aty, *Analytical discussion for the mixed integral equations*, J. Fixed Point Theory Appl., **20** (2018), 19 pages. 1
[32] M. E. Nasr, M. A. Abdel-Aty, *A new techniques applied to Volterra-Fredholm integral equations with discontinuous kernel*, J. Comput. Anal. Appl., **29** (2021), 11–24. 1
[33] S. Noeiaghdam, S. Micula, *A novel method for solving second kind Volterra integral equations with discontinuous kernel*, Mathematics, **9** (2021), 12 pages. 1
[34] S. Paul, M. M. Panja, B. N. Mandal, *Use of Legendre multiwavelets to solve Carleman type singular integral equations*, Appl. Math. Model., **55** (2018), 522–535. 1

[35] G. Ya. Popov, *Contact problems for a linearly deformable foundation*, Vishcha Schola., Kyiv-Odesa,, (1982). 1
[36] A. M. Rocha, J. S. Azevedo, S. P. Oliveira, M. R. Correa, *Numerical analysis of a collocation method for functional integral equations*, Appl. Numer. Math., **134** (2018), 31–45. 1
[37] N. H. Sweilam, A. M. Nagy, I. K. Youssef, M. M. Mokhtar, *New spectral second kind Chebyshev wavelets scheme for solving systems of integro-differential equations*, Int. J. Appl. Comput. Math., **3** (2017), 333–345. 1
[38] A. N. Tikhonov, V. Y. Arsenin, *Solutions of ill-posed problems*, Wiley, Washington, (1977). 1
[39] K. Wang, Q. Wang, *Taylor polynomial method and error estimation for a kind of mixed Volterra-Fredholm integral equations*, Appl. Math. Comput., **229** (2014), 53–59. 1
[40] K. F. Warnick, *Numerical analysis for electromagnetic integral equations*, Artech House, Norwood, (2008). 1
[41] A.-M. Wazwaz, *Linear and Nonlinear integral equations: Methods and Applications*, Springer, Heidelberg, (2011). 1