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Qualitative analysis of a coupled system with nonlinear mixed fractional integro-differential equation involving Caputo fractional derivative

Muhammad Asif[a], Akbar Zada[a,*], Alexandru Avram[b]

[a] Department of Mathematics, University of Peshawar, 25000, Pakistan.

[b] Department of Computing, Mathematics and Engineering, "1 Decembrie 1918" University of Alba Iulia, Alba Iulia, 510 0 09, Romania.

Abstract

This paper introduces a novel model involving an integro-differential equation of fractional order characterized by multiple nonlinear terms with Caputo fractional derivatives. We derive conditions ensuring the existence, uniqueness, and stability of the mild solution. Furthermore, we explore a coupled system that includes the proposed equations and extend our analysis to an equation featuring three nonlinearities. The study utilizes Banach's, Krasnoselskii's, and Schaefer's fixed point theorems. Additionally, we investigate various forms of Hyers-Ulam stability for the equation and the coupled system. To demonstrate the practical implications of our findings, illustrative examples are provided.

Keywords: Caputo derivative, Riemann-Liouville fractional integral, existence, fixed point, Ulam stability.

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1. Introduction

The description of the fractional order of integral and derivative operators is the main focus of fractional calculus. It has a vast mathematical history that predates differential calculus. The derivative of the fractional order was presented by Leibnitz. The fractional differential and integral operators have many applications in applied mathematics. The fractional differential equations was utilized in the tautochronous problem [10], ultrasonic wave propagation [28], and electrical networks of real-world problems [11].

A fractional integro-differential equation is a kind of differential equation that incorporates both fractional order derivatives and integrals. By including fractional derivatives and integrals, it broadens the meaning of the concept of ordinary and partial differential equations. A fractional integro-differential equation can be written in general form as:

$$D^{\eta}u(\omega) = F(\omega, u(\omega), I^{\psi}u(\omega)),$$

*Corresponding Author

Email addresses: asiffatih123@gmail.com (Muhammad Asif), akbarzada@uop.edu.pk;zadababo@yahoo.com (Akbar Zada), alex.avram@uab.ro (Alexandru Avram)

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where D^{η} is the derivative of fractional order η. Due to the non-local and non-linear nature of fractional derivatives and integrals, solving fractional integro-differential equations can be difficult. To solve these problems, numerous numerical and analytical approaches can be applied [18–20]. Numerous writers have researched fractional-order initial and boundary value problems that involve diverse types of derivatives, including Riemann-Liouville, Caputo, and Hadamard types. Over the past few years, a range of intriguing and helpful results have been added to the literature on the subject, rapidly expanding it [5, 6, 17, 21, 29]. The existence as well as uniqueness of the solution for fractional differential equations and fractional integro differential equations is an important aspect. A number of valuable research works has been made in this regard in the past years [3, 4, 9, 12–14]. A coupled system consists of two or more interdependent components that interact and influence each other's behaviors. These can be found in diverse domains, such as mechanical systems like gears and motors, or more complex environmental and social systems where human activities interact with natural processes. In such systems, changes in one component affect the others, leading to feedback loops and emergent properties that are not apparent when the components are studied in isolation. The study of coupled systems spans various disciplines, including physics, engineering, and environmental science, and helps to improve our understanding of system-wide dynamics and optimization strategies. A number of helpful results on systems have been added [1, 7, 23, 24, 27, 30, 31].

Hyers-Ulam stability is a concept that provides a gap between an approximate solution and an exact solution for the problem in consideration. A number of valuable generalizations towards Ulam's stability have been made in the past decade [22, 25, 32, 35–37]. In [2], using anti-periodic boundary conditions, Ahmad et al. explored a multi-term nonlinear fractional integro-differential equation:

$$\begin{cases}(\sigma_1^c D^{\eta}+\sigma_2^c D^{\psi})\rho(\omega)=\zeta_1(\omega,\rho(\omega))+I^{\nu}\zeta_2(\omega,\rho(\omega)), \omega\in\mathcal{J},\\ \rho(0)=-\rho(\Omega), \rho'(0)=-\rho'(\Omega),\end{cases}\tag{1.1}$$

where ${}^cD^{\eta}$ and ${}^cD^{\psi}$ are Caputo derivatives of order η and ψ, respectively. ζ_1, ζ_2 are continuous functions defined as $\zeta_1,\zeta_2:\mathcal{J}\times\mathbb{R}\to\mathbb{R}$, $\eta\in(1,2]$, $\psi\in(1,\eta)$, where $\mathcal{J}=[0,\Omega]$ and $\Omega>0$.

In [34], the multi-point boundary value problem with non-instantaneous impulses for sequential fractional differential equations has been studied for the following model:

$$\begin{cases}{}^cD^{\psi}(D+\lambda)\rho(\omega)=\gamma(\omega,\rho(\omega)), \omega\in(\omega_n,s_n], \text{for} n=0,1,\dots,k, 0<\psi\leqslant 1,\\ \rho(\omega)=\zeta_n(\omega,\rho(\omega)), \omega\in(s_{n-1},\omega_n], n=1,2,\dots,k,\\ \rho(0)=0, \rho(s_n)=0, n=0,1,2,\dots,k,\end{cases}$$

where D represents ordinary derivative and ${}^cD^{\psi}$ denotes Caputo derivative of order ψ and $\mu\in\mathbb{R}^+$. The functions γ and ζ_n are continuous and defined as $\gamma:\mathcal{J}\times\mathbb{R}\to\mathbb{R}$ and $\zeta_n:[s_{n-1},t_n]\times\mathbb{R}\to\mathbb{R}$ for all $n=1,2,\dots,k$.

In this article, our main objective is to analyze the ideas of existence, uniqueness, and several types of Ulam stability for the mild solution of a coupled system of the form:

$$\begin{cases}(\wp_1^c D^{\eta_1}+\wp_2^c D^{\xi_1})(D+\wp_3)\rho(\omega)=\zeta_1(\omega,\rho(\omega)),\Upsilon(\omega))+I^{w_1}\zeta_2(\omega,\rho(\omega)),\Upsilon(\omega)), \ \omega\in\mathcal{J},\\ (\wp_4^c D^{\eta_2}+\wp_5^c D^{\xi_2})(D+\wp_6)\Upsilon(\omega)=\zeta_3(\omega,\rho(\omega),\Upsilon(\omega))+I^{w_2}\zeta_4(\omega,\rho(\omega)),\Upsilon(\omega)), \ \omega\in\mathcal{J},\\ \rho(0)=\rho'(0)=\rho''(0)=0,\\ \Upsilon(0)=\Upsilon'(0)=\Upsilon''(0)=0,\end{cases}\tag{1.2}$$

where $\wp_1$, $\wp_2,\wp_3,\wp_4,\wp_5,\wp_6\in\mathbb{R}$, $\wp_1,\wp_4>0$, ${}^cD^{\eta_1}$,${}^cD^{\eta_2}$ and ${}^cD^{\xi_1}$, ${}^cD^{\xi_2}$, respectively denote the Caputo derivative of order η_1, η_2 and ξ_1,ξ_2, where $\eta_i\in[1,2)$ and $\xi_i\in(1,\eta_i)$. I^{ω_i}denotes the fractional integral (Riemann-Liouville) of order $\omega_i>0$, $\zeta_1,\zeta_2,\zeta_3,\zeta_4:\mathcal{J}\times\mathbb{R}\to\mathbb{R}$ are suitable continuous functions.

We have organized the rest of the paper as follows. In Section 2, we provide some basic definitions and lemmas that are associated with our results. The existence and uniqueness results are presented in Section 3. In Section 4, mixed nonlinearities cases are discussed. In Section 5, stability result is presented and examples are given in Section 6. At the end in Section 7, we conclude the paper.

2. Preliminaries and notations

First, we define the basics of fractional calculus from [15], which will be used later on. Let $\boldsymbol{C} = C(\mathcal{J}, R)$ and $\boldsymbol{C}^1 = C^1(\mathcal{J}, R)$.

Definition 2.1. Let $\psi \in L^1[a, b]$, then the fractional integral (Riemann-Liouville) for ψ is:

$$I^{\alpha}\psi(\omega) = \frac{1}{\Gamma(\alpha)} \int_a^{\omega} (\omega - s)^{\alpha-1}\psi(s)ds,$$

where $\alpha > 0$ and Euler Gamma function is denoted by Γ, $\forall\ \omega \in [a, b]$.

Definition 2.2. Let $\psi \in AC^n[a, b]$, then the Caputo fractional derivative for the function ψ is given by:

$${}^cD^q\psi(\omega) = \frac{1}{\Gamma(n-q)} \int_a^{\omega} (\omega - s)^{n-q-1}\psi^n(s)ds, \forall\omega \in [a, b],$$

where q $\in$ (n-1, n), and n $\in$ N.

Definition 2.3. A continuous function ψ is called the mild solution of Cauchy problem $\frac{d}{d\omega}\rho(\omega) = \zeta(\omega)$ with $\zeta(\omega_0) = \zeta_0$, if

$$\psi(\omega) = \zeta(\omega_0) + \int_{\omega_0}^{\omega} \zeta(s)ds.$$

Lemma 2.4 ([34])**.** *Consider the fractional differential equation ${}^cD^{\eta}\rho(\omega) = 0$, where ${}^cD^{\eta}$ is the Caputo derivative of order $\eta > 0$. The aforementioned equation has a solution of the form*

$$\rho(\omega) = d_0 + d_1\omega + d_2\omega^2 + \cdots + d_{k-1}\omega^{k-1},$$

here $d_m \in R$, $m = 0, 1, \ldots, k-1$, and $k = [\eta] + 1$.

Lemma 2.5 ([34])**.** *Let $\eta > 0$, then*

$$I^{\eta}({}^cD^{\eta}\rho(\omega)) = \rho(\omega) + d_0 + d_1\omega + d_2\omega^2 + \cdots + d_{k-1}\omega^{k-1},$$

here $d_m \in R$, $m = 0, 1, \cdots, k-1$, and $k = [\eta] + 1$.

Theorem 2.6 ([16, Krasnoselskii's fixed point theorem])**.** *Let $Q \neq \emptyset$ be a closed and convex set, K be a Banach space with $Q \subseteq K$, and M and P are operators that map Q into K and,*

(i) $Mx + Py \in Q$, $\forall(x, y \in Q)$;
(ii) *M is compact and continuous;*
(iii) *P is a contraction mapping.*

Then there exists $z \in Q$ such that $Mx + Py = z$.

Theorem 2.7 ([33, Banach fixed point theorem])**.** *If K is a non-empty closed set in a Banach space χ, then any contraction mapping ϕ from K to K has a fixed point (unique).*

Theorem 2.8 ([33, Schaefer fixed point theorem])**.** *Let $(K, \|.\|)$ be a normed space, for every bounded subset χ of K, ϕ represents a compact and continuous mapping from K into K. Then either*

(i) *the equation $\omega = \delta\phi\omega$ has a solution for $\delta = 1$;*
(ii) *for any $0 < \delta < 1$, the set of all such solutions, denoted as ω, is unbounded.*

Remark 2.9. We obtain only the mild solution due to Lemma 2 and Example 1 in [8], along with Example 3.1 and Fact 2 from [26]. These sources imply that the presence of continuous (even Holderian) solutions for fractional-type integral expressions does not necessarily indicate the existence of solutions to the related Caputo-type fractional differential problems.

Lemma 2.10. *Let $\zeta \in \mathbf{C}$ and $\sigma_1 \neq 0$, then multi-term linear fractional differential equation*

$$(\sigma_1^c D^{\eta} + \sigma_2^c D^{\xi}).(D+\sigma_3)\rho(\omega) = \zeta(\omega), \tag{2.1}$$

with the initial condition $\rho(0) = \rho'(0) = \rho''(0) = 0$ has a unique mild solution provided by

$$\begin{aligned}\rho(\omega) = &\frac{1}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)} I^{\eta}\zeta(s)ds - \frac{\sigma_2}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)} I^{\eta-\xi-1}\rho(s)ds \\ &- \frac{\sigma_3.\sigma_2}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)} I^{\eta-\xi}\rho(s)ds - \frac{1}{\sigma_1} I^{\eta}\zeta(0)\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right) \\ &- \left[\frac{\sigma_3}{\sigma_1} I^{\eta}\zeta(0) + \frac{1}{\sigma_1} I^{\eta-1}\zeta(0)\right]\left(\frac{\omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right).\end{aligned}$$

Proof. $(D+\sigma_3)(\sigma_1^c D^{\eta} + \sigma_2^c D^{\xi})\rho(\omega) = \zeta(\omega)$, implies that

$${}^cD^{\eta}(D+\sigma_3)\rho(\omega) = \frac{1}{\sigma_1}\zeta(\omega) - \frac{1}{\sigma_1}\sigma_2^c D^{\xi}(D+\sigma_3)\rho(\omega).$$

By applying the integral of order η, we get

$$(D+\sigma_3)\rho(\omega) = \frac{1}{\sigma_1} I^{\eta}\zeta(\omega) - \frac{\sigma_2}{\sigma_1} I^{\eta-\xi}(D+\sigma_3)\rho(\omega) + d_0 + d_1\omega.$$

Equivalently

$$D\rho(\omega) + \sigma_3\rho(\omega) = \frac{1}{\sigma_1} I^{\eta}\zeta(\omega) - \frac{\sigma_2}{\sigma_1} I^{\eta-\xi}(D+\sigma_3)\rho(\omega) + d_0 + d_1\omega.$$

Now multiplying by integrating factor $e^{\sigma_3\omega}$ and then integrating from 0 to ω, we have

$$\begin{aligned}\rho(\omega) = &\frac{1}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)} I^{\eta}\zeta(s)ds - \frac{\sigma_2}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)} I^{\eta-\xi-1}\rho(s)ds \\ &- \frac{\sigma_3.\sigma_2}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)} I^{\eta-\xi}\rho(s)ds + d_0\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right) + d_1\left(\frac{\omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right) + d_2 e^{-\sigma_3\omega}.\end{aligned}$$

Using condition $\rho(0) = \rho'(0) = \rho''(0) = 0$, $d_0 = -\frac{1}{\sigma_1} I^{\nu}\zeta(0)$, $d_1 = \left(-\frac{\sigma_3}{\sigma_1}\right) I^{\nu}\zeta(0) - \frac{1}{\sigma_1} I^{\nu-1}\zeta(0)$, $d_2 = 0$. Putting values of d_0, d_1, and d_2, we get

$$\begin{aligned}\rho(\omega) = &\frac{1}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)} I^{\eta}\zeta(s)ds - \frac{\sigma_2}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)} I^{\eta-\xi-1}\rho(s)ds \\ &- \frac{\sigma_3.\sigma_2}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)} I^{\eta-\xi}\rho(s)ds - \frac{1}{\sigma_1} I^{\eta}\zeta(0)\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right) \\ &- \left[\frac{\sigma_3}{\sigma_1} I^{\eta}\zeta(0) + \frac{1}{\sigma_1} I^{\eta-1}\zeta(0)\right]\left(\frac{\omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right),\end{aligned}$$

which is the required proof. □

3. The existence and uniqueness of the mild solution

The equation 2.1 can be transform into nonlinear form that contains Caputo derivative operator of order $\eta \in [1,2)$, $\xi \in (1,\eta)$ given by

$$\begin{cases}(\sigma_1^c D^{\eta} + \sigma_2^c D^{\xi})(D+\sigma)\rho(\omega) = \zeta_1(\omega,\rho(\omega)) + I^{\nu}\zeta_2(\omega,\rho(\omega)), \ \omega \in \mathcal{J}, \\ \rho(0) = \rho'(0) = \rho''(0) = 0,\end{cases} \tag{3.1}$$

where σ, σ_1, $\sigma_2 \in \mathbb{R}$, $\sigma_1 > 0$, I^{ν} denotes the fractional integral (Riemann-Liouville) of order ν, and ${}^cD^{\eta}$ and ${}^cD^{\xi}$ denote the Caputo derivative of order η and ξ, respectively. ζ_1, ζ_2 are continuous functions defined as $\zeta_1, \zeta_2 : \mathfrak{J} \times \mathbb{R} \to \mathbb{R}$. It is possible to transform the given problem into a fixed point problem as $\mathfrak{R}^*(\omega) = \omega$ and $\mathfrak{R}^* : \boldsymbol{C} \to \boldsymbol{C}$ is defined as:

$$\begin{aligned}(\mathfrak{R}^*\rho)(\omega) = &\frac{1}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)} I^{\eta}\zeta_1(s,\rho(s))ds + \frac{1}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)} I^{\eta+\nu}\zeta_2(s,\rho(s))ds \\ &- \frac{\sigma_2}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)} I^{\eta-\xi-1}\rho(s)ds - \frac{\sigma_3.\sigma_2}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)} I^{\eta-\xi}\rho(s)ds \\ &- \frac{1}{\sigma_1} I^{\eta}\zeta_1(0,0)\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right) - \frac{1}{\sigma_1} I^{\eta+\nu}\zeta_2(0,0)\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right) \\ &- \frac{\sigma_3}{\sigma_1} I^{\eta}\zeta_1(0,0)\left(\frac{\omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right) - \frac{\sigma_3}{\sigma_1} I^{\eta+\nu}\zeta_2(0,0)\left(\frac{\omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right) \\ &- \frac{1}{\sigma_1} I^{\eta-1}\zeta_1(0,0)\left(\frac{\omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right) - \frac{1}{\sigma_1} I^{\eta+\nu-1}\zeta_2(0,0)\left(\frac{\omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right).\end{aligned} \tag{3.2}$$

We assume the following necessary assumptions for obtaining our results.

($\bar{G}1$) Let $\zeta_1, \zeta_2 : \mathfrak{J} \times \mathfrak{R} \to \mathfrak{R}$ are continuous functions, and A_1^*, A_2^*, N_1^*, and N_2^* are positive constants and $\forall \omega \in \mathfrak{J}$ such that

$$\begin{aligned}&|\zeta_1(\omega,\rho_1(\omega)) - \zeta_1(\omega,\rho_2(\omega))| \leqslant A_1^*|\rho_1(\omega) - \rho_2(\omega)|,\\ &|\zeta_2(\omega,\rho_1(\omega)) - \zeta_2(\omega,\rho_2(\omega))| \leqslant A_2^*|\rho_1(\omega) - \rho_2(\omega)|,\\ &\qquad |\zeta_1(0,0)| \leqslant N_1^*,\ |\zeta_2(0,0)| \leqslant N_2^*,\ N^* = \max\{N_1^*, N_2^*\}, \forall \omega \in \mathfrak{J}.\end{aligned}$$

($\bar{G}2$) $(A^*\varphi_1 + \varphi_2) < 1$, where $A^* = \max\{A_1^*, A_2^*\}$, and

$$\begin{aligned}\varphi_1 &= \frac{1}{\sigma_1}\left(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\right)\left[\frac{\Omega^{\eta}}{\Gamma(\eta+1)} + \frac{\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)}\right],\\ \varphi_2 &= \frac{\sigma_2}{\sigma_1}\left(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\right)\left[\frac{\Omega^{\eta-\xi-1}}{\Gamma(\eta-\xi)} + \sigma_3\frac{\Omega^{\eta-\xi}}{\Gamma(\eta-\xi+1)}\right],\\ \varphi_3 &= \frac{N^*}{\sigma_1}\left(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\right)\left[\frac{\Omega^{\eta}}{\Gamma(\eta+1)} + \frac{\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)}\right]\\ &\quad + \frac{N^*}{\sigma_1}\left[\sigma_3\frac{\Omega^{\eta}}{\Gamma(\eta+1)} + \sigma_3\frac{\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)} + \frac{\Omega^{\eta-1}}{\Gamma(\eta)} + \frac{\Omega^{\eta+\nu-1}}{\Gamma(\eta+\nu)}\right]\left(\frac{\Omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\Omega}}{\sigma_3^2}\right).\end{aligned}$$

($\bar{G}3$) Let us suppose that there exist $\varpi_1, \varpi_2 \in \boldsymbol{C}$ with $\|\varpi\| = \max\{\|\varpi_1\|, \|\varpi_2\|\}$ such that $|\zeta_1(\omega,\rho(\omega))| \leqslant \|\varpi_1\|$ and $|\zeta_2(\omega,\rho(\omega))| \leqslant \|\varpi_2\|$, $\forall(\omega,\rho) \in (\mathfrak{J} \times \mathbb{R})$.

Theorem 3.1. *The problem* (3.1) *has a unique mild solution, if the conditions* ($\bar{G}1$)-($\bar{G}2$) *are satisfied.*

Proof. Consider a closed ball $B_{\vartheta} = \{\rho \in C, \|\rho\| \leqslant \vartheta\}$ and we show that $\mathfrak{R}^*B_{\vartheta} \subset B_{\vartheta}$, $\mathfrak{R}^* : \mathbf{C} \to \mathbf{C}$ is previously defined in (3.2) and $\vartheta \geqslant (N^*\varphi_1 + \varphi_3)(1 - A^*\varphi_1 - \varphi_2)^{-1}$; $\varphi_i(i = 1, 2, 3)$ are described in ($\bar{G}2$). According to ($\bar{G}1$), for any $\rho \in B_{\vartheta}$, we can write

$$|\zeta_1(\omega,\rho(\omega))| = |\zeta_1(\omega,\rho(\omega)) - \zeta_1(\omega,0) + \zeta_1(\omega,0)| \leqslant A_1^*\|\rho\| + N_1^* \leqslant A^*\vartheta + N_1^*.$$

By similar way

$$|\zeta_2(\omega,\rho(\omega))| \leqslant A_2^*\|\rho\| + N_2^* \leqslant A^*\vartheta + N_2^*.$$

Utilizing the preceding inequalities and the assumptions specified in ($\bar{G}2$), we attain the following results

$$\|\mathfrak{R}^*\rho\| = \sup_{\omega\in I}|(\mathfrak{R}^*\rho)(\omega)|$$

$$
\begin{aligned}
&\leqslant \sup_{\omega\in\mathcal{J}}\Big\{\frac{1}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)}I^{\eta}|\zeta_1(s,\rho(s))|ds+\frac{1}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)}I^{\eta+\nu}|\zeta_2(s,\rho(s))|ds\\
&\quad+\frac{\sigma_2}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)}I^{\eta-\xi-1}|\rho(s)|ds+\frac{\sigma_3.\sigma_2}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)}I^{\eta-\xi}|\rho(s)|ds\\
&\quad+\frac{1}{\sigma_1}I^{\eta}|\zeta_1(0,0)|\Big(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\Big)+\frac{1}{\sigma_1}I^{\eta+\nu}|\zeta_2(0,0)|\Big(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\Big)\\
&\quad+\frac{\sigma_3}{\sigma_1}I^{\eta}|\zeta_1(0,0)|\Big(\frac{\omega}{\sigma_3}-\frac{1}{\sigma_3^2}+\frac{e^{-\sigma_3\omega}}{\sigma_3^2}\Big)+\frac{\sigma_3}{\sigma_1}I^{\eta+\nu}|\zeta_2(0,0)|\Big(\frac{\omega}{\sigma_3}-\frac{1}{\sigma_3^2}+\frac{e^{-\sigma_3\omega}}{\sigma_3^2}\Big)\\
&\quad+\frac{1}{\sigma_1}I^{\eta-1}|\zeta_1(0,0)|\Big(\frac{\omega}{\sigma_3}-\frac{1}{\sigma_3^2}+\frac{e^{-\sigma_3\omega}}{\sigma_3^2}\Big)+\frac{1}{\sigma_1}I^{\eta+\nu-1}|\zeta_2(0,0)|\Big(\frac{\omega}{\sigma_3}-\frac{1}{\sigma_3^2}+\frac{e^{-\sigma_3\omega}}{\sigma_3^2}\Big)\Big\}.\\
&\leqslant \frac{1}{\sigma_1}\Big(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\Big)\Big((A^*\vartheta+N_1^*)\frac{\Omega^{\eta}}{\Gamma(\eta+1)}+(A^*\vartheta+N_2^*)\frac{\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)}\Big)\\
&\quad+\frac{\sigma_2}{\sigma_1}\Big(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\Big)\Big(\frac{\Omega^{\eta-\psi-1}}{\Gamma(\eta-\xi)}+\sigma_3\frac{\Omega^{\eta-\xi}}{\Gamma(\eta-\xi+1)}\Big)\vartheta+\frac{N^*}{\sigma_1}\Big(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\Big)\Big(\frac{\Omega^{\eta}}{\Gamma(\eta+1)}+\frac{\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)}\Big)\\
&\quad+\frac{N^*}{\sigma_1}\Big(\sigma_3\frac{\Omega^{\eta}}{\Gamma(\eta+1)}+\sigma_3\frac{\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)}+\frac{\Omega^{\eta-1}}{\Gamma(\eta)}+\frac{\Omega^{\eta+\nu-1}}{\Gamma(\eta+\nu)}\Big)+\Big(\frac{\Omega}{\sigma_3}-\frac{1}{\sigma_3^2}+\frac{e^{-\sigma_3\Omega}}{\sigma_3^2}\Big)\\
&\leqslant (A^*\vartheta+N^*)\frac{1}{\sigma_1}\Big(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\Big)\Big(\frac{\Omega^{\eta}}{\Gamma(\eta+1)}+\frac{\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)}\Big)\\
&\quad+\frac{\sigma_2}{\sigma_1}\Big(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\Big)\Big(\frac{\Omega^{\eta-\xi-1}}{\Gamma(\eta-\xi)}+\sigma_3\frac{\Omega^{\eta-\xi}}{\Gamma(\eta-\xi+1)}\Big)\vartheta+\frac{N^*}{\sigma_1}\Big(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\Big)\Big(\frac{\Omega^{\eta}}{\Gamma(\eta+1)}+\frac{\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)}\Big)\\
&\quad+\frac{N^*}{\sigma_1}\Big(\sigma_3\frac{\Omega^{\eta}}{\Gamma(\eta+1)}+\sigma_3\frac{\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)}+\frac{\Omega^{\eta-1}}{\Gamma(\eta)}+\frac{\Omega^{\eta+\nu-1}}{\Gamma(\eta+\nu)}\Big)+\Big(\frac{\Omega}{\sigma_3}-\frac{1}{\sigma_3^2}+\frac{e^{-\sigma_3\Omega}}{\sigma_3^2}\Big)\\
&=(A^*\vartheta+N^*)\varphi_1+\vartheta\varphi_2+\varphi_3\leqslant\vartheta.
\end{aligned}
$$

It shows that for any $\rho\in B_{\vartheta}$, $\mathfrak{R}^*\rho\in B_{\vartheta}$. Hence $\mathfrak{R}^*B_{\vartheta}\subset B_{\vartheta}$. Let's now demonstrate the contraction nature of the $\mathfrak{R}^*$ operator. For this,

$$
\begin{aligned}
\|\mathfrak{R}^*\rho_1-\mathfrak{R}^*\rho_2\| &= \sup_{\omega\in\mathcal{J}}|(\mathfrak{R}^*\rho_1)(\omega)-(\mathfrak{R}^*\rho_2)(\omega)|\\
&\leqslant \sup_{\omega\in\mathcal{J}}\Big\{\frac{1}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)}I^{\eta}|\zeta_1(s,\rho_1(s))-\zeta_1(s,\rho_2(s))|ds\\
&\quad+\frac{1}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)}I^{\eta+\nu}|\zeta_2(s,\rho_1(s))-\zeta_2(s,\rho_2(s))|ds\\
&\quad+\frac{\sigma_2}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)}I^{\eta-\xi-1}|\rho_1(s)-\rho_2(s)|ds+\frac{\sigma_3.\sigma_2}{\sigma_1}\int_0^{\omega} e^{-\sigma_3(\omega-s)}I^{\eta-\xi}|\rho_1(s)-\rho_2(s)|ds\Big\}\\
&\leqslant\Big\{\frac{A_1^*}{\sigma_1}\Big(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\Big)\Big(\frac{\Omega^{\eta}}{\Gamma(\eta+1)}\Big)+\frac{A_2^*}{\sigma_1}\Big(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\Big)\Big(\frac{\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)}\Big)\\
&\quad+\frac{\sigma_2}{\sigma_1}\Big(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\Big)\Big(\frac{\Omega^{\eta-\xi-1}}{\Gamma(\eta-\xi)}+\sigma_3\frac{\Omega^{\eta-\xi}}{\Gamma(\eta-\xi+1)}\Big)\Big\}\|\rho_1-\rho_2\|\\
&\leqslant\Big\{\frac{A^*}{\sigma_1}\Big(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\Big)\Big(\frac{\Omega^{\eta}}{\Gamma(\eta+1)}\Big)+\frac{A^*}{\sigma_1}\Big(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\Big)\Big(\frac{\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)}\Big)\\
&\quad+\frac{\sigma_2}{\sigma_1}\Big(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\Big)\Big(\frac{\Omega^{\eta-\xi-1}}{\Gamma(\eta-\xi)}+\sigma_3\frac{\Omega^{\eta-\xi}}{\Gamma(\eta-\xi+1)}\Big)\Big\}\|\rho_1-\rho_2\|\\
&\leqslant(A^*\varphi_1+\varphi_2)\|\rho_1-\rho_2\|.
\end{aligned}
$$

As $A^*\varphi_1+\varphi_2\leqslant 1$, by ($\bar{G}2$) implies that the operator $\mathfrak{R}^*$ is contraction. The operator $\mathfrak{R}^*$ has a distinct fixed point according to the Banach contraction mapping principle, suggesting that the problem (3.1) has a unique mild solution. □

Theorem 3.2. *There is atleast one mild solution for the problem* (3.1) *on the interval I, if* $\varphi_2 < 1$ *and the conditions* ($\bar{G}1$)-($\bar{G}3$) *are satisfied, while* φ_2 *is given in* ($\bar{G}2$).

Proof. Consider $k_\varkappa = \{\rho \in C : \|\rho\| \leqslant \varkappa\}$ with $\varkappa \geqslant (\|\varpi\|\varphi_1 + \varphi_3)(1-\varphi_2)^{-1}$. Define $\mathfrak{R}_1^*$ and $\mathfrak{R}_2^*$ on $K_\varkappa \to \mathbf{C}$ as follow.

$$\begin{aligned}
\mathfrak{R}_1^*\rho(\omega) &= \frac{1}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^\eta\zeta_1(s,\rho(s))ds + \frac{1}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta+\nu}\zeta_2(s,\rho(s))ds\\
&\quad - \frac{1}{\sigma_1}I^\eta\zeta_1(0,0)\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right) - \frac{1}{\sigma_1}I^{\eta+\nu}\zeta_2(0,0)\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right)\\
&\quad - \frac{\sigma_3}{\sigma_1}I^\eta\zeta_1(0,0)\left(\frac{\omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right) - \frac{\sigma_3}{\sigma_1}I^{\eta+\nu}\zeta_2(0,0)\left(\frac{\omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right)\\
&\quad - \frac{1}{\sigma_1}I^{\eta-1}\zeta_1(0,0)\left(\frac{\omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right) - \frac{1}{\sigma_1}I^{\eta+\nu-1}\zeta_2(0,0)\left(\frac{\omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right),\\
\mathfrak{R}_2^*\rho(\omega) &= -\frac{\sigma_2}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta-\xi-1}\rho(s)ds - \frac{\sigma_3.\sigma_2}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta-\xi}\rho(s)ds.
\end{aligned}$$

Here $\mathfrak{R}^* = \mathfrak{R}_1^* + \mathfrak{R}_2^*$. Let's investigate the Kresnoselskii FPT hypothesis.

(i) For $\rho_1, \rho_2 \in K_\varkappa$, we have

$$\begin{aligned}
\|\mathfrak{R}_1^*\rho_1 + \mathfrak{R}_2^*\rho_2\| &= \sup_{\omega\in\mathcal{J}}|(\mathfrak{R}_1^*\rho_1)(\omega) + (\mathfrak{R}_2^*\rho_2)(\omega)|\\
&\leqslant \sup_{\omega\in\mathcal{J}}\Bigg\{\frac{1}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^\eta|\zeta_1(s,\rho_1(s))|ds + \frac{1}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta+\nu}|\zeta_2(s,\rho_1(s))|ds\\
&\quad + \frac{\sigma_2}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta-\xi-1}|\rho_2(s)|ds + \frac{\sigma_3.\sigma_2}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta-\xi}|\rho_2(s)|ds\\
&\quad + \frac{1}{\sigma_1}I^\eta|\zeta_1(0,0)|\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right) + \frac{1}{\sigma_1}I^{\eta+\nu}|\zeta_2(0,0)|\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right)\\
&\quad + \frac{\sigma_3}{\sigma_1}I^\eta|\zeta_1(0,0)|\left(\frac{\omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right) + \frac{\sigma_3}{\sigma_1}I^{\eta+\nu}|\zeta_2(0,0)|\left(\frac{\omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right)\\
&\quad + \frac{1}{\sigma_1}I^{\eta-1}|\zeta_1(0,0)|\left(\frac{\omega}{\sigma_3} + \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right) + \frac{1}{\sigma_1}I^{\eta+\nu-1}|\zeta_2(0,0)|\left(\frac{\omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right)\Bigg\}\\
&\leqslant \frac{\|\varpi\|}{\sigma_1}\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right)\left[\frac{\Omega^\eta}{\Gamma(\eta+1)} + \frac{\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)}\right] + \frac{\sigma_2}{\sigma_1}\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right)\left[\frac{\Omega^{\eta-\xi-1}}{\Gamma(\eta-\xi)}\right.\\
&\quad \left. + \sigma_3\frac{\Omega^{\eta-\xi}}{\Gamma(\eta-\xi+1)}\right] + \frac{1}{\sigma_1}\left(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\right)\left[\frac{N_1^*\Omega^\eta}{\Gamma(\eta+1)} + \frac{N_2^*\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)}\right]\\
&\quad + \frac{1}{\sigma_1}\left[\sigma_3\frac{N_1^*\Omega^\eta}{\Gamma(\eta+1)} + \sigma_3\frac{N_2^*\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)} + \frac{N_1^*\Omega^{\eta-1}}{\Gamma(\eta)} + \frac{N_2^*\Omega^{\eta+\nu-1}}{\Gamma(\eta+\nu)}\right]\left(\frac{\Omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\Omega}}{\sigma_3^2}\right)\\
&\leqslant \frac{\|\varpi\|}{\sigma_1}\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right)\left[\frac{\Omega^\eta}{\Gamma(\eta+1)} + \frac{\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)}\right] + \frac{\sigma_2}{\sigma_1}\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right)\left[\frac{\Omega^{\eta-\xi-1}}{\Gamma(\eta-\xi)}\right.\\
&\quad \left. + \sigma_3\frac{\Omega^{\eta-\xi}}{\Gamma(\eta-\xi+1)}\right] + \frac{N^*}{\sigma_1}\left(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\right)\left[\frac{\Omega^\eta}{\Gamma(\eta+1)} + \frac{\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)}\right]\\
&\quad + \frac{N^*}{\sigma_1}\left[\sigma_3\frac{\Omega^\eta}{\Gamma(\eta+1)} + \sigma_3\frac{\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)} + \frac{\Omega^{\eta-1}}{\Gamma(\eta)} + \frac{\Omega^{\eta+\nu-1}}{\Gamma(\eta+\nu)}\right]\left(\frac{\Omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\Omega}}{\sigma_3^2}\right)\\
&= \|\varpi\|\varphi_1 + \varkappa\varphi_2 + \varphi_3 \leqslant \varkappa,
\end{aligned}$$

implies that $\mathfrak{R}_1^* + \mathfrak{R}_2^* \in K_{\varkappa}$.

(ii) Here we need to demonstrate that $\mathfrak{R}_1^*$ is compact and continuous. Continuity of $\mathfrak{R}_1^*$ is from that of ζ_1 and ζ_2. Furthermore, $\|\mathfrak{R}_1^*\| \leqslant \|\varpi\|\Phi_1 + \Phi_3$ implies that $\mathfrak{R}_1^*$ is uniformly bounded on $K_{\varkappa}$. Also, we need to show that $\mathfrak{R}_1^*$ is compact. Let $\sup_{(\omega,\rho(\omega))\in \mathfrak{I}\times R}|\zeta_1(\omega,\rho(\omega))|=$ “$_1$ and $\sup_{(\omega,\rho(\omega))\in \mathfrak{I}\times R}|\zeta_2(\omega,\rho(\omega))|=$ “$_2$. Then for $\omega_1, \omega_2 \in \mathfrak{I}, \omega_1 < \omega_2$, we have

$$
\begin{aligned}
&|(\mathfrak{R}_1^*\rho)(\omega_2) - (\mathfrak{R}_1^*\rho)(\omega_1)|\\
&= \Bigg| \frac{1}{\sigma_1}\int_0^{\omega_2} e^{-\sigma_3(\omega_2 - s)} I^{\eta}\zeta_1(s,\rho(s))ds + \frac{1}{\sigma_1}\int_0^{\omega_2} e^{-\sigma_3(\omega_2 - s)} I^{\eta+\nu}\zeta_2(s,\rho(s))ds\\
&\quad - \frac{1}{\sigma_1} I^{\eta}\zeta_1(0,0)\left(\frac{1-e^{-\sigma_3\omega_2}}{\sigma_3}\right) - \frac{1}{\sigma_1} I^{\eta+\nu}\zeta_2(0,0)\left(\frac{1-e^{-\sigma_3\omega_2}}{\sigma_3}\right)\\
&\quad - \frac{\sigma_3}{\sigma_1} I^{\eta}\zeta_1(0,0)\left(\frac{\omega_2}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega_2}}{\sigma_3^2}\right) - \frac{\sigma_3}{\sigma_1} I^{\eta+\nu}\zeta_2(0,0)\left(\frac{\omega_2}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega_2}}{\sigma_3^2}\right)\\
&\quad - \frac{1}{\sigma_1} I^{\eta-1}\zeta_1(0,0)\left(\frac{\omega_2}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega_2}}{\sigma_3^2}\right) - \frac{1}{\sigma_1} I^{\eta+\nu-1}\zeta_2(0,0)\left(\frac{\omega_2}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega_2}}{\sigma_3^2}\right)\\
&\quad - \frac{1}{\sigma_1}\int_0^{\omega_1} e^{-\sigma_3(\omega_1 - s)} I^{\eta}\zeta_1(s,\rho(s))ds - \frac{1}{\sigma_1}\int_0^{\omega_1} e^{-\sigma_3(\omega_1 - s)} I^{\eta+\nu}\zeta_2(s,\rho(s))ds\\
&\quad + \frac{1}{\sigma_{1/.}} I^{\eta}\zeta_1(0,0)\left(\frac{1-e^{-\sigma_3\omega_1}}{\sigma_3}\right) + \frac{1}{\sigma_1} I^{\eta+\nu}\zeta_2(0,0)\left(\frac{1-e^{-\sigma_3\omega_1}}{\sigma_3}\right)\\
&\quad + \frac{\sigma_3}{\sigma_1} I^{\eta}\zeta_1(0,0)\left(\frac{\omega_1}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega_1}}{\sigma_3^2}\right) + \frac{\sigma_3}{\sigma_1} I^{\eta+\nu}\zeta_2(0,0)\left(\frac{\omega_1}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega_1}}{\sigma_3^2}\right)\\
&\quad + \frac{1}{\sigma_1} I^{\eta-1}\zeta_1(0,0)\left(\frac{\omega_1}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega_1}}{\sigma_3^2}\right) + \frac{1}{\sigma_1} I^{\eta+\nu-1}\zeta_2(0,0)\left(\frac{\omega_1}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3\omega_1}}{\sigma_3^2}\right)\Bigg|.
\end{aligned}
$$

In above, if $\omega_1 \to \omega_2$, we have

$$|\mathfrak{R}_1^*\rho(\omega_2) - (\mathfrak{R}_1^*\rho)(\omega_1)| \to 0.$$

So $\mathfrak{R}_1^*$ is equicontinuous, applying the conclusion of Arzela-Ascoli theorem, suggests that $\mathfrak{R}_1^*$ is relatively compact on $K_{\varkappa}$.

(iii) As $\varphi_2 < 1$, $\mathfrak{R}_2^*$ is contraction. Therefore, by confirming the conditions of the Krasnoselskii fixed point theorem, we insure that the problem has at least one mild solution within the specified interval $\mathfrak{I}$. This completes the proof. □

For the existence and uniqueness of the mild solution to the problem (1.2), we assume the following necessary assumptions for obtaining our results.

(H_1^*) $\zeta_1, \zeta_2, \zeta_3, \zeta_4 : \mathfrak{I}\times R \to R$ are continuous functions, there exists positive constants B_1^*, B_2^*, B_3^*, B_4^*, B_5^*, B_6^*, B_7^*, B_8^* such that $\forall \omega \in \mathfrak{I}$,

$$
\begin{gathered}
|\zeta_i(\mathbf{0},0,0)| \leqslant F_i^*, \ \forall\, i = 1,2,3,4,\\
\max\{F_i^*\} = F^*, \ \max\{B_i^*\} = B^*,\\
|\zeta_1(\omega,\rho(\omega),\Upsilon(\omega)) - \zeta_1(\omega,\bar{\mathbf{j}}(\omega),\bar{\Upsilon}(\omega))| \leqslant B_1^*|\rho(\omega)-\bar{\rho}(\omega)| + B_2^*|\Upsilon(\omega)-\bar{\Upsilon}(\omega)|,\\
|\zeta_2(\omega,\rho(\omega),\Upsilon(\omega)) - \zeta_2(\omega,\bar{\rho}(\omega),\bar{\Upsilon}(\omega))| \leqslant B_3^*|\rho(\omega)-\bar{\rho}(\omega)| + B_4^*|\Upsilon(\omega)-\bar{\Upsilon}(\omega)|,\\
|\zeta_3(\omega,\rho(\omega),\Upsilon(\omega)) - \zeta_3(\omega,\bar{\rho}(\omega),\bar{\Upsilon}(\omega))| \leqslant B_5^*|\rho(\omega)-\bar{\rho}(\omega)| + B_6^*|\Upsilon(\omega)-\bar{\Upsilon}(\omega)|,\\
|\zeta_4(\omega,\rho(\omega),\Upsilon(\omega)) - \zeta_4(\omega,\bar{\rho}(\omega),\bar{\Upsilon}(\omega))| \leqslant B_7^*|\rho(\omega)-\bar{\rho}(\omega)| + B_8^*|\Upsilon(\omega)-\bar{\Upsilon}(\omega)|.
\end{gathered}
$$

(H_2^*) $G^* < 1$, where $G^* = \max\{G_1*, G_2*\}$, also we have $(B^*\Phi_1 + \Phi_2) = G_1*$, $(B^*\theta_1 + \theta_2) = G_2*$, and

$$\theta_1 = \frac{1}{\wp_1}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_1}}{\Gamma(\eta_1+1)} + \frac{\Omega^{\eta_1+\nu_1}}{\Gamma(\eta_1+w_1+1)}\right],$$

$$\theta_2 = \frac{\wp_2}{\wp_1}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_1-\xi_1-1}}{\Gamma(\eta_1-\xi_1)} + \wp_3\frac{\Omega^{\eta_1-\xi_1}}{\Gamma(\eta_1-\xi_1+1)}\right],$$
$$\theta_3 = \frac{F^*}{\wp_1}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_1}}{\Gamma(\eta_1+1)} + \frac{\Omega^{\eta_1+w_1}}{\Gamma(\eta_1+w_1+1)}\right]$$
$$+\frac{F^*}{\wp_1}\left[\wp_3\frac{\Omega^{\eta_1}}{\Gamma(\eta_1+1)} + \wp_3\frac{\Omega^{\eta_1+w_1}}{\Gamma(\eta_1+w_1+1)} + \frac{\Omega^{\eta_1-1}}{\Gamma(\eta_1)} + \frac{\Omega^{\eta_1+w_1-1}}{\Gamma(\eta_1+w_1)}\right]\left(\frac{\Omega}{\wp_3} - \frac{1}{\wp_3^2} + \frac{e^{-\wp_3\Omega}}{\wp_3^2}\right),$$

also

$$\phi_1 = \frac{1}{\wp_4}\left(\frac{1-e^{-\wp_6\Omega}}{\wp_6}\right)\left[\frac{\Omega^{\eta_2}}{\Gamma(\eta_2+1)} + \frac{\Omega^{\eta_2+w_2}}{\Gamma(\eta_2+w_2+1)}\right],$$
$$\phi_2 = \frac{\wp_5}{\wp_4}\left(\frac{1-e^{-\wp_6\Omega}}{\wp_6}\right)\left[\frac{\Omega^{\eta_2-\xi_2-1}}{\Gamma(\eta_2-\xi_2)} + \wp_6\frac{\Omega^{\eta_2-\xi_2}}{\Gamma(\eta_2-\xi_2+1)}\right],$$
$$\phi_3 = \frac{F^*}{\wp_4}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_2}}{\Gamma(\eta_2+1)} + \frac{\Omega^{\eta_2+w_2}}{\Gamma(\eta_2+w_2+1)}\right]$$
$$+\frac{F^*}{\wp_4}\left[\wp_6\frac{\Omega^{\eta_2}}{\Gamma(\eta_2+1)} + \wp_6\frac{\Omega^{\eta_2+w_2}}{\Gamma(\eta_2+w_2+1)} + \frac{\Omega^{\eta_2-1}}{\Gamma(\eta_2)} + \frac{\Omega^{\eta_2+w_2-1}}{\Gamma(\eta_2+w_2)}\right]\left(\frac{\Omega}{\wp_6} - \frac{1}{\wp_6^2} + \frac{e^{-\wp_6\Omega}}{\wp_6^2}\right).$$

(H_3^*) $|\zeta_i(\omega,\rho(\omega),\Upsilon(\omega))| \leqslant l_{i-1}(\omega) + l_i(\omega)|\rho(\omega)| + l_{i+1}(\omega)|\Upsilon(\omega)|$, $l_i^* = \sup(l_i(\omega))$, and $(l_0^* + l_1^* q^* + l_2^* q^*) = \beta_1^*$, $(l_1^* + l_2^* q^* + l_3^* q) = \beta_2^*$, $\max\{\beta_1^*, \beta_2^*\} = \beta^*$.

Theorem 3.3. *The problem* (1.2) *has a unique mild solution, if the conditions* (H_1^*)-(H_2^*) *are satisfied.*

Proof. Consider an operator $\mathfrak{R}^*$ defined from $\mathfrak{R}^* : \mathbf{X}\times\mathbf{Y} \to \mathbf{X}\times\mathbf{Y}$ as:

$$\mathfrak{R}^*(\rho,\Upsilon)(\omega) = (\mathfrak{R}_1^*(\rho,\Upsilon)(\omega), \mathfrak{R}_2^*(\rho,\Upsilon)(\omega)).$$

Using Lemma 2.10, we have

$$\begin{aligned}(\mathfrak{R}_1^*)(\omega) = &\frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1}\zeta_1(s,\rho(s),\Upsilon(s))ds \\
&+ \frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1+w_1}\zeta_2(s,\rho(s),\Upsilon(s))ds \\
&- \frac{\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1-\xi_1-1}\rho(s)ds - \frac{\wp_3\cdot\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1-\xi_1}\rho(s)ds \\
&- \frac{1}{\wp_1}\left[I^{\eta_1}\zeta_1(0,0,0) + I^{\eta_1+w_1}\zeta_2(0,0,0)\right]\left(\frac{1-e^{-\wp_3\omega}}{\wp_3}\right) - \left\{\frac{\wp_3}{\wp_1}\left[I^{\eta_1}\zeta_1(0,0,0) + I^{\eta_1+w_1}\zeta_2(0,0,0)\right]\right. \\
&\left.+ \frac{1}{\wp_1}\left[I^{\eta_1-1}\zeta_1(0,0,0) + I^{\eta_1+w_1-1}\zeta_2(0,0,0)\right]\right\}\left(\frac{\omega}{\wp_3} - \frac{1}{\wp_3^2} + \frac{e^{-\wp_3\omega}}{\wp_3^2}\right),\end{aligned}$$

and

$$\begin{aligned}(\mathfrak{R}_2^*)(\omega) = &\frac{1}{\wp_4}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_2}\zeta_3(s,\rho(s),\Upsilon(s))ds \\
&+ \frac{1}{\wp_4}\int_0^\omega e^{-\wp_6(\omega-s)} I^{\eta_2+w_2}\zeta_4(s,\rho(s),\Upsilon(s))ds \\
&- \frac{\wp_5}{\wp_4}\int_0^\omega e^{-\wp_6(\omega-s)} I^{\eta_2-\xi_2-1}\Upsilon(s)ds - \frac{\wp_6\cdot\wp_5}{\wp_4}\int_0^\omega e^{-\wp_6(\omega-s)} I^{\eta_2-\xi_2}\Upsilon(s)ds \\
&- \frac{1}{\wp_4}\left[I^{\eta_2}\zeta_3(0,0,0) + I^{\eta_2+w_2}\zeta_4(0,0,0)\right]\left(\frac{1-e^{-\wp_6\omega}}{\wp_6}\right) - \left\{\frac{\wp_6}{\wp_4}\left[I^{\eta_2}\zeta_3(0,0,0) + I^{\eta_2+w_2}\zeta_4(0,0,0)\right]\right. \\
&\left.+ \frac{1}{\wp_4}\left[I^{\eta_2-1}\zeta_3(0,0,0) + I^{\eta_2+w_2-1}\zeta_4(0,0,0)\right]\right\}\left(\frac{\omega}{\wp_6} - \frac{1}{\wp_6^2} + \frac{e^{-\wp_6\omega}}{\wp_6^2}\right).\end{aligned}$$

Let $(\rho_1,\Upsilon_1),(\rho_2,\Upsilon_2)\in \mathbf{X}\times\mathbf{Y}$, then

$$
\begin{aligned}
&|\mathfrak{R}_1^*(\rho_1,\Upsilon_1)-\mathfrak{R}_1^*(\rho_2,\Upsilon_2)|\\
&\leqslant \frac{1}{\wp_1}\int_0^{\omega} e^{-\wp_3(\omega-s)}\mathrm{I}^{\eta_1}|\zeta_1(s,\rho_1(s),\Upsilon_1(s))-\zeta_1(s,\rho_2(s),\Upsilon_2(s))|ds\\
&\quad+\frac{1}{\wp_1}\int_0^{\omega} e^{-\wp_3(\omega-s)}\mathrm{I}^{\eta_1+w_1}|\zeta_2(s,\rho_1(s),\Upsilon_1(s))-\zeta_1(s,\rho_2(s),\Upsilon_2(s))|ds\\
&\quad-\frac{\wp_2}{\wp_1}\int_0^{\mathbf{t}} e^{-\wp_3(\omega-s)}\mathrm{I}^{\eta_1-\xi_1-1}|\mathbf{J_1}(s)-\mathbf{J_2}(s)|ds+\frac{\wp_3\cdot\wp_2}{\wp_1}\int_0^{\omega} e^{-\wp_3(\omega-s)}\mathrm{I}^{\eta_1-\xi_1}|\mathbf{J_1}(s)-\mathbf{J_2}(s)|ds|\\
&\leqslant \frac{1}{\wp_1}\int_0^{\omega} e^{-\wp_3(\omega-s)}\mathrm{I}^{\eta_1+w_1}|\Big(B_1^*\|\rho_1-\rho_2\|+B_2^*\|\Upsilon_1-\Upsilon_2\|\Big)|ds\\
&\quad+\frac{1}{\wp_1}\int_0^{\omega} e^{-\wp_3(\omega-s)}\mathrm{I}^{\eta_1}|\Big(B_3^*\|\rho_1-\rho_2\|+B_4^*\|\Upsilon_1-\Upsilon_2\|\Big)|ds\\
&\quad-\frac{\wp_2}{\wp_1}\int_0^{\omega} e^{-\wp_3(\omega-s)}\mathrm{I}^{\eta_1-\xi-1}|\mathbf{J_1}(s)-\mathbf{J_2}(s)|ds+\frac{\wp_3\cdot\wp_2}{\wp_1}\int_0^{\omega} e^{-\wp_3(\omega-s)}\mathrm{I}^{\eta_1-\xi_1}|\mathbf{J_1}(s)-\mathbf{J_2}(s)|ds\\
&\leqslant \frac{B^*}{\wp_1}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_1}}{\Gamma(\eta_1+1)}+\frac{\Omega^{\eta_1+w_1}}{\Gamma(\eta_1+w_1+1)}\right]\Big(\|\rho_1-\rho_2\|+\|\Upsilon_1-\Upsilon_2\|\Big)\\
&\quad+\frac{\wp_2}{\wp_1}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_1-\xi_1-1}}{\Gamma(\eta_1-\xi_1)}+\wp_3\frac{\Omega^{\eta_1-\xi_1}}{\Gamma(\eta_1-\xi_1+1)}\right]\|\rho_1-\rho_2\|\\
&\leqslant B^*\theta_1\Big(\|\rho_1-\rho_2\|+\|\Upsilon_1-\Upsilon_2\|\Big)+\theta_2\|\rho_1-\rho_2\|\\
&\leqslant (B^*\theta_1+\theta_2)\Big(\|\rho_1-\rho_2\|+\|\Upsilon_1-\Upsilon_2\|\Big)\leqslant (B^*\theta_1+\theta_2)\Big(\|\rho_1-\rho_2\|+\|\Upsilon_1-\Upsilon_2\|\Big),
\end{aligned}
$$

and

$$|\mathfrak{R}_1^*(\rho_1,\Upsilon_1)-\mathfrak{R}_1^*(\rho_2,\Upsilon_2)|\leqslant G_1^*\left(\|\rho_1-\rho_2\|+\|\Upsilon_1-\Upsilon_2\|\right).$$

Similarly

$$|\mathfrak{R}_2^*(\rho_1,\Upsilon_1)-\mathfrak{R}_2^*(\rho_2,\Upsilon_2)|\leqslant G_2^*\left(\|\rho_1-\rho_2\|+\|\Upsilon_1-\Upsilon_2\|\right).$$

Let $\max\{G_1*,G_2*\}=G*$, so we can write

$$\|\mathfrak{R}^*(\rho_1,\Upsilon_1)-\mathfrak{R}^*(\rho_2,\Upsilon_2)\|\leqslant G^*\left(\|\rho_1-\rho_2\|+\|\Upsilon_1-\Upsilon_2\|\right).$$

The operator $\mathfrak{R}^*$ is a contractive, supported by $(G^*\leqslant 1)$ as per (H_2^*). This conclusion is drawn from the given fact. Applying the Banach fixed point theorem, the operator $\mathfrak{R}^*$ has a unique fixed point, implying the existence of a mild solution to the problem (1.2). □

Theorem 3.4. *Let the conditions* (H_1^*)-(H_3^*) *are satisfied, then the problem* (1.2) *has at least one mild solution.*

Proof. $\mathfrak{R}_1^*$ is continuous since ζ_1,ζ_2 are continuous. Additionally, ζ_3,ζ_4 are continuous, meaning that $\mathfrak{R}_2^*$ is continuous as well, implying the continuity of $\mathfrak{R}^*$. Consider a set

$$Q_{q^*}=\{(\rho,\Upsilon)\in X\times Y:\|(\rho,\Upsilon)\|\leqslant q^*\}.$$

For any $\omega\in\mathcal{I}$, we have

$$
\begin{aligned}
|\mathfrak{R}_1^*(\rho,\Upsilon,\omega)|=&\left|\frac{1}{\wp_1}\int_0^{\omega} e^{-\wp_3(\omega-s)}\mathrm{I}^{\eta_1}\zeta_1(s,\rho(s),\Upsilon(s))ds+|\frac{1}{\wp_1}\int_0^{\omega} e^{-\wp_3(\omega-s)}\mathrm{I}^{\eta_1+w_1}\zeta_2(s,\rho(s),\Upsilon(s))ds\right.\\
&-\frac{\wp_2}{\wp_1}\int_0^{\omega} e^{-\wp_3(\omega-s)}\mathrm{I}^{\eta_1-\xi-1}\rho(s)ds-\frac{\wp_3\cdot\wp_2}{\wp_1}\int_0^{\omega} e^{-\wp_3(\omega-s)}\mathrm{I}^{\eta_1-\xi_1}\rho(s)ds
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{1}{\wp_1}\left[I^{\eta_1}\zeta_1(0,0,0) + I^{\eta_1+w_1}\zeta_2(0,0,0)\right]\left(\frac{1-e^{-\wp_3\omega}}{\wp_3}\right) - \left\{\frac{\wp_3}{\wp_1}\left[I^{\eta_1}\zeta_1(0,0,0) + I^{\eta_1+w_1}\zeta_2(0,0,0)\right]\right. \\
& \left. + \frac{1}{\wp_1}\left[I^{\eta_1-1}\zeta_1(0,0,0) + I^{\eta_1+w_1-1}\zeta_2(0,0,0)\right]\right\}\left(\frac{\omega}{\wp_3} - \frac{1}{\wp_3^2} + \frac{e^{-\wp_3\omega}}{\wp_3^2}\right)\Bigg| \\
\leqslant\ & \frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1}|\zeta_1(s,\rho(s),\Upsilon(s))|ds + \frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1+w_1}|\zeta_2(s,\rho(s),\Upsilon(s))|ds \\
& + \frac{\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1-\psi-1}|\rho(s)|ds - \frac{\wp_3\cdot\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1-\xi_1}|\rho(s)|ds \\
& - \frac{1}{\wp_1}\left[I^{\eta_1}|\zeta_1(0,0,0)| + I^{\eta_1+w_1}|\zeta_2(0,0,0)|\right]\left(\frac{1-e^{-\lambda_3^*\omega}}{\wp_3}\right) \\
& - \left\{\frac{\wp_3}{\wp_1}\left[I^{\eta_1}|\zeta_1(0,0,0)| + I^{\eta_1+w_1}|\zeta_2(0,0,0)|\right]\right. \\
& \left. + \frac{1}{\wp_1}\left[I^{\eta_1-1}|\zeta_1(0,0,0)| + I^{\eta_1+w_1-1}|\zeta_2(0,0,0)|\right]\right\}\left(\frac{\omega}{\wp_3} - \frac{1}{\wp_3^2} + \frac{e^{-\wp_3\omega}}{\wp_3^2}\right) \\
\leqslant\ & \frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1}(l_0(\omega) + A_1^*(\omega)|\rho| + A_2^*(\omega)|\Upsilon|)ds \\
& + \frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1+w_1}(A_1^*(\omega) + A_2^*(\omega)|\rho| + A_3^*(\omega)|\Upsilon|)ds \\
& + \frac{\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1-\xi-1}q^* ds + \frac{\wp_3\cdot\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1-\xi_1}q^* ds \\
& + \frac{1}{\wp_1}\left[I^{\eta_1}F_1^* + I^{\eta_1+w_1}F_2^*\right]\left(\frac{1-e^{-\wp_3\omega}}{\wp_3}\right) - \left[\frac{\wp_3}{\wp_1}\left(I^{\eta_1}F_1^* + I^{\eta_1+w_1}F_2^*\right)\right. \\
& \left. + \frac{1}{\wp_1}\left(I^{\eta_1-1}F_1^* + I^{\eta_1+w_1-1}F_2^*\right)\right]\left(\frac{\omega}{\wp_3} - \frac{1}{\wp_3^2} + \frac{e^{-\wp_3\omega}}{\wp_3^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
|\mathfrak{R}_1^*(\rho,\Upsilon,\omega)| \leqslant\ & \frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1}(l_0^* + l_1^* q^* + l_2^* q^*)ds + \frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1+w_1}(l_1^* + l_2^* q^* + l_3^* q^*)ds \\
& + \frac{\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1-\xi-1}q^* ds - \frac{\wp_3\cdot\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1-\xi_1}q^* ds \\
& + \frac{1}{\wp_1}\left[I^{\eta_1} + I^{\eta_1+w_1}\right]F^*\left(\frac{1-e^{-\wp_3\omega}}{\wp_3}\right) \\
& - \left[\frac{\wp_3}{\wp_1}\left(I^{\eta_1} + I^{\eta_1+w_1}\right)F^* + \frac{1}{\wp_1}\left(I^{\eta_1-1} + I^{\eta_1+w_1-1}\right)F^*\right]\left(\frac{\omega}{\wp_3} - \frac{1}{\wp_3^2} + \frac{e^{-\wp_3\omega}}{\wp_3^2}\right) \\
\leqslant\ & \frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1}\beta_1^* ds + \frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1+w_1}\beta_2^* ds \\
& + \frac{\wp_2}{\wp_1}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_1-\psi_1-1}}{\Gamma(\eta_1-\xi_1)} + \wp_3\frac{\Omega^{\eta_1-\xi_1}}{\Gamma(\eta_1-\xi_1+1)}\right]q^* \\
& + \frac{F^*}{\wp_1}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_1}}{\Gamma(\eta_1+1)} + \frac{\Omega^{\eta_1+w_1}}{\Gamma(\eta_1+w_1+1)}\right] \\
& + \frac{F^*}{\wp_1}\left[\wp_3\frac{\Omega^{\eta_1}}{\Gamma(\eta_1+1)} + \lambda_3\frac{\Omega^{\eta_1+w_1}}{\Gamma(\eta_1+w_1+1)} + \frac{\Omega^{\eta_1-1}}{\Gamma(\eta_1)} + \frac{\Omega^{\eta_1+w_1-1}}{\Gamma(\eta_1+w_1)}\right]\left(\frac{\Omega}{\wp_3} - \frac{1}{\wp_3^2} + \frac{e^{-\wp_3\Omega}}{\wp_3^2}\right) \\
\leqslant\ & \frac{\beta^*}{\wp_1}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_1}}{\Gamma(\eta_1+1)} + \frac{\Omega^{\eta_1+w_1}}{\Gamma(\eta_1+w_1+1)}\right] \\
& + \frac{\wp_2}{\wp_1}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_1-\xi_1-1}}{\Gamma(\eta_1-\xi_1)} + \wp_3\frac{\Omega^{\eta_1-\xi_1}}{\Gamma(\eta_1-\xi_1+1)}\right]q^*
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{F^*}{\wp_1}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_1}}{\Gamma(\eta_1+1)}+\frac{\Omega^{\eta_1+w_1}}{\Gamma(\eta_1+w_1+1)}\right]\\
&+\frac{F^*}{\wp_1}\left[\wp_3\frac{\Omega^{\eta_1}}{\Gamma(\eta_1+1)}+\lambda_3\frac{\Omega^{\eta_1+w_1}}{\Gamma(\eta_1+w_1+1)}+\frac{\Omega^{\eta_1-1}}{\Gamma(\eta_1)}+\frac{\Omega^{\eta_1+w_1-1}}{\Gamma(\eta_1+w_1)}\right]\left(\frac{\Omega}{\wp_3}-\frac{1}{\wp_3^2}+\frac{e^{-\wp_3\Omega}}{\wp_3^2}\right).
\end{aligned}
$$

Therefore $\|\mathfrak{R}_1^*(\rho,\Upsilon,\omega)\|\leqslant\mu_1$. In the same manner, we can prove that $\|\mathfrak{R}_2^*(\rho,\Upsilon,\omega)\|\leqslant\mu_2$, where

$$
\begin{aligned}
\mu_2=&\frac{\beta^*}{\wp_4}\left(\frac{1-e^{-\wp_6\Omega}}{\wp_6}\right)\left[\frac{\Omega^{\eta_2}}{\Gamma(\eta_1+1)}+\frac{\Omega^{\eta_2+w_2}}{\Gamma(\eta_2+w_2+1)}\right]+\frac{\lambda_5^*}{\wp_4}\left(\frac{1-e^{-\wp_6\Omega}}{\wp_6}\right)\left[\frac{\Omega^{\eta_2-\xi_2-1}}{\Gamma(\eta_2-\psi_2)}\right.\\
&\left.+\wp_6\frac{\Omega^{\eta_2-\xi_2}}{\Gamma(\eta_2-\xi_2+1)}\right]q^*+\frac{F^*}{\wp_4}\left(\frac{1-e^{-\wp_6\Omega}}{\wp_6}\right)\left[\frac{\Omega^{\eta_2}}{\Gamma(\eta_2+1)}+\frac{\Omega^{\eta_2+w_2}}{\Gamma(\eta_2+w_2+1)}\right]\\
&+\frac{F^*}{\wp_4}\left[\wp_6\frac{\Omega^{\eta_2}}{\Gamma(\eta_2+1)}+\wp_6\frac{\Omega^{\eta_2+w_2}}{\Gamma(\eta_2+w_2+1)}+\frac{\Omega^{\eta_2-1}}{\Gamma(\eta_2)}+\frac{\Omega^{\eta_2+w_2-1}}{\Gamma(\eta_2+w_2)}\right]\left(\frac{\Omega}{\wp_6}-\frac{1}{\wp_6^2}+\frac{e^{-\wp_6\Omega}}{\wp_6^2}\right).
\end{aligned}
$$

Let $\max\{\mu_1,\mu_2\}=\mu$, then we have $\|\mathfrak{R}^*(\rho,\Upsilon,\omega)\|\leqslant\mu$. The boundedness of the operator $\mathfrak{R}^*$ is shown by the aforementioned inequality. Now we have to show that the operator $\mathfrak{R}^*$ is equicontinuous . For this let $\mathbf{w_1},\mathbf{w_2}\in\mathfrak{J}_k$ such that $\mathbf{w_1}<\mathbf{w_2}$, where $k=0,1,2,\dots,p$. Let $(\rho,\Upsilon)\in Q_q^*$, then we have

$$
\begin{aligned}
&|(\mathfrak{R}_1^*\rho)(\mathbf{w_2})-(\mathfrak{R}_1^*\rho)(\mathbf{w_1})|\\
&\leqslant\left|\frac{1}{\wp_1}\int_0^{\mathbf{w_2}}e^{-\wp_3(\mathbf{w_2}-s)}I^{\eta_1}\zeta_1(s,\rho(s),\Upsilon(s))ds+\frac{1}{\wp_1}\int_0^{\mathbf{w_2}}e^{-\wp_3(\mathbf{w_2}-s)}I^{\eta_1+w_1}\zeta_2(s,\rho(s),\Upsilon(s))ds\right.\\
&\quad-\frac{\wp_2}{\wp_1}\int_0^{\mathbf{w_2}}e^{-\wp_3(\mathbf{w_2}-s)}I^{\eta_1-\xi-1}\rho(s)ds-\frac{\wp_3\cdot\wp_2}{\wp_1}\int_0^{\mathbf{w_2}}e^{-\wp_3(\mathbf{w_2}-s)}I^{\eta_1-\xi_1}\rho(s)ds\\
&\quad-\frac{1}{\wp_1}\left[I^{\eta_1}\zeta_1(0,0,0)+I^{\eta_1+w_1}\zeta_2(0,0,0)\right]\left(\frac{1-e^{-\wp_3\mathbf{w_2}}}{\wp_3}\right)-\left\{\frac{\wp_3}{\wp_1}\left[I^{\eta_1}\zeta_1(0,0,0)+I^{\eta_1+w_1}\zeta_2(0,0,0)\right]\right.\\
&\quad\left.+\frac{1}{\wp_1}\left[I^{\eta_1-1}\zeta_1(0,0,0)+I^{\eta_1+w_1-1}\zeta_2(0,0,0)\right]\right\}\left(\frac{\mathbf{w_2}}{\wp_3}-\frac{1}{\wp_3^2}+\frac{e^{-\wp_3\mathbf{w_2}}}{\wp_3^2}\right)\\
&\quad-\frac{1}{\wp_1}\int_0^{\mathbf{w_1}}e^{-\wp_3(\mathbf{w_1}-s)}I^{\eta_1}\zeta_1(s,\rho(s),\Upsilon(s))ds-\frac{1}{\wp_1}\int_0^{\mathbf{w_1}}e^{-\wp_3(\mathbf{w_1}-s)}I^{\eta_1+w_1}\zeta_2(s,\rho(s),\Upsilon(s))ds\\
&\quad+\frac{\wp_2}{\wp_1}\int_0^{\mathbf{w_1}}e^{-\wp_3(\mathbf{w_1}-s)}I^{\eta_1-\xi-1}\rho(s)ds+\frac{\wp_3\cdot\wp_2}{\wp_1}\int_0^{\mathbf{w_1}}e^{-\wp_3(\mathbf{w_1}-s)}I^{\eta_1-\psi_1}\rho(s)ds\\
&\quad-\frac{1}{\wp_1}\left[I^{\eta_1}\zeta_1(0,0,0)+I^{\eta_1+w_1}\zeta_2(0,0,0)\right]\left(\frac{1-e^{-\wp_3\mathbf{w_1}}}{\wp_3}\right)-\left\{\frac{\wp_3}{\wp_1}\left[I^{\eta_1}\zeta_1(0,0,0)+I^{\eta_1+w_1}\zeta_2(0,0,0)\right]\right.\\
&\quad\left.\left.+\frac{1}{\wp_1}\left[I^{\eta_1-1}\zeta_1(0,0,0)+I^{\eta_1+w_1-1}\zeta_2(0,0,0)\right]\right\}\left(\frac{\mathbf{w_1}}{\wp_3}-\frac{1}{\wp_3^2}+\frac{e^{-\wp_3\mathbf{w_1}}}{\wp_3^2}\right)\right|.
\end{aligned}
$$

From above inequality, if $\mathbf{w_1}\to\mathbf{w_2}$, we deduce that

$$|(\mathfrak{R}_1^*\rho)(\mathbf{w_2})-(\mathfrak{R}_1^*\rho)(\mathbf{w_1})|\to 0,$$

also we can show that

$$|(\mathfrak{R}_2^*\rho)(\mathbf{w_2})-(\mathfrak{R}_2^*\rho)(\mathbf{w_1})|\to 0.$$

Hence $\mathfrak{R}_1^*$ and $\mathfrak{R}_2^*$ are equicontinuous by the Arzila-Ascoli theorem, implying that $\mathfrak{R}^*$ is equicontinuous. Now let us define a set

$$G=\{(\rho,\Upsilon)\in X\times Y;(\rho,\Upsilon)=\delta\mathfrak{R}^*(\rho,\Upsilon);0<\delta<1\}.$$

We demonstrate the boundedness of the set G. For this, let $z\in\mathfrak{J}$ and $(\rho,\Upsilon)\in G$, we have $(\rho,\Upsilon)=\delta\mathfrak{R}^*(\rho,\Upsilon)$, i.e., $\rho(z)=\delta\mathfrak{R}_1^*(\rho,\Upsilon)$ and $\Upsilon(z)=\delta\mathfrak{R}_2^*(\rho,\Upsilon)$. Now

$$|\rho(z)|=|\delta\mathfrak{R}_1^*(\rho,\Upsilon)||\delta|\frac{1}{\wp_1}\int_0^{\omega}e^{-\wp_3(\omega-s)}I^{\eta_1}\zeta_1(s,\rho(s),\Upsilon(s))ds$$

$$
\begin{aligned}
&+|\frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)}I^{\eta_1+w_1}\zeta_2(s,\rho(s),\Upsilon(s))ds\\
&-\frac{\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)}I^{\eta_1-\xi-1}\rho(s)ds-\frac{\wp_3\cdot\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)}I^{\eta_1-\psi_1}\rho(s)ds\\
&-\frac{1}{\wp_1}\left[I^{\eta_1}\zeta_1(0,0,0)+I^{\eta_1+w_1}\zeta_2(0,0,0)\right]\left(\frac{1-e^{-\wp_3\omega}}{\wp_3}\right)-\left\{\frac{\wp_3}{\wp_1}\left[I^{\eta_1}\zeta_1(0,0,0)+I^{\eta_1+w_1}\zeta_2(0,0,0)\right]\right.\\
&\left.+\frac{1}{\wp_1}\left[I^{\eta_1-1}\zeta_1(0,0,0)+I^{\eta_1+w_1-1}\zeta_2(0,0,0)\right]\right\}\left(\frac{\omega}{\wp_3}-\frac{1}{\wp_3^2}+\frac{e^{-\wp_3\omega}}{\wp_3^2}\right)|\\
\leqslant{}&|\delta|\frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)}I^{\eta_1}|\zeta_1(s,\rho(s),\Upsilon(s))|ds+\frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)}I^{\eta_1+w_1}|\zeta_2(s,\rho(s),\Upsilon(s))|ds\\
&+\frac{\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)}I^{\eta_1-\xi-1}|\rho(s)|ds-\frac{\wp_3\cdot\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)}I^{\eta_1-\psi_1}|\rho(s)|ds\\
&-\frac{1}{\wp_1}\left[I^{\eta_1}|\zeta_1(0,0,0)|+I^{\eta_1+w_1}|\zeta_2(0,0,0)|\right]\left(\frac{1-e^{-\wp_3\omega}}{\wp_3}\right)-\left\{\frac{\wp_3}{\wp_1}\left[I^{\eta_1}|\zeta_1(0,0,0)|+I^{\eta_1+w_1}|\zeta_2(0,0,0)|\right]\right.\\
&\left.+\frac{1}{\wp_1}\left[I^{\eta_1-1}|\zeta_1(0,0,0)|+I^{\eta_1+w_1-1}|\zeta_2(0,0,0)|\right]\right\}\left(\frac{\omega}{\wp_3}-\frac{1}{\wp_3^2}+\frac{e^{-\wp_3\omega}}{\wp_3^2}\right)\\
\leqslant{}&|\delta|\frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)}I^{\eta_1}(l_0(\omega)+A_1^*(\omega)|\rho|+A_2^*(\omega)|\Upsilon|)ds\\
&+\frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)}I^{\eta_1+w_1}(A_1^*(\omega)+A_2^*(\omega)|\rho|+A_3^*(\omega)|\Upsilon|)ds\\
&+\frac{\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)}I^{\eta_1-\xi-1}q^*ds+\frac{\wp_3\cdot\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)}I^{\eta_1-\xi_1}q^*ds\\
&+\frac{1}{\wp_1}\left[I^{\eta_1}F_1^*+I^{\eta_1+w_1}F_2^*\right]\left(\frac{1-e^{-\wp_3\omega}}{\wp_3}\right)-\left[\frac{\wp_3}{\wp_1}\left(I^{\eta_1}F_1^*+I^{\eta_1+w_1}F_2^*\right)\right.\\
&\left.+\frac{1}{\wp_1}\left(I^{\eta_1-1}F_1^*+I^{\eta_1+w_1-1}F_2^*\right)\right]\left(\frac{\omega}{\wp_3}-\frac{1}{\wp_3^2}+\frac{e^{-\wp_3\omega}}{\wp_3^2}\right).\\
\leqslant{}&|\delta|\frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)}I^{\eta_1}(l_0^*+l_1^*q^*+l_2^*q^*)ds+\frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)}I^{\eta_1+w_1}(l_1^*+l_2^*q^*+l_3^*q^*)ds\\
&+\frac{\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)}I^{\eta_1-\xi-1}q^*ds-\frac{\wp_3\cdot\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)}I^{\eta_1-\xi_1}q^*ds\\
&+\frac{1}{\wp_1}\left[I^{\eta_1}+I^{\eta_1+w_1}\right]F^*\left(\frac{1-e^{-\wp_3\omega}}{\wp_3}\right)\\
&-\left[\frac{\wp_3}{\wp_1}\left(I^{\eta_1}+I^{\eta_1+w_1}\right)F^*+\frac{1}{\wp_1}\left(I^{\eta_1-1}+I^{\eta_1+w_1-1}\right)F^*\right]\left(\frac{\omega}{\wp_3}-\frac{1}{\wp_3^2}+\frac{e^{-\wp_3\omega}}{\wp_3^2}\right)\\
\leqslant{}&|\delta|\frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)}I^{\eta_1}\beta_1^*ds+\frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)}I^{\eta_1+w_1}\beta_2^*ds+\frac{\wp_2}{\wp_1}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\\
&\times\left[\frac{\Omega^{\eta_1-\xi_1-1}}{\Gamma(\eta_1-\xi_1)}+\wp_3\frac{\Omega^{\eta_1-\xi_1}}{\Gamma(\eta_1-\xi_1+1)}\right]q^*+\frac{F^*}{\wp_1}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_1}}{\Gamma(\eta_1+1)}+\frac{\Omega^{\eta_1+w_1}}{\Gamma(\eta_1+w_1+1)}\right]\\
&+\frac{F^*}{\wp_1}\left[\wp_3\frac{\Omega^{\eta_1}}{\Gamma(\eta_1+1)}+\wp_3\frac{\Omega^{\eta_1+w_1}}{\Gamma(\eta_1+w_1+1)}+\frac{\Omega^{\eta_1-1}}{\Gamma(\eta_1)}+\frac{\Omega^{\eta_1+w_1-1}}{\Gamma(\eta_1+w_1)}\right]\left(\frac{\Omega}{\wp_3}-\frac{1}{\wp_3^2}+\frac{e^{-\wp_3\Omega}}{\wp_3^2}\right)\\
\leqslant{}&|\delta|\frac{\beta^*}{\wp_1}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_1}}{\Gamma(\eta_1+1)}+\frac{\Omega^{\eta_1+w_1}}{\Gamma(\eta_1+w_1+1)}\right]+\frac{\wp_2}{\wp_1}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\\
&\times\left[\frac{\Omega^{\eta_1-\xi_1-1}}{\Gamma(\eta_1-\xi_1)}+\wp_3\frac{\Omega^{\eta_1-\xi_1}}{\Gamma(\eta_1-\xi_1+1)}\right]q^*+\frac{F^*}{\wp_1}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_1}}{\Gamma(\eta_1+1)}+\frac{\Omega^{\eta_1+w_1}}{\Gamma(\eta_1+w_1+1)}\right]
\end{aligned}
$$

$$+\frac{F^*}{\wp_1}\left[\wp_3\frac{\Omega^{\eta_1}}{\Gamma(\eta_1+1)}+\wp_3\frac{\Omega^{\eta_1+w_1}}{\Gamma(\eta_1+w_1+1)}+\frac{\Omega^{\eta_1-1}}{\Gamma(\eta_1)}+\frac{\Omega^{\eta_1+w_1-1}}{\Gamma(\eta_1+w_1)}\right]\left(\frac{\Omega}{\wp_3}-\frac{1}{\wp_3^2}+\frac{e^{-\wp_3\Omega}}{\wp_3^2}\right),$$
$$|\rho(z)|=|\delta\mathfrak{R}_1^*(\rho,\Upsilon)|\leqslant\delta\mu_1.$$

Likewise, we can demonstrate that $\|\mathfrak{R}_2^*(\rho,\Upsilon,\omega)\|\leqslant\delta\mu_2$. Let $\max\{\delta\mu_1,\delta\mu_2\}=\Lambda$. Then we have

$$\|\mathfrak{R}^*(\rho,\Upsilon,\omega)\|\leqslant\Lambda.$$

Because of this, the set G is bounded and by Schaefer's fixed point theorem, we can say that the problem (1.2) has at least one mild solution. □

4. Mixed nonlinearities

We examine the subsequent problem encompassing three nonlinearities:

$$\begin{cases}\sigma_1^c D^\eta(D+\sigma_3)\rho(\omega)+\sigma_2^c D^\xi(D+\sigma_3)\zeta_3(\omega,\rho(\omega))=\zeta_1(\omega,\rho(\omega)),\\ +I^\nu\zeta_2(\omega,\rho(\omega)),\ \forall\omega\in\mathfrak{I},\\ \rho(0)=\rho'(0)=\rho''(0)=0.\end{cases}\tag{4.1}$$

All other quantities are defined previously, except ζ_3, which is continuous and defined as $\zeta_3:\mathfrak{I}\times R\to R$. Considering the operator $\mathfrak{I}:C\to C$ in connection with the problem (4.1), presented below

$$\begin{aligned}(\mathfrak{I}\rho)(\omega)=&\frac{1}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^\eta\zeta_1(s,\rho(s))ds+\frac{1}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta+\nu}\zeta_2(s,\rho(s))ds\\&-\frac{\sigma_2}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta-\xi-1}\zeta_3(s,\rho(s))ds-\frac{\sigma_3.\sigma_2}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta-\xi}\zeta_3(s,\rho(s))ds\\&-\left[\frac{1}{\sigma_1}I^\eta\zeta_1(0,0)+\frac{1}{\sigma_1}I^{\eta+\nu}\zeta_2(0,0)-\frac{\sigma_2}{\sigma_1}I^{\eta-\xi-1}\zeta_3(0,0)-\frac{\sigma_3.\sigma_2}{\sigma_1}I^{\eta-\xi}\zeta_3(0,0)\right]\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right)\\&+\left[-\frac{1}{\sigma_1}I^{\eta-1}\zeta_1(0,0)-\frac{1}{\sigma_1}I^{\eta+\nu-1}\zeta_2(0,0)+\frac{\sigma_2}{\sigma_1}I^{\eta-\xi-2}\zeta_3(0,0)+\frac{\sigma_3.\sigma_2}{\sigma_1}I^{\eta-\xi-1}\zeta_3(0,0)\right]\\&\times\left(\frac{\omega}{\sigma_3}-\frac{1}{\sigma_3^2}+\frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right)-\sigma_3\left[-\frac{1}{\sigma_1}I^\eta\zeta_1(0,0)-\frac{1}{\sigma_1}I^{\eta+\nu}\zeta_2(0,0)+\frac{\sigma_2}{\sigma_1}I^{\eta-\xi-1}\zeta_3(0,0)\right.\\&\left.+\frac{\sigma_3.\sigma_2}{\sigma_1}I^{\eta-\xi}\zeta_3(0,0)\right]\left(\frac{\omega}{\sigma_3}-\frac{1}{\sigma_3^2}+\frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right).\end{aligned}$$

A theorem concerning the uniqueness of mild solutions will be discussed below.

($\bar{G}4$) ζ_1,ζ_2,ζ_3 are continuous functions defined from $\zeta_1,\zeta_2,\zeta_3:\mathfrak{I}\times R\to R$ and there exist $A_1^*,A_2^*,A_3^*>0$ such that $|\zeta_1(\omega,\rho_1)-\zeta_1(\omega,\rho_2)|\leqslant A_1^*|\rho_1-\rho_2|$, $|\zeta_2(\omega,\rho_1)-\zeta_2(\omega,\rho_2)|\leqslant A_2^*|\rho_1-\rho_2|$, $|\zeta_3(\omega,\rho_1)-\zeta_3(\omega,\rho_2)|\leqslant A_3^*|\rho_1-\rho_2|$, $\forall\omega\in\mathfrak{I}$, and $\rho_1,\rho_2\in R$. And $|\zeta_1(0,0)|\leqslant N_1^*<\infty$, $|\zeta_2(0,0)|\leqslant N_2^*<\infty$, $|\zeta_3(0,0)|\leqslant N_3^*<\infty$, and let $\max\{N_1^*,N_2^*,N_3^*\}=N^*$.

($\bar{G}5$) $A^*(\theta_1+\theta_2)<1$, where $A^*=\max\{A_1^*,A_2^*,A_3^*\}$, and φ_1 and φ_2 are defined in ($\bar{G}2$).

Theorem 4.1. *There is a unique mild solution for the problem (4.1) on the interval $\mathfrak{I}$, if the conditions ($\bar{G}4$) and ($\bar{G}5$) are satisfied.*

Proof. We skip the proof since it is comparable to the proof of the Theorem 3.1. □

Now to study the mixed nonlinearities case for coupled system. Consider the couple system with nonlinear functions given by:

$$\begin{cases} \wp_1^c D^{\eta_1}(D+\wp_3)\rho(\omega) + \wp_2^c D^{\xi_1}(D+\wp_3)\bar{H}_1(\omega,\rho(\omega)),\Upsilon(\omega)) \\ \quad = \zeta_1(\omega,\rho(\omega),\Upsilon(\omega)) + I^{\nu_1}\zeta_2(\omega,\rho(\omega)),\Upsilon(\omega)), \ \omega\in\mathcal{J}, \\ \wp_4^c D^{\eta_2}(D+\wp_6)\Upsilon(\omega) + \wp_5^c D^{\xi_2}(D+\wp_6)\bar{H}_2(\omega,\rho(\omega)),\Upsilon(\omega)) \\ \quad = \zeta_3(\omega,\rho(\omega),\Upsilon(\omega)) + I^{\nu_2}\zeta_4(\omega),\rho(\omega)),\Upsilon(\omega)), \ \omega\in\mathcal{J}, \\ \rho(0) = \rho_1'(0) = \rho_1''(0) = 0, \\ \Upsilon(0) = \Upsilon'(0) = \Upsilon''(0) = 0. \end{cases} \tag{4.2}$$

In this case, the continuous functions $\bar{H}_1$ and $\bar{H}_2$ are defined as follows: $\bar{H}_1, \bar{H}_2 : \mathcal{J}\times R \to R$. Regarding the problem (4.2), we define an operator $\mathfrak{fi} : \mathbf{C} \to \mathbf{C}$ given as

$$\mathfrak{fi}(\rho,\Upsilon)(\omega) = (\mathfrak{fi}_1(\rho,\Upsilon)(\omega), \mathfrak{fi}_2(\rho,\Upsilon)(\omega)),$$

where

$$\begin{aligned}(\mathfrak{fi}_1)(\omega) &= \frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1}\zeta_1(s,\rho(s),\Upsilon(s))ds \\ &+ \frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1+w_1}\zeta_2(s,\rho(s),\Upsilon(s))ds - \frac{\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1-\xi_1-1}\bar{H}_1(s)ds \\ &- \frac{\wp_3.\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1-\xi_1}\bar{H}_1(s)ds - \frac{1}{\wp_1}\Big[I^{\eta_1}\zeta_1(0,0,0) + I^{\eta_1+w_1}\zeta_2(0,0,0) \\ &+ \frac{\wp_2}{\wp_1}I^{\eta_1-\xi_1-1}\bar{H}_1(0,0,0) + \frac{\wp_2.\wp_3}{\wp_1}I^{\eta_1-\xi_1}\bar{H}_1(0,0,0)\Big]\left(\frac{1-e^{-\wp_3\omega}}{\wp_3}\right) \\ &- \Big\{\frac{\wp_3}{\wp_1}\Big[I^{\eta_1}\zeta_1(0,0,0) + I^{\eta_1+w_1}\zeta_2(0,0,0) + \frac{\wp_2}{\wp_1}I^{\eta_1-\xi_1-1}\bar{H}_1(0,0,0) \\ &+ \frac{\wp_2.\bar{\lambda}_3}{\wp_1}I^{\eta_1-\xi_1}\bar{H}_1(0,0,0)\Big] + \frac{1}{\wp_1}\Big[I^{\eta_1-1}\zeta_1(0,0,0) + I^{\eta_1+w_1-1}\zeta_2(0,0,0) \\ &+ \frac{\wp_2}{\wp_1}I^{\eta_1-\xi_1-1}\bar{H}_1(0,0,0) + \frac{\wp_2.\wp_3}{\wp_1}I^{\eta_1-\xi_1}\bar{H}_1(0,0,0)\Big]\Big\}\left(\frac{\omega}{\wp_3} - \frac{1}{\wp_3^2} + \frac{e^{-\wp_3\omega}}{\wp_3^2}\right),\end{aligned}$$

and

$$\begin{aligned}(\mathfrak{fi}_2)(\omega) &= \frac{1}{\wp_4}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_2}\zeta_3(s,\rho(s),\Upsilon(s))ds \\ &+ \frac{1}{\wp_4}\int_0^\omega e^{-\wp_6(\omega-s)} I^{\eta_2+w_2}\zeta_4(s,\rho(s),\Upsilon(s))ds - \frac{\wp_5}{\wp_4}\int_0^\omega e^{-\wp_6(\omega-s)} I^{\eta_2-\xi_2-1}\bar{H}_2(s)ds \\ &- \frac{\wp_6.\wp_5}{\wp_4}\int_0^\omega e^{-\wp_6(\omega-s)} I^{\eta_2-\xi_2}\bar{H}_2(s)ds - \frac{1}{\wp_4}\Big[I^{\eta_2}\zeta_3(0,0,0) + I^{\eta_2+w_2}\zeta_4(0,0,0) \\ &+ \frac{\wp_5}{\wp_4}I^{\eta_2-\xi_2-1}\bar{H}_2(0,0,0) + \frac{\wp_5.\wp_6}{\wp_4}I^{\eta_2-\xi_2}\bar{H}_2(0,0,0)\Big]\left(\frac{1-e^{-\wp_6\omega}}{\wp_6}\right) \\ &- \Big\{\frac{\wp_6}{\wp_4}\Big[I^{\eta_2}\zeta_3(0,0,0) + I^{\eta_2+w_2}\zeta_4(0,0,0) + \frac{\wp_5}{\wp_4}I^{\eta_2-\xi_2-1}\bar{H}_2(0,0,0) \\ &+ \frac{\wp_5.\wp_6}{\wp_4}I^{\eta_2-\xi_2}\bar{H}_2(0,0,0)\Big] + \frac{1}{\wp_4}\Big[I^{\eta_2-1}\zeta_3(0,0,0) + I^{\eta_2+w_2-1}\zeta_4(0,0,0) \\ &+ \frac{\wp_5}{\wp_4}I^{\eta_2-\xi_2-1}\bar{H}_2(0,0,0) + \frac{\wp_5.\wp_6}{\wp_4}I^{\eta_2-\xi_2}\bar{H}_2(0,0,0)\Big]\Big\}\left(\frac{\omega}{\wp_6} - \frac{1}{\wp_6^2} + \frac{e^{-\wp_6\omega}}{\wp_6^2}\right).\end{aligned}$$

A theorem relating to the uniqueness of the problem (4.2) will be discussed below.

(H_4^*) $\bar{H}_1, \bar{H}_2$ are continuous functions defined by: $\bar{H}_1, \bar{H}_2 : \mathcal{J} \times R \to R$, and there exists positive constants B_9^*, B_{10}^*, B_{11}^*, B_{12}^* such that

$$|\bar{H}_1(\omega, \rho(\omega), \Upsilon(\omega)) - \bar{H}_1(\omega, \bar{\rho}(\omega), \bar{\Upsilon}(\omega))| \leqslant B_9^*|\rho(\omega) - \bar{\rho}(\omega)| + B_{10}^*|\Upsilon(\omega) - \bar{\Upsilon}(\omega)|,$$
$$|\bar{H}_2(\omega, \rho(\omega), \Upsilon(\omega)) - \bar{H}_2(\omega, \bar{\rho}(\omega), \bar{\Upsilon}(\omega)| \leqslant B_{11}^*|\rho(\omega) - \bar{\rho}(\omega)| + B_{12}^*|\Upsilon(\omega) - \bar{\Upsilon}(\omega)|,$$
$$|\bar{H}_i(\mathbf{0}, 0, 0)| \leqslant K, \ \forall\, i = 9, 10, 11, 12,$$
$$|\bar{H}_i(\omega, \rho(\omega), \Upsilon(\omega))| \leqslant h_{i-1}(\omega) + h_i(\omega)|\rho(\omega)| + h_{i+1}(\omega)|\Upsilon(\omega)|.$$

(H_5^*) $\max\{S_1*, S_2*\} = S^* < 1$, where $B^*(\theta_1 + \theta_2) = S_1*$ and $B^*(\phi_1 + \phi_2) = S_2*$, where $\max\{B_i^*\} = B^*$.

Theorem 4.2. *There is a unique mild solution for the problem* (4.2) *on the interval* $\mathcal{J}$, *if the conditions* (H_1^*)-(H_5^*) *are satisfied.*

Proof. We skip the proof of this theorem since it is comparable to the proof of the Theorem (3.3). □

5. Ulam stability

We provide the Hyers-Ulam stability idea for the problem (3.1). Let $\delta > 0$ and ζ_1, ζ_2 are continuos functions defined as $\zeta_1, \zeta_2 : \mathcal{J} \times R \to R$. Let us consider a Banach space $\Theta = \boldsymbol{C}$ with a norm define by

$$\|\rho\| = \sup_{\omega \in I} |\rho(\omega)|, \ \omega \in \mathcal{J}.$$

Definition 5.1. The problem (3.1) is Hyers-Ulam stable if there is a number $\breve{n} > 0$, where $\breve{n} \in R$ exists such that for each $\epsilon \geqslant 0$ and for every mild solution $\rho_1 \in \boldsymbol{C}^1$ of the inequality,

$$|(\sigma_1^c D^\eta + \sigma_2^c D^\xi)(D + \sigma_3)\rho_1(\omega) - \zeta_1(\omega, \rho_1(\omega)) + I^\gamma \zeta_2(\omega, \rho_1(\omega))| \leqslant \delta, \tag{5.1}$$

then, a mild solution $\rho_2 \in \boldsymbol{C}^1$ occurs with $|\rho_1(\omega) - \rho_2(\omega)| \leqslant \breve{n}\delta$.

Definition 5.2. The problem (3.1) is generalized Hyers-Ulam stable if X is a function that belongs to $\mathbf{C}[R_+, R_+]$ exists and $X(0) = 0$, such that for each $\delta \geqslant 0$ and for each mild solution $\rho_1 \in \boldsymbol{C}$ of the inequality

$$|(\sigma_1^c D^\eta + \sigma_2^c D^\xi)(D + \sigma_3)\rho_1(\omega) - \zeta_1(\omega, \rho_1(\omega)) + I^\gamma \zeta_2(\omega, \rho_1(\omega))| \leqslant \delta, \tag{5.2}$$

then, a mild solution $\rho_2 \in \boldsymbol{C}$ occurs with $|\rho_1(\omega) - \rho_2(\omega)| \leqslant X(\delta)$.

Remark 5.3. A function $\rho_1 \in \boldsymbol{C}^1$ is a mild solution of of the inequality (5.2) if and only if a function exists $\vartheta(\omega) \in \boldsymbol{C}$ (which depends on ρ_2) such that

(1) $|\vartheta(\omega)| \leqslant \delta$, $\forall \omega \in \mathcal{J}$;
(2) $(\sigma_1^c D^\eta + \sigma_2^c D^\xi)(D + \sigma_3)\rho_1(\omega) = \zeta_1(\omega, \rho_2(\omega)) + I^\gamma \zeta_2(\omega, \rho_1(\omega)) + \vartheta(\omega)$, $\omega \in \mathcal{J}$.

Theorem 5.4. *If* $\rho_1 \in \Theta$ *is solution of inequality* (5.1), *then* ρ_1 *will be the solution of the following inequality:*

$$|\rho_1(\omega) - \rho_0(\omega)| \leqslant \delta D,$$

where

$$D = \frac{2}{\sigma_1}\left(\frac{1 - e^{-\sigma_3 \Omega}}{\sigma_3}\right)\frac{\Omega^\eta}{\Gamma(\eta+1)} + \frac{\sigma_3}{\sigma_1}\left[\frac{\Omega^\eta}{\Gamma(\eta+1)} - \frac{\Omega^{\eta-1}}{\Gamma(\eta)}\right]\left(\frac{\omega}{\sigma_3} - \frac{1}{\sigma_3^2} + \frac{e^{-\sigma_3 \omega}}{\sigma_3^2}\right).$$

Proof. If ρ_1 is the solution of inequality (5.1), then ρ_1 will also be the solution of

$$\begin{cases}(\sigma_1^c D^\eta + \sigma_2^c D^\xi)(D+\sigma_3)\rho_1(\omega) = \zeta_1(\omega,\rho_1(\omega)) + I^{\eta+\nu}\zeta_2(\omega,\rho_1(\omega)) + \vartheta(\omega),\\ \rho_1(0)=\rho_1'(0)=\rho_1''(0)=0,\end{cases}$$

i.e.,

$$\begin{aligned}\rho_1(\omega) = {} & \frac{1}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)} I^\eta(\zeta_1(s,\rho_1(s))+\vartheta(s))ds + \frac{1}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)} I^{\eta+\nu}\zeta_2(s,\rho_1(s))ds\\ & - \frac{\mu_2}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)} I^{\eta-\xi-1}\rho_1(s)ds - \frac{\sigma_3.\sigma_2}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)} I^{\eta-\xi}\rho_1(s)ds\\ & - \frac{1}{\sigma_1}I^\eta(\zeta_1(0,0)+\vartheta(0))\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right) - \frac{1}{\sigma_1}I^{\eta+\nu}\zeta_2(0,0)\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right)\\ & - \frac{\sigma_3}{\sigma_1}I^\eta(\zeta_1(0,0)+\vartheta(0))\left(\frac{\omega}{\sigma_3}-\frac{1}{\sigma_2}+\frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right) - \frac{\sigma_3}{\sigma_1}I^{\eta+\nu}\zeta_2(0,0)\left(\frac{\omega}{\sigma_3}-\frac{1}{\sigma_2}+\frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right)\\ & - \frac{1}{\sigma_1}I^{\eta-1}(\zeta_1(0,0)+\vartheta(0))\left(\frac{\omega}{\sigma_3}-\frac{1}{\sigma_2}+\frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right) - \frac{1}{\sigma_1}I^{\eta+\nu-1}\zeta_2(0,0)\left(\frac{\omega}{\sigma_3}-\frac{1}{\sigma_2}+\frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right).\end{aligned}$$

For convenience, let $\rho_0(\omega)$ denotes those terms of $\rho_1(\omega)$ free from ϑ, i.e.,

$$\begin{aligned}\rho_0(\omega) = {} & \frac{1}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)} I^\eta\zeta_1(s,\rho_1(s))ds + \frac{1}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)} I^{\eta+\nu}\zeta_2(s,\rho_1(s))ds\\ & - \frac{\sigma_2}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)} I^{\eta-\xi-1}\rho_1(s)ds - \frac{\sigma_3.\sigma_2}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)} I^{\eta-\xi}\rho_1(s)ds\\ & - \frac{1}{\sigma_1}I^\eta\zeta_1(0,0)\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right) - \frac{1}{\sigma_1}I^{\eta+\nu}\zeta_2(0,0)\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right)\\ & - \frac{\sigma_3}{\sigma_1}I^\eta\zeta_1(0,0)\left(\frac{\omega}{\sigma_3}-\frac{1}{\sigma_2}+\frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right) - \frac{\sigma_3}{\sigma_1}I^{\eta+\nu}\zeta_2(0,0)\left(\frac{\omega}{\sigma_3}-\frac{1}{\sigma_2}+\frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right)\\ & - \frac{1}{\sigma_1}I^{\eta-1}\zeta_1(0,0)\left(\frac{\omega}{\sigma_3}-\frac{1}{\sigma_2}+\frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right) - \frac{1}{\sigma_1}I^{\eta+\nu-1}\zeta_2(0,0)\left(\frac{\omega}{\sigma_3}-\frac{1}{\sigma_2}+\frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right).\end{aligned}$$

So we can write

$$\begin{aligned}|\rho_1(\omega)-\rho_0(\omega)| \leqslant {} & |\frac{1}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)} I^\eta\vartheta(s)ds + \frac{1}{\sigma_1}I^\eta\vartheta(0))\left(\frac{1-e^{-\sigma_3\omega}}{\sigma_3}\right)\\ & + \frac{\sigma_3}{\sigma_1}I^\eta\vartheta(0)\left(\frac{\omega}{\sigma_3}-\frac{1}{\sigma_2}+\frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right) + \frac{1}{\sigma_1}I^{\eta-1}\vartheta(0)\left(\frac{\omega}{\sigma_3}-\frac{1}{\sigma_2}+\frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right)|\\ \leqslant {} & \delta.\frac{2}{\sigma_1}\left(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\right)\frac{\Omega^\eta}{\Gamma(\eta+1)} + \frac{\sigma_3}{\sigma_1}\left[\frac{\Omega^\eta}{\Gamma(\eta+1)}-\frac{\Omega^{\eta-1}}{\Gamma(\eta)}\right]\left(\frac{\omega}{\sigma_3}-\frac{1}{\sigma_3^2}+\frac{e^{-\sigma_3\omega}}{\sigma_3^2}\right) = \delta D.\end{aligned}$$

□

Theorem 5.5. *Let the conditions* $(\bar{G}1)$-$(\bar{G}2)$ *are satisfied, then the the system is Hyers-Ulam stable.*

Proof. Consider $\rho_1 \in C^1$ be the solution of inequality (5.1),

$$|(\sigma_1^c D^\eta + \sigma_2^c D^\xi)(D+\sigma_3)\rho_1(\omega) - \zeta_1(\omega,\rho_1(\omega)) + I^\nu\zeta_2(\omega,\rho_1(\omega))| \leqslant \delta,\ \omega\in\mathcal{J},$$

and let $\rho_2 \in C^1$ is the exact solution of the problem (3.1),

$$\begin{cases}(\sigma_1^c D^\eta + \sigma_2^c D^\xi)(D+\sigma_3)\rho_2(\omega) = \zeta_1(\omega,\rho_2(\omega)) + I^\nu\zeta_2(\omega,\rho_1(\omega)),\\ \rho_2(0)=\rho_2'(0)=\rho_2''(0)=0.\end{cases}$$

Let

$$|\rho_1(\omega)-\rho_2(\omega)|=|\rho_1(\omega)-\rho_0(\omega)+\rho_0(\omega)-\rho_2(\omega)|.$$

Using Theorem 5.4, we have

$$\begin{aligned}
|\rho_1(\omega)-\rho_2(\omega)| &\leqslant \delta.D+|\rho_0(\omega)-\rho_2(\omega)|\\
&\leqslant \delta.D+\frac{1}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta}|\zeta_1(s,\rho_1(s))-\zeta_1(s,\rho_2(s))|ds\\
&\quad+\frac{1}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta+\nu}|\zeta_2(s,\rho_1(s))-\zeta_2(s,\rho_2(s))|ds\\
&\quad+\frac{\sigma_2}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta-\xi-1}|\rho_1(s)-\rho_2(s)|ds+\frac{\sigma_3.\sigma_2}{\mu_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta-\xi}|\rho_1(s)-\rho_2(s)|ds\\
&\leqslant \delta.D+\frac{A_1^*}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta}|\rho_1(s)-\rho_2(s)|ds+\frac{A_2^*}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta+\nu}|\rho_1(s)-\rho_2(s)|ds\\
&\quad+\frac{\sigma_2}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta-\xi-1}|\rho_1(s)-\rho_2(s)|ds+\frac{\sigma_3.\sigma_2}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta-\xi}|\rho_1(s)-\rho_2(s)|ds\\
&\leqslant \delta.D+\frac{(A^*)}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta}|\rho_1(s)-\rho_2(s)|ds+\frac{(A^*)}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta+\nu}|\rho_1(s)-\rho_2(s)|ds\\
&\quad+\frac{\sigma_2}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta-\xi-1}|\rho_1(s)-\rho_2(s)|ds+\frac{\sigma_3.\sigma_2}{\sigma_1}\int_0^\omega e^{-\sigma_3(\omega-s)}I^{\eta-\xi}|\rho_1(s)-\rho_2(s)|ds,
\end{aligned}$$

$$\|\rho_1-\rho_2\|_\Theta\leqslant \delta.D+(A^*\theta_1+\theta_2)\|\rho_1-\rho_2\|_\Theta,\quad \|\rho_1-\rho_2\|_\Theta\leqslant\frac{\delta.D}{1-(A^*\theta_1+\theta_2)},\quad \|\rho_1-\rho_2\|_\Theta\leqslant\delta\breve{n},$$

where

$$\breve{n}=\frac{D}{1-(A^*\theta_1+\theta_2)}.$$

Consequently, the problem (3.1) is Hyers-Ulam stable. If we put $X(\delta)=\delta.\breve{n}$, $X(0)=0$ suggests the generalized Hyers-Ulam stability of the problem (3.1). □

Now for the Hyers-Ulam stability of the coupled system, we state the following.

Definition 5.6. The problem (1.2) is Hyers-Ulam stable if we can find a real number $\breve{n}>0$ such that for every $\epsilon\geqslant 0$ and for every solution $(\bar\rho,\bar\Upsilon)\in X\times Y$ of the inequality,

$$\begin{cases}|(\wp_1^cD^{\eta_1}+\wp_2^cD^{\xi_1})(D+\wp_3)\bar\rho(\omega)-\zeta_1(\omega,\bar\rho(\omega)),\bar\Upsilon(\omega))-I^{w_1}\zeta_2(\omega,\bar\rho(\omega)),\bar\Upsilon(\omega))|\leqslant\bar\delta_1,\\ |(\wp_4^cD^{\eta_1}+\wp_5^cD^{\xi_1})(D+\wp_6)\bar\Upsilon(\omega)-\zeta_3(\omega,\bar\rho(\omega)),\bar\Upsilon(\omega))-I^{w_2}\zeta_4(\omega,\bar\rho(\omega)),\bar\Upsilon(\omega))|\leqslant\bar\delta_2.\end{cases}\tag{5.3}$$

Then, there exists a solution (ρ,Υ) of (1.2) with

$$\|(\rho,\Upsilon)-(\bar\rho,\bar\Upsilon)\|\leqslant\breve{n}\bar\delta,\omega\in\mathfrak{I}.$$

Remark 5.7. $(\bar\rho,\bar\Upsilon)$ is a solution of of the inequality (5.3) if and only if there exists $\gamma_1(\omega)$ and $\gamma_2(\omega)$ such that

(1) $|\gamma_1(\omega)|\leqslant\bar\delta_1, |\gamma_2(\omega)|\leqslant\bar\delta_2, \forall\omega\in I$; and

(2)
$$\begin{cases}(\wp_1^cD^{\eta_1}+\wp_2^cD^{\xi_1})(D+\wp_3)\bar\rho(\omega)=\zeta_1(\omega,\bar\rho(\omega)),\bar\Upsilon(\omega))+I^{w_1}\zeta_2(\omega,\bar\rho(\omega)),\bar\Upsilon(\omega))+\gamma_1(\omega),\\ (\wp_4^cD^{\eta_1}+\wp_5^cD^{\xi_1})(D+\lambda_6)\bar\Upsilon(\omega)=\zeta_3(\omega,\bar\rho(\omega)),\bar\Upsilon(\omega))+I^{w_2}\zeta_4(\omega,\bar\rho(\omega)),\bar\Upsilon(\omega))+\gamma_2(\omega).\end{cases}$$

Theorem 5.8. *The coupled problem* (1.2) *is Hyers-Ulam stable, if the conditions* (H_1^*) -(H_2^*) *are satisfied.*

Proof. Let $(\bar{\rho}, \bar{\Upsilon}) \in X \times Y$ be the solution of the inequality

$$\begin{cases} |(\wp_1^c D^{\eta_1} + \wp_2^c D^{\xi_1})(D + \wp_3)\bar{\rho}(\omega) - \zeta_1(\omega, \bar{\rho}(\omega)), \bar{\Upsilon}(\omega)) - I^{w_1}\zeta_2(\omega, \bar{\rho}(\omega)), \bar{\Upsilon}(\omega))| \leqslant \bar{\delta}_1, \\ |(\wp_4^c D^{\eta_1} + \wp_5^c D^{\xi_1})(D + \wp_6)\bar{\Upsilon}(\omega) - \zeta_3(\omega, \bar{\rho}(\omega)), \bar{\Upsilon}(\omega)) - I^{w_2}\zeta_4(\omega, \bar{\rho}(\omega)), \bar{\Upsilon}(\omega))| \leqslant \bar{\delta}_2. \end{cases}$$

From the above inequality, using Remark 5.7, we can write

$$\begin{cases} (\wp_1^c D^{\eta_1} + \wp_2^c D^{\xi_1})(D + \lambda_3^*)\bar{\rho}(\omega) = \zeta_1(\omega, \bar{\rho}(\omega)), \bar{\Upsilon}(\omega)) + I^{w_1}\zeta_2(\omega, \bar{\rho}(\omega)), \bar{\Upsilon}(\omega)) + \gamma_1(\omega), \\ (\wp_4^c D^{\eta_1} + \wp_5^c D^{\xi_1})(D + \wp_6)\bar{\Upsilon}(\omega) = \zeta_3(\omega, \bar{\rho}(\omega)), \bar{\Upsilon}(\omega)) + I^{w_2}\zeta_4(\omega, \bar{\rho}(\omega)), \bar{\Upsilon}(\omega)) + \gamma_2(\omega). \end{cases}$$

Using Lemma 2.10, we have

$$\begin{aligned} \bar{\rho}(\omega) = {} & \frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1}(\zeta_1(\omega, \bar{\rho}(\mathbf{s})), \bar{\Upsilon}(\mathbf{s})) + I^{\eta_1+w_1}\zeta_2(\omega, \bar{\rho}(\mathbf{s})), \bar{\Upsilon}(\mathbf{s})) + \gamma_1(\omega))ds \\ & - \frac{\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1-\xi-1}\bar{\rho}(s)ds - \frac{\wp_3\cdot\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1-\xi_1}\bar{\rho}(s)ds \\ & - \frac{1}{\wp_1}\left[I^{\eta_1}\zeta_1(0,0,0) + I^{\eta_1+w_1}\zeta_2(0,0,0) + \gamma_1(\omega)\right]\left(\frac{1-e^{-\wp_3\omega}}{\wp_3}\right) \\ & - \left\{\frac{\wp_3}{\wp_1}\left[I^{\eta_1}\zeta_1(0,0,0) + I^{\eta_1+w_1}\zeta_2(0,0,0) + \gamma_1(\omega)\right]\right. \\ & \left. + \frac{1}{\wp_1}\left[I^{\eta_1-1}\zeta_1(0,0,0) + I^{\eta_1+w_1-1}\zeta_2(0,0,0) + \gamma_1(\omega)\right]\right\}\left(\frac{\omega}{\wp_3} - \frac{1}{\wp_3^2} + \frac{e^{-\wp_3\omega}}{\wp_3^2}\right), \\ \bar{\Upsilon}(\omega) = {} & \frac{1}{\wp_4}\int_0^\omega e^{-\wp_6(\omega-s)} I^{\eta_2}(\zeta_3(\omega, \bar{\rho}(\mathbf{s})), \bar{\Upsilon}(\mathbf{s})) + I^{\eta_2+w_2}\zeta_4(\omega, \bar{\rho}(\mathbf{s})), \bar{\Upsilon}(\mathbf{s})) + \gamma_2(\omega))ds \\ & - \frac{\wp_5}{\wp_4}\int_0^\omega e^{-\wp_6(\omega-s)} I^{\eta_2-\xi_2-1}\bar{\Upsilon}(s)ds - \frac{\wp_6\cdot\wp_5}{\wp_4}\int_0^\omega e^{-\wp_6(\omega-s)} I^{\eta_2-\xi_2}\bar{\Upsilon}(s)ds \\ & - \frac{1}{\wp_4}\left[I^{\eta_2}\zeta_3(0,0,0) + I^{\eta_2+w_2}\zeta_4(0,0,0) + \gamma_2(\omega))\left(\frac{1-e^{-\wp_6\omega}}{\wp_6}\right)\right. \\ & - \frac{1}{\wp_4}\left[I^{\eta_2}\zeta_3(0,0,0) + I^{\eta_2+w_2}\zeta_4(0,0,0) + \gamma_2(\omega)\right]\left(\frac{1-e^{-\wp_6\omega}}{\wp_6}\right) \\ & - \left\{\frac{\wp_6}{\wp_1}\left[I^{\eta_2}\zeta_3(0,0,0) + I^{\eta_2+w_2}\zeta_4(0,0,0) + \gamma_2(\omega)\right]\right. \\ & \left.\left. + \frac{1}{\wp_4}\left[I^{\eta_2-1}\zeta_3(0,0,0) + I^{\eta_2+w_2-1}\zeta_4(0,0,0) + \gamma_2(\omega)\right]\right\}\right]\left(\frac{\omega}{\wp_6} - \frac{1}{\wp_6^2} + \frac{e^{-\wp_6\omega}}{\wp_6^2}\right). \end{aligned}$$

Let (ρ, Υ) be the solution of (1.2), then

$$\begin{aligned} \|\rho - \bar{\rho}\| \leqslant {} & \sup_{(\omega)\in I}\frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1}|\zeta_1(s, \rho(s), \Upsilon(s)) - \zeta_1(s, \bar{\rho}(s), \bar{\Upsilon}(s))|ds \\ & + \frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1+w_1}|\zeta_2(s, \rho(s), \Upsilon(s)) - \zeta_2(s, \bar{\rho}(s), \bar{\Upsilon}(s))|ds \\ & - \frac{\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1-\xi-1}\rho(s) - \bar{\rho}(s)ds + \frac{\wp_3\cdot\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1-\xi_1}\rho(s) - \bar{\rho}(s)ds| + \bar{\delta}_1\bar{W}_1 \\ \leqslant {} & \sup_{(\omega)\in I}\frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1}|\Big(B_1^*\|\rho - \bar{\rho}\| + B_2^*\|\Upsilon - \bar{\Upsilon}\|\Big)|ds \\ & + \frac{1}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1}|\Big(B_3^*\|\rho - \bar{\rho}\| + B_4^*\|\Upsilon - \bar{\Upsilon}\|\Big)|ds \\ & - \frac{\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1-\xi-1}|\rho(s) + \bar{\rho}(s)|ds + \frac{\wp_3\cdot\wp_2}{\wp_1}\int_0^\omega e^{-\wp_3(\omega-s)} I^{\eta_1-\xi_1}|\rho(s) - \bar{\rho}(s)|ds + \bar{\delta}_1\bar{W}_1 \end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{B^*}{\wp_1}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_1}}{\Gamma(\eta_1+1)}+\frac{\Omega^{\eta_1+w_1}}{\Gamma(\eta_1+w_1+1)}\right]\Big(\|\rho-\bar\rho\|+\|\Upsilon-\bar\Upsilon\|\Big)\\
&\quad+\frac{\wp_2}{\wp_1}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_1-\xi_1-1}}{\Gamma(\eta_1-\xi_1)}+\wp_3\frac{\Omega^{\eta_1-\xi_1}}{\Gamma(\eta_1-\xi_1+1)}\right]\|\rho-\bar\rho\|+\bar\delta_1\bar W_1\\
&\leqslant B^*\theta_1\Big(\|\rho-\bar\rho\|+\|\Upsilon-\bar\Upsilon\|\Big)+\theta_2\|\rho-\bar\rho\|+\bar\delta_1\bar W_1\\
&\leqslant (B^*\theta_1+\theta_2)\Big(\|\rho-\bar\rho\|+\|\Upsilon-\bar\Upsilon\|\Big)+\bar\delta_1\bar W_1\leqslant S_1^*\Big(\|\rho-\bar\rho\|+\|\Upsilon-\bar\Upsilon\|\Big)+\bar\delta_1\bar W_1,
\end{aligned}$$

$$\|\rho-\bar\rho\|\leqslant S_1^*\Big(\|\rho-\bar\rho\|+\|\Upsilon-\bar\Upsilon\|\Big)+\bar\delta_1 W_1.$$

Similarly

$$\|\Upsilon-\bar\Upsilon\|\leqslant S_2^*\left(\|\rho-\bar\rho\|+\|\Upsilon-\bar\Upsilon\|\right)+\bar\delta_2\bar W_2.$$

Let $\max\{\bar\delta_1,\bar\delta_2\}=\bar\delta$, $\max\{\bar W_1,\bar W_2\}=\bar W$, and $\max\{S_1*,S_2*\}=S*$, then

$$\|\rho-\bar\rho\|\leqslant S^*\left(\|\rho-\bar\rho\|+\|\Upsilon-\bar\Upsilon\|\right)+\bar\delta\bar W.$$

Similarly

$$\|\Upsilon-\bar\Upsilon\|\leqslant S^*\left(\|\rho-\bar\rho\|+\|\Upsilon-\bar\Upsilon\|\right)+\bar\delta\bar W,$$

which implies that

$$\|\rho-\bar\rho\|+\|\Upsilon-\bar\Upsilon\|\leqslant S^*\left(\|\rho-\bar\rho\|+\|\Upsilon-\bar\Upsilon\|\right)+\bar\delta\bar W,$$

which implies that

$$\|\rho-\bar\rho\|+\|\Upsilon-\bar\Upsilon\|\leqslant\frac{\bar\delta\bar W}{1-S^*},\quad \|\rho-\bar\rho\|+\|\Upsilon-\bar\Upsilon\|\leqslant\bar\delta\breve{n},$$

where $\breve{n}=\frac{\bar W}{1-S^*}$. Consequently, the coupled system is Hyers-Ulam stable. By putting $\bar\delta=0$ in $X(\bar\delta)=\bar\delta.\breve{n}$, i.e., $X(0)=0$, we suggest the generalized Hyers-Ulam stability of the problem (1.2). □

6. Examples

To demonstrate the previously obtained results, let's look at some examples.

Example 6.1. Consider the problem:

$$\begin{cases}(10^cD^{1.25}+3^cD^{1.15})(D+1)\rho(\omega)=\zeta_1(\omega,\rho(\omega))+I^{0.75}\zeta_2(\omega,\rho(\omega)),\ \omega\in[0,1],\\ \rho(0)=\rho'(0)=\rho''(0)=0,\end{cases}\tag{6.1}$$

where $\eta=1.25,\xi=1.15,\nu=0.75,\sigma_3=1,\sigma_1=10,\sigma_2=3$, and

$$\zeta_1(\omega,\rho(\omega))=\left(\frac{e^{-2\omega}}{(\omega+5)^2}\right)\left(\frac{|\rho(\omega)|}{1+|\rho(\omega)|}\right)+\frac{1}{2},\quad \zeta_2(\omega,\rho(\omega))=\frac{\sin\omega}{\sqrt{49+\omega}}.\tan^{-1}\rho+\frac{1}{7}.$$

By calculation, we found that $\varphi_1=0.0874$, $\varphi_2=0.3066$. Also

$$|\zeta_1(\omega,\rho_1(\omega))-\zeta_1(\omega,\rho_2(\omega))|\leqslant\frac{e^{-2\omega}}{(\omega+5)^2}|\rho_1-\rho_2|\leqslant\frac{1}{25}|\rho_1-\rho_2|,$$

$$|\zeta_2(\omega,\rho_1(\omega))-\zeta_2(\omega,\rho_2(\omega))|\leqslant \frac{\sin\omega}{\sqrt{(49+\omega)}}|\rho_1-\rho_2|\leqslant \frac{1}{7}|\rho_1-\rho_2|,$$

we have $A_1^*=\frac{1}{25}$, $A_2^*=\frac{1}{7}$, and $A^*=\max\{A_1^*,A_2^*\}=\frac{1}{7}$. Moreover

$$\begin{aligned}&A^*\frac{1}{\sigma_1}\left(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\right)\left[\frac{\Omega^\eta}{\Gamma(\eta+1)}+\frac{\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)}\right]\\&\quad+\frac{\sigma_2}{\sigma_1}\left(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\right)\left[\frac{\Omega^{\eta-\xi-1}}{\Gamma(\eta-\xi)}+\sigma_3\frac{\Omega^{\eta-\xi}}{\Gamma(\eta-\xi+1)}\right]=A^*\varphi_1+\varphi_2\approx 0.3100<1.\end{aligned}$$

The problem (6.1) has a unique mild solution and is Hyers-Ulam stable because all the requirements of Theorems 3.1 and 5.5 are fulfilled.

Example 6.2. Consider the problem:

$$\begin{cases}5{}^cD^{1.75}(D+1)\rho(\omega)-{}^cD^{1.60}(D+1)\rho(\omega)=\zeta_1(\omega,\rho(\omega))+I^{\frac{1}{2}}\zeta_2(\omega,\rho(\omega)),\\ \rho(0)=\rho'(0)=\rho''(0)=0,\end{cases}\tag{6.2}$$

where $\omega\in[0,5]$, $\eta=1.75$, $\xi=1.60$, $\nu=\frac{1}{2}$, $\sigma_3=1$, $\sigma_1=5$, $\sigma_2=1$, also

$$\zeta_1(\omega,\rho(\omega))=\frac{\cos\omega}{(2\omega+7)}(\sin\rho+\tan^{-1}\rho),\quad \zeta_2(\omega,\rho(\omega))=\frac{\tan^{-1}\omega}{15}\left(\frac{|\rho(\omega)|}{(1+\rho(\omega))}+\cot^{-1}\rho\right)+e^{-5\omega}.$$

By calculation, we found that $\theta_2\approx 0.27915<1$, $\|\varpi_1\|=\frac{(46+9\pi)}{126}$, and $|\varpi_2\|=\frac{(\pi^2+\pi+30)}{30}$, implying that the problem (6.2) has a mild solution, since the conditions of Theorem 3.2 hold.

Example 6.3. Consider the problem:

$$\begin{cases}7{}^cD^{1.95}(D+1)\rho(\omega)+{}^cD^{1.55}(D+1)\zeta_3(\omega,\rho(\omega))=\zeta_1(\omega,\rho(\omega))+I^{3/2}\zeta_2(\omega,\rho(\omega)),\\ \rho(0)=\rho'(0)=\rho''(0)=0,\end{cases}\tag{6.3}$$

where $\omega\in[0,2]$, $\eta=1.95$, $\xi=1.55$, $\nu=3/2$, $\sigma_3=1$, $\sigma_1=7$, $\sigma_2=1$, $\zeta_1(\omega,\rho(\omega))=\frac{1}{(\omega^2+2)}\tan^{-1}\rho+\frac{1}{4}$, also

$$\zeta_2(\omega,\rho(\omega))=\frac{e^{-2\omega}|\rho|}{(4+\omega)(1+|\rho|)}+\frac{1}{3},\quad \zeta_3(\omega,\rho(\omega))=\frac{e^{-3\omega}(\sin\rho+\rho(\omega))}{\sqrt{(\omega^4+36)}}+\frac{1}{\sqrt{(\omega^2+25)}}.$$

With the given data, it is found that $A_1^*=\frac{1}{2}$, $A_2^*=\frac{1}{4}$, $A_3^*=\frac{1}{3}$, $A^*=\max\{A_1^*,A_2^*,A_3^*\}$, and

$$\begin{aligned}&A^*\left(\frac{1}{\sigma_1}\left(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\right)\left[\frac{\Omega^\eta}{\Gamma(\eta+1)}+\frac{\Omega^{\eta+\nu}}{\Gamma(\eta+\nu+1)}\right]\right.\\&\quad\left.+\frac{\sigma_2}{\sigma_1}\left(\frac{1-e^{-\sigma_3\Omega}}{\sigma_3}\right)\left[\frac{\Omega^{\eta-\xi-1}}{\Gamma(\eta-\xi)}+\sigma_3\frac{\Omega^{\eta-\xi}}{\Gamma(\eta-\xi+1)}\right]\right)=A^*(\varphi_1+\varphi_2)\approx 0.5598<1.\end{aligned}$$

The problem (6.3) has a mild solution, since all the requirements of Theorem 4.1 are fulfilled.

Example 6.4. Consider the coupled problem:

$$\begin{cases}(3{}^cD^{1.5}+2{}^cD^{1.25})(D-2)\rho(\omega)=\zeta_1(\omega,\rho(\omega)),\Upsilon(\omega))+I^{1.3}\zeta_2(\omega,\rho(\omega)),\Upsilon(\omega)),\\ (5{}^cD^{1.6}+3{}^cD^{1.35})(D-3)\Upsilon(\omega)=\zeta_3(\omega,\rho(\omega),\Upsilon(\omega))+I^{3/2}\zeta_4(\omega,\rho(\omega)),\Upsilon(\omega)),\\ \rho(0)=\rho_1'(0)=\rho_1''(0)=0,\\ \Upsilon(0)=\Upsilon'(0)=\Upsilon''(0)=0,\end{cases}\tag{6.4}$$

where $\omega \in [0,1]$, $\zeta_1(\omega,\rho(\omega)),\Upsilon(\omega)) = \sin(\rho+\Upsilon)/15$, $\zeta_2(\omega,\rho(\omega)),\Upsilon(\omega)) = \rho+\Upsilon$, $\zeta_3(\omega,\rho(\omega)),\Upsilon(\omega)) = \cos(\rho+\Upsilon)/5$, $\zeta_4(\omega,\rho(\omega)),\Upsilon(\omega)) = (\rho+\Upsilon)/6$, $\wp_1 = 3$, $\wp_2 = 2$, $\wp_3 = -2$, $\wp_4 = 5$, $\wp_5 = 3$, $\wp_6 = -3$, $B_1^* = B_2^* = 1/15$, $B_3^* = B_4^* = 1$, $B_5^* = B_6^* = 1/5$, $B_7^* = B_8^* = 1/6$,

$$G_1* = B^* \frac{1}{\bar{\wp_1}}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_1}}{\Gamma(\eta_1+1)} + \frac{\Omega^{\eta_1+w_1}}{\Gamma(\eta_1+w_1+1)}\right] + \frac{\wp_2}{\bar{\wp_1}}\left(\frac{1-e^{-\wp_3\Omega}}{\wp_3}\right)\left[\frac{\Omega^{\eta_1-\xi_1-1}}{\Gamma(\eta_1-\xi_1)} + \wp_3\frac{\Omega^{\eta_1-\xi_1}}{\Gamma(\eta_1-\xi_1+1)}\right],$$

by calculating $G_1* = 0.4373$,

$$G_2* = B^* \frac{1}{\wp_4}\left(\frac{1-e^{-\wp_6\Omega}}{\wp_6}\right)\left[\frac{\Omega^{\eta_2}}{\Gamma(\eta_2+1)} + \frac{\Omega^{\eta_2+w_2}}{\Gamma(\eta_2+w_2+1)}\right] + \frac{\wp_5}{\wp_4}\left(\frac{1-e^{-\wp_6\Omega}}{\wp_6}\right)\left[\frac{\Omega^{\eta_2-\xi_2-1}}{\Gamma(\eta_2-\xi_2)} + \wp_6\frac{\Omega^{\eta_2-\xi_2}}{\Gamma(\eta_2-\xi_2+1)}\right],$$

$G_2* = 0.1904$, $\max\{G_1*, G_2*\} = G_2* = 0.4373 < 1$. Because of this, problem (6.4) has a unique mild solution and is Hyers-Ulam stable, satisfying the requirements of Theorems 3.3 and 5.8.

Example 6.5. Consider the coupled problem:

$$\begin{cases} 2^cD^{1.9}(D-1)\rho(\omega) + 3^cD^{1.45}(D-1)H_1(\omega,\rho(\omega)),\Upsilon(\omega)) \\ \quad = \zeta_1(\omega,\rho(\omega),\Upsilon(\omega)) + I^{0.45}\zeta_2(\omega,\rho(\omega)),\Upsilon(\omega)), \\ 2^cD^{1.65}(D-4)\Upsilon(\omega) + 2^cD^{1.3}(D-4)H_2(\omega,\rho(\omega)),\Upsilon(\omega)) \\ \quad = \zeta_3(\omega,\rho(\omega),\Upsilon(\omega)) + I^{1.1}\zeta_4(\omega),\rho(\omega)),\Upsilon(\omega)), \\ \rho(0) = \rho_1'(0) = \rho_1''(0) = 0, \\ \Upsilon(0) = \Upsilon'(0) = \Upsilon''(0) = 0, \end{cases} \tag{6.5}$$

where $\omega \in [0,1]$, $\zeta_1(\omega,\rho(\omega)),\Upsilon(\omega)) = \sin(\rho+\Upsilon)/15$, $\zeta_2(\omega,\rho(\omega)),\Upsilon(\omega)) = \rho+\Upsilon/9$, $\zeta(\omega,\rho(\omega)),\Upsilon(\omega)) = \cos(\rho+\Upsilon)/3$, $\zeta_4(\omega,\rho(\omega)),\Upsilon(\omega)) = (\rho+\Upsilon)/6$, $H_1(\omega,\rho(\omega)),\Upsilon(\omega)) = \rho-\Upsilon/2$, $H_2(\omega,\rho(\omega)),\Upsilon(\omega)) = \rho-\Upsilon/6$ $\bar{\lambda_1} = 2$, $\wp_2 = 3$, $\wp_3 = -1$, $\wp_4 = 2$, $\wp_5 = 2$, $\wp_6 = -4$, $B_1^* = B_2^* = 1/15$, $B_3^* = B_4^* = 1/9$, $B_5^* = B_6^* = 1/3$, $B_7^* = B_8^* = 1/6$, $B_9^* = B_{10}^* = 1/2$, $B_{11}^* = B_{12}^* = 1/6$, by calculating $S_1* = B^*(\theta_1+\theta_2) = 0.44647$, $S_2* = B^*(\phi_1+\phi_2) = 0.63739$, $\max\{G_1*, G_2*\} = G_2* = 0.63739 < 1$. Consequently, problem (6.5) has a unique mild solution on the interval $[0,1]$, since the criteria of Theorem 4.2 is satisfied.

7. Conclusion

In this work, we established criteria for the existence, uniqueness, and various forms of Hyers-Ulam stability of mild solutions for problems (3.1) and (1.2) using fixed point methods. Banach's fixed point theorem confirmed the uniqueness of mild solutions, while Krasnoselskii's and Schaefer's theorems ensured the existence of at least one solution for problems (3.1) and (1.2), respectively. We also analyzed cases with mixed nonlinearities and demonstrated Hyers-Ulam and generalized Hyers-Ulam stability using Theorems 5.5 and 5.8. Illustrative examples were provided to validate our results. Future research could explore more complex integro-differential systems with varied fractional derivatives, boundary conditions, and perturbations, as well as develop numerical methods to approximate solutions and extend stability analysis to other forms of Ulam stability and variable order fractional derivatives, broadening the scope and applicability of these findings.

Authors' contributions

All authors contributed equally and significantly to this paper. All authors have read and approved the final version of the manuscript.

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