J. Math. Computer Sci., 38 (2025), 80–97

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

New common fixed point theorems for quartet mappings on orthogonal $\mathcal{S}$-metric spaces with applications

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Abstract

In this article, we extend the scope of fixed point theory by proving a common fixed point theorem applicable to quartet mappings defined on orthogonal $\mathcal{S}$-metric spaces. Our theorems establish conditions under which the quartet mappings $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are orthogonal preserving, orthogonal continuous, and pairwise compatible mappings, possess a unique common fixed point. To elucidate the practical implications of our theoretical result, we present a concrete example illustrating its application. Finally, we demonstrate the versatility of our theorem by applying it to establish the existence and uniqueness of solutions for Volterra-type integral system, production-consumption equilibrium and fractional differential equations.

Keywords: Compatible mappings, $\mathcal{S}$-metric space, orthogonal metric spaces, orthogonal $\mathcal{S}$-metric space, common fixed point.

2020 MSC: 47H10, 45D05.

1. Introduction

Fixed point theorems are fundamental results in mathematics, particularly in the study of functional analysis and metric spaces. These theorems provide conditions under which mappings from a set to itself must have at least one point that remains unchanged after the mapping operation. Among these theorems, common fixed point theorems are particularly interesting as they establish conditions under which multiple mappings share a fixed point. Orthogonal S-metric spaces are a generalization of metric spaces that incorporate the notions of orthogonality and symmetry. These spaces provide a rich framework for

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doi: 10.22436/jmcs.038.01.06

Received: 2024-06-29 *Revised:* 2024-09-02 *Accepted:* 2024-09-17

studying mappings and their fixed points in a broader setting, allowing for the exploration of complex relationships between multiple mappings.

In 1922, Banach [5] introduced Banach fixed point theorem to establish that integral equations and nonlinear operator equations possess solutions. Four maps in metric spaces have a common fixed point (CFP) defined in 2007 by Sedghi et al. [36]. In 2012, Sedghi et al. [37], established the notion of $\mathcal{S}$-metric spaces (SMS). Gupta [15], proved the generalized fixed point (FP) theorem for cyclic contraction in SMS. Chouhan [7], proved an unique CFP theorem for expansive mappings in SMS. Sedghi and Dung [34], established a general FP theorem in SMS. Sedghi et al. [33], proved the CFP theorem for multivalued maps on complete SMS. Sedghi et al. [38], introduced more FP theorems in SMS. Double contractive mappings, which are expansions of $(\alpha - \psi)$-contractive mappings in SMS, were introduced by Mojaradi et al. [3]. Sedghi et al. [35], established a CFP theorem in $\mathcal{S}_b$-metric spaces. Van Dunga et al. [8], investigated some FP theorems for g-monotone mappings. Kim et al. [20], derived FP theorems for two maps on complete SMS. Further, integration-type contractive mapping of ordered SMS was first introduced by Gholidahneh et al. [11]. Moreover, on an SMS, Mlaiki et al. [28] used the set of simulation functions to prove new FP theorems. For weakly compatible mapping that satisfies a more generalized contractive condition, Tiwari et al. [43], demonstrated a CFP theorem. The concept of soft SMS was first developed and several significant properties were examined by Khandait et al. [18]. Adewale et al. [2], established the concept of rectangular SMS, which generalizes Branciari's rectangular metric spaces.

Gordji et al. [14], introduced the concepts of orthogonal set as well as orthogonal metric spaces (OMS) in 2017. They also demonstrated the FP theorem on contraction mappings in metric spaces. Further, they demonstrated that the first-order ordinary differential equation has a unique solution and that the Banach Contraction mapping is inapplicable to this particular situation. FP theorems in a generalized OMS were proved by Gordji and Habibi [10]. The notions of an orthogonal Kannan F-contractive type, an orthogonal F-expanding type & an orthogonal F-contractive type mappings were first presented by Gunaseelan et al. [24]. Joseph et al. [13], proved the FP theorem using orthogonal triangular α-admissibility on OMS. The notion of generalized orthogonal-Suzuki contraction mapping was developed by Ismat et al. [6]. Gunaseelan et al. [23] demonstrated the coupled FP theorem in OMS. Oliac et al. [9] developed FP theorems on orthogonal $\mathcal{S}$-metric spaces (OSMS). Sedghi et al. [39], introduced CFP theorem on OSMS. Later, many authors have proved unique solution for fractional differential and integral equations [1, 4, 12, 16, 17, 19, 21, 22, 25–27, 29–32, 40–42].

In this article, we prove the CFP theorem for four mappings on OSMS. The CFP theorem for four mappings on OSMS establishes conditions under which four mappings, defined on a common orthogonal S-metric space, have a single point that is simultaneously fixed under all four mappings. This theorem extends the classical Banach contraction principle to the setting of orthogonal S-metric spaces and provides a powerful tool for analyzing the behavior of mappings in these spaces. The result is illustrated by a suitable example. As a consequence, applications of the CFP theorem for four mappings on OSMS for the Volterra-type integral system of the second kind, the fractional differential equations and in production-consumption equilibrium is given.

2. Preliminaries

We start this part with some fundamental definitions, theorems, and examples, which will be used in the sequel.

Definition 2.1 ([37])**.** Let Σ be a non-void set. A function $\mathcal{S} : \Sigma^3 \to [0, \infty)$ is said to be an $\mathcal{S}$-metric on Σ, if for each $\vartheta, w, \eta, \mathfrak{a} \in \Sigma$, one has

1. $\mathcal{S}(\vartheta, w, \eta) \geqslant 0$;
2. $\mathcal{S}(\vartheta, w, \eta) = 0$ iff $\vartheta = w = \eta$;
3. $\mathcal{S}(\vartheta, w, \eta) \leqslant \mathcal{S}(\vartheta, \vartheta, \mathfrak{a}) + \mathcal{S}(w, w, \mathfrak{a}) + \mathcal{S}(\eta, \eta, \mathfrak{a})$.

Then, $(\Sigma, \mathcal{S})$ is called an SMS.

Example 2.2. Let Σ be a non-void set and $\mathfrak{d}_1, \mathfrak{d}_2$ be two usual metrics on Σ. Then

$$\mathcal{S}(\vartheta, w, \eta) = \mathfrak{d}_1(\vartheta, \eta) + \mathfrak{d}_2(w, \eta),$$

is an SMS on Σ.

Lemma 2.3 ([33]). *Let $(\Sigma, \mathcal{S})$ be an SMS. Then, we have $\mathcal{S}(\vartheta, \vartheta, w) = \mathcal{S}(w, w, \vartheta)$, $\forall \vartheta, w \in \Sigma$.*

Definition 2.4 ([34]). Let $(\Sigma, \mathcal{S})$ be an SMS.

1. A sequence $\{\vartheta_\varphi\}$ in Σ converges to ϑ if $\mathcal{S}(\vartheta_\varphi, \vartheta_\varphi, \vartheta) \to 0$ as $\varphi \to \infty$, i.e., $\forall \epsilon > 0$, $\exists \varphi_0 \in \mathbb{N}$ such that $\varphi \geqslant \varphi_0$, $\mathcal{S}(\vartheta_\varphi, \vartheta_\varphi, \vartheta) < \epsilon$.
2. A sequence $\{\vartheta_\varphi\}$ in Σ is called Cauchy, if $\forall \epsilon > 0$, $\exists \varphi_0 \in \mathbb{N}$, such that $\mathcal{S}(\vartheta_\varphi, \vartheta_\varphi, \vartheta_\ell) < \epsilon$, for each $\varphi, \ell \geqslant \varphi_0$.
3. The SMS $(\Sigma, \mathcal{S})$ is said to be complete, if every Cauchy sequence is convergent.

Lemma 2.5 ([34]). *Let $(\Sigma, \mathcal{S})$ be an SMS. If $\exists$ sequences $\{\vartheta_\varphi\}$ and $\{w_\varphi\}$ such that $\lim_{\varphi\to\infty} \vartheta_\varphi = \vartheta$, and $\lim_{\varphi\to\infty} w_\varphi = w$, then $\lim_{\varphi\to\infty} \mathcal{S}(\vartheta_\varphi, \vartheta_\varphi, w_\varphi) = \mathcal{S}(\vartheta, \vartheta, w)$.*

Definition 2.6 ([39]). Let $(\Sigma, \mathcal{S})$ be an SMS. A pair $\{\Phi, \Psi\}$ is said to be compatible iff

$$\lim_{\varphi\to\infty} \mathcal{S}(\Phi\Psi\vartheta_\varphi, \Phi\Psi\vartheta_\varphi, \Phi\Psi\vartheta_\varphi) = 0,$$

whenever $\{\vartheta_\varphi\}$ is a sequence in Σ, such that $\lim_{\varphi\to\infty} \Phi\vartheta_\varphi = \lim_{\varphi\to\infty} \Psi\vartheta_\varphi = \theta$, for some $\theta \in \Sigma$.

Example 2.7. Let $\Sigma = \mathbb{R}$, and define the $\mathcal{S}$-metric $\mathcal{S} : \mathbb{R}^3 \to \mathbb{R}^+$ by $\mathcal{S}(\vartheta, w, \eta) = |\vartheta - w| + |w - \eta| + |\eta - \vartheta|$, $\forall \vartheta, w, \eta \in \mathbb{R}$. Consider the mappings $\Phi, \Psi : \mathbb{R} \to \mathbb{R}$ are defined as $\Phi\vartheta = \frac{\vartheta}{2}$, $\Psi\vartheta = \frac{\vartheta}{3}$. To show that $\{\Phi, \Psi\}$ is compatible, consider a sequence $\{\vartheta_\varphi\}$ such that $\lim_{\varphi\to\infty} \vartheta_\varphi = \theta$, for some $\theta \in \mathbb{R}$, then

$$\lim_{\varphi\to\infty} \Phi\vartheta_\varphi = \lim_{\varphi\to\infty} \frac{\vartheta_\varphi}{2} = \frac{\theta}{2}, \quad \lim_{\varphi\to\infty} \Psi\vartheta_\varphi = \lim_{\varphi\to\infty} \frac{\vartheta_\varphi}{3} = \frac{\theta}{3}.$$

Since $\Phi\Psi\vartheta_\varphi = \frac{\vartheta_\varphi}{6}$, $\lim_{\varphi\to\infty} \Phi\Psi\vartheta_\varphi = \frac{\theta}{6}$. Now,

$$\lim_{\varphi\to\infty} \mathcal{S}(\Phi\Psi\vartheta_\varphi, \Phi\Psi\vartheta_\varphi, \Phi\Psi\vartheta_\varphi) = |\frac{\vartheta_\varphi}{6} - \frac{\vartheta_\varphi}{6}| + |\frac{\vartheta_\varphi}{6} - \frac{\vartheta_\varphi}{6}| + |\frac{\vartheta_\varphi}{6} - \frac{\vartheta_\varphi}{6}| = 0.$$

Therefore $\lim_{\varphi\to\infty} \mathcal{S}(\Phi\Psi\vartheta_\varphi, \Phi\Psi\vartheta_\varphi, \Phi\Psi\vartheta_\varphi) = 0$. Hence, the pair of mappings $\{\Phi, \Psi\}$ is compatible.

Definition 2.8 ([14]). Let a binary relation $\perp$ ($\mathtt{br}_\perp$), defined on a non-void set Σ. If $\mathtt{br}_\perp$ satisfies the following criteria:

$$\exists \vartheta_0, \quad (\forall w \in \Sigma, w \perp \vartheta_0) \quad \text{or} \quad (\forall w \in \Sigma, \vartheta_0 \perp w),$$

then $(\Sigma, \perp)$ is known as an orthogonal set ($\perp$-set), and element ϑ_0 is called an orthogonal element ($\perp$-element).

Example 2.9. Let $\Sigma = 2\mathbb{Z}$, and set a $\mathtt{br}_\perp$ on $2\mathbb{Z}$, as $\mathfrak{p} \perp \mathfrak{q}$, if $\mathfrak{p}.\mathfrak{q} = 0$, where $\mathfrak{p}, \mathfrak{q} \in \mathbb{Z}$. Then, $(2\mathbb{Z}, \perp)$ is an $\perp$-set with 0, as an $\perp$-element.

Definition 2.10 ([14]). Let $(\Sigma, \perp)$ be an $\perp$-set. A sequence $\{\vartheta_\varphi\}_{\varphi\in\mathbb{N}}$ is said to be an orthogonal sequence ($\perp$-seq) if

$$(\forall \varphi \in \mathbb{N}; \vartheta_\varphi \perp \vartheta_{\varphi+1}) \quad \text{or} \quad (\forall \varphi \in \mathbb{N}; \vartheta_{\varphi+1} \perp \vartheta_\varphi).$$

Definition 2.11 ([14]). Consider a $\mathtt{br}_\perp$ on a non-void set Σ with metric Δ, defined on a set Σ, then $(\Sigma, \perp, \Delta)$ is called OMS, if every Cauchy $\perp$-seq converges in Σ, then Σ is called orthogonal complete.

Definition 2.12 ([14])**.** Let $(\Sigma, \perp, \Delta)$ be an OMS, and a self-map $\Phi : \Sigma \to \Sigma$. If for each $\perp$-seq $\{\vartheta_\varphi\}_{\varphi \in \mathbb{N}} \to \vartheta$ implies that $\Phi(\vartheta_\varphi) \to \Phi(\vartheta)$ as $\varphi \to \infty$, then Φ is called $\perp$-continuous $(\perp_{\mathcal{C}})$ at ϑ.

Definition 2.13 ([14])**.** Consider a $\mathtt{br}_\perp$ on a non-void set Σ, and let $(\Sigma, \perp)$ be a $\perp$-set. A mapping $\Phi : \Sigma \to \Sigma$ is called $\perp$-preserving $(\perp_{\mathcal{P}})$, if $\Phi(\vartheta) \perp \Phi(w)$, whenever $\vartheta \perp w$.

Definition 2.14 ([9])**.** $(\Sigma, \mathcal{S}, \perp)$ is called an OSMS, if $(\Sigma, \perp)$ is an $\perp$-set and $(\Sigma, \mathcal{S})$ is an SMS.

In 2018, Sedghi et al. [39] proved the following important theorem.

Theorem 2.15. *Suppose that $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are self maps of a complete SMS $(\Sigma, \mathcal{S})$ with $\Phi(\Sigma) \subseteq \mathcal{K}(\Sigma)$, $\Psi(\Sigma) \subseteq \mathcal{H}(\Sigma)$, and that the pairs $\{\Phi, \mathcal{H}\}$ and $\{\Psi, \mathcal{K}\}$ are compatible. If*

$$\mathcal{S}(\Phi\vartheta, \Phi w, \Phi\eta) \leqslant \tau \max\left\{\mathcal{S}(\mathcal{H}\vartheta, \mathcal{H}w, \mathcal{K}\eta), \mathcal{S}(\Phi\vartheta, \Phi w, \mathcal{H}\vartheta), \mathcal{S}(\Psi\eta, \Psi\eta, \mathcal{K}\eta), \mathcal{S}(\Phi w, \Phi w, \Psi\eta)\right\},$$

for each $\vartheta, w, \eta \in \Sigma$, with $0 < \tau < 1$, then $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ have a unique CFP in Σ, provided that $\mathcal{H}$ and $\mathcal{K}$ are continuous.

Motivated by the aforementioned work, here we establish CFP theorem for four mappings on OSMS with an application.

3. Main results

Throughout this part, we establish CFP theorems for four maps on OSMS. To end this, we first prove some basic results.

Lemma 3.1. *Let $(\Sigma, \mathcal{S}, \perp)$ be an OSMS. If there exists two $\perp$-sequences $\{\vartheta_\varphi\}$ and $\{w_\varphi\}$, such that $\lim_{\varphi\to\infty} \mathcal{S}(\vartheta_\varphi, \vartheta_\varphi, w_\varphi) = 0$, whenever $\{\vartheta_\varphi\}$ is an $\perp$-seq in Σ, where $\lim_{\varphi\to\infty} \vartheta_\varphi = \theta$, for some $\theta \in \Sigma$, then $\lim_{\varphi\to\infty} w_\varphi = \theta$.*

Proof. By the triangle inequality in OSMS, one has

$$\mathcal{S}(w_\varphi, w_\varphi, \theta) \leqslant \mathcal{S}(w_\varphi, w_\varphi, \vartheta_\varphi) + \mathcal{S}(w_\varphi, w_\varphi, \vartheta_\varphi) + \mathcal{S}(\theta, \theta, \vartheta_\varphi) \leqslant 2\mathcal{S}(w_\varphi, w_\varphi, \vartheta_\varphi) + \mathcal{S}(\theta, \theta, \vartheta_\varphi).$$

Taking upper limit when $\varphi \to \infty$ in above inequality, we find

$$\limsup_{\varphi\to\infty} \mathcal{S}(w_\varphi, w_\varphi, \theta) \leqslant 2 \limsup_{\varphi\to\infty} \mathcal{S}(w_\varphi, w_\varphi, \vartheta_\varphi) + \limsup_{\varphi\to\infty} \mathcal{S}(\theta, \theta, \vartheta_\varphi) = 0.$$

Hence, $\lim_{\varphi\to\infty} w_\varphi = \theta$. □

In this position, we state and prove our first result.

Theorem 3.2. *Let $(\Sigma, \mathcal{S}, \perp)$ be a complete OSMS, such that $\exists\, \vartheta_0 \in \Sigma$ and $\vartheta_0 \perp \Phi\vartheta$, $\forall \vartheta \in \Sigma$. Let $\Phi, \Psi, \mathcal{H}, \mathcal{K} : \Sigma \to \Sigma$ are $\perp_{\mathcal{P}}$, $\perp_{\mathcal{C}}$ mappings of a complete OSMS with $\Phi(\Sigma) \subseteq \mathcal{K}(\Sigma)$, $\Psi(\Sigma) \subseteq \mathcal{H}(\Sigma)$, and that the pairs $\{\Phi, \mathcal{H}\}$ and $\{\Psi, \mathcal{K}\}$ are compatible. If*

$$\mathcal{S}(\Phi\vartheta, \Phi w, \Phi\eta) \leqslant \tau \max\left\{\mathcal{S}(\mathcal{H}\vartheta, \mathcal{H}w, \mathcal{K}\eta), \mathcal{S}(\Phi\vartheta, \Phi\vartheta, \mathcal{H}\vartheta), \mathcal{S}(\Psi\eta, \Psi\eta, \mathcal{K}\eta), \mathcal{S}(\Phi w, \Phi w, \Psi\eta)\right\}, \tag{3.1}$$

for each $\vartheta, w, \eta \in \Sigma$, with $\vartheta \perp w \perp \eta$, and $0 < \tau < 1$, then, $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ have a unique CFP in Σ, provided that $\mathcal{H}$ and $\mathcal{K}$ are continuous.

Proof. The orthogonality of a non-void set implies that $\exists\vartheta_0 \in \Sigma$, fulfilling $(\forall w \in \Sigma, \vartheta_0 \perp w)$ or $(\forall w \in \Sigma, w \perp \vartheta_0)$. It follows that $\vartheta_0 \perp \Phi\vartheta_0$ or $\Phi\vartheta_0 \perp \vartheta_0$. Since $\Phi(\Sigma) \subseteq \mathcal{K}(\Sigma)$, $\exists\, \vartheta_1 \in \Sigma$, such that $\Phi\vartheta_0 = \mathcal{K}\vartheta_1$, and also as $\Psi\vartheta_1 \in \mathcal{H}(\Sigma)$, we choose $\vartheta_2 \in \Sigma$ such that $\Psi\vartheta_1 = \mathcal{H}\vartheta_2$. In general, $\vartheta_{2\varphi+1} \in \Sigma$, with $\Phi\vartheta_{2\varphi} = \mathcal{K}\vartheta_{2\varphi+1}$, and $\vartheta_{2\varphi+2} \in \Sigma$, with $\Psi\vartheta_{2\varphi+1} = \mathcal{H}\vartheta_{2\varphi+2}$, we obtain an $\perp$-seq $\{w_\varphi\}$ in Σ, such that

$$w_{2\varphi} = \Phi\vartheta_{2\varphi} = \mathcal{K}\vartheta_{2\varphi+1}, \quad w_{2\varphi+1} = \Psi\vartheta_{2\varphi+1} = \mathcal{H}\vartheta_{2\varphi+2}, \quad \text{for} \quad \varphi \geqslant 0.$$

Next, we show that $\{w_\varphi\}$ is a Cauchy $\perp$-seq. Regarding this, one has

$$\begin{aligned}
&\mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi+1})\\
&\quad= \mathcal{S}(\Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi}, \Psi\vartheta_{2\varphi+1})\\
&\quad\leqslant \tau \max\big\{\mathcal{S}(\mathcal{H}\vartheta_{2\varphi}, \mathcal{H}\vartheta_{2\varphi}, \mathcal{K}\vartheta_{2\varphi+1}), \mathcal{S}(\Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi}, \mathcal{H}\vartheta_{2\varphi}),\\
&\qquad\qquad \mathcal{S}(\Psi\vartheta_{2\varphi+1}, \Psi\vartheta_{2\varphi+1}, \mathcal{K}\vartheta_{2\varphi+1}), \mathcal{S}(\Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi+1})\big\}\\
&\quad= \tau \max\big\{\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi}), \mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-1}), \mathcal{S}(w_{2\varphi+1}, w_{2\varphi+1}, w_{2\varphi}), \mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi+1})\big\}\\
&\quad= \tau \max\big\{\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi}), \mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi+1})\big\}.
\end{aligned}$$

Now, if $\mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi+1}) > \mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi})$, then by above inequality, we get

$$\mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi+1}) < \tau\, \mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi+1}), \tag{3.2}$$

which is a contradiction. Hence, $\mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi+1}) \leqslant \mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi})$, therefore by equation (3.2), we find

$$\mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi+1}) \leqslant \tau\, \mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi}). \tag{3.3}$$

Similarly, we have

$$\begin{aligned}
\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi}) &= \mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-1})\\
&= \mathcal{S}(\Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi}, \Psi\vartheta_{2\varphi-1})\\
&\leqslant \tau \max\big\{\mathcal{S}(\mathcal{H}\vartheta_{2\varphi}, \mathcal{H}\vartheta_{2\varphi}, \mathcal{K}\vartheta_{2\varphi-1}), \mathcal{S}(\Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi}, \mathcal{H}\vartheta_{2\varphi}),\\
&\qquad\qquad \mathcal{S}(\Psi\vartheta_{2\varphi-1}, \Psi\vartheta_{2\varphi-1}, \mathcal{K}\vartheta_{2\varphi-1}), \mathcal{S}(\Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi-1})\big\}\\
&= \tau \max\big\{\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi-2}), \mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-1}),\\
&\qquad\qquad \mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi-2}), \mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-1})\big\}\\
&= \tau \max\big\{\mathcal{S}(w_{2\varphi-2}, w_{2\varphi-2}, w_{2\varphi-1}), \mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-1})\big\}.
\end{aligned}$$

Next, if $\mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-1}) > \mathcal{S}(w_{2\varphi-2}, w_{2\varphi-2}, w_{2\varphi-1})$, then it implies that

$$\mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-1}) < \tau\, \mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-1}), \tag{3.4}$$

which is a contradiction. Hence, $\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi}) \leqslant \mathcal{S}(w_{2\varphi-2}, w_{2\varphi-2}, w_{2\varphi-1})$, therefore by equation (3.4), we get

$$\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi}) \leqslant \tau\, \mathcal{S}(w_{2\varphi-2}, w_{2\varphi-2}, w_{2\varphi-1}). \tag{3.5}$$

Also, from equations (3.3) and (3.5), we obtain

$$\mathcal{S}(w_\varphi, w_\varphi, w_{\varphi-1}) \leqslant \tau\mathcal{S}(w_{\varphi-1}, w_{\varphi-1}, w_{\varphi-2}), \quad \varphi \geqslant 2,$$

where $0 < \tau < 1$. Hence for $\varphi \geqslant 2$, it follows that

$$\mathcal{S}(w_\varphi, w_\varphi, w_{\varphi-1}) \leqslant \cdots \leqslant \tau^{\varphi-1}\mathcal{S}(w_1, w_1, w_0). \tag{3.6}$$

By the triangle inequality in OSMS, for $\varphi > \ell$, we have

$$\begin{aligned}\mathcal{S}(w_\varphi, w_\varphi, w_\ell) &\leqslant 2\mathcal{S}(w_\ell, w_\ell, w_{\ell+1}) + 2\mathcal{S}(w_{\ell+1}, w_{\ell+1}, w_{\ell+2}) + \cdots + \mathcal{S}(w_{\varphi-1}, w_{\varphi-1}, w_\varphi)\\ &< 2\mathcal{S}(w_\ell, w_\ell, w_{\ell+1}) + 2\mathcal{S}(w_{\ell+1}, w_{\ell+1}, w_{\ell+2}) + \cdots + 2\mathcal{S}(w_{\varphi-1}, w_{\varphi-1}, w_\varphi).\end{aligned}$$

Hence, from equation (3.6), and as $0 < \tau < 1$, we find

$$\begin{aligned}\mathcal{S}(w_\varphi, w_\varphi, w_\ell) &\leqslant 2(\tau^\ell + \tau^{\ell+1} + \cdots + \tau^{\varphi-1})\mathcal{S}(w_1, w_1, w_0)\\ &\leqslant 2\tau^\ell(1 + \tau + \tau^2 + \cdots)\mathcal{S}(w_1, w_1, w_0) \leqslant 2\frac{\tau^\ell}{1-\tau}\mathcal{S}(w_1, w_1, w_0) \to 0, \quad \text{as} \quad \ell \to \infty.\end{aligned}$$

$\Rightarrow \{w_\varphi\}$ is a Cauchy $\perp$-seq. Since Σ is a complete OSMS, then there is some w in Σ, where

$$\lim_{\varphi\to\infty} \Phi\vartheta_{2\varphi} = \lim_{\varphi\to\infty} \vartheta_{2\varphi+1} = \lim_{\varphi\to\infty} \Psi\vartheta_{2\varphi+1} = \lim_{\varphi\to\infty} \mathcal{H}\vartheta_{2\varphi+2} = w.$$

We show that w is a CFP of $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$. Since $\mathcal{H}$ is continuous, one has

$$\lim_{\varphi\to\infty} \mathcal{H}^2\vartheta_{2\varphi+2} = \mathcal{H}w, \quad \lim_{\varphi\to\infty} \mathcal{H}\Phi\vartheta_{2\varphi} = \mathcal{H}w.$$

Also, since Φ and $\mathcal{H}$ are compatible, then $\lim_{\varphi\to\infty} \mathcal{S}(\Phi\mathcal{H}\vartheta_{2\varphi}, \Phi\mathcal{H}\vartheta_{2\varphi}, \mathcal{H}\Phi\vartheta_{2\varphi}) = 0$. Thus, by Lemma 3.1, we have $\lim_{\varphi\to\infty} \Phi\mathcal{H}\vartheta_{2\varphi} = \mathcal{H}w$. By putting $\vartheta = w = \mathcal{H}\vartheta_{2\varphi}$ and $\eta = \vartheta_{2\varphi+1}$ in equation (3.1), we obtain

$$\begin{aligned}\mathcal{S}(\Phi\mathcal{H}\vartheta_{2\varphi}, \Phi\mathcal{H}\vartheta_{2\varphi}, \mathcal{H}\vartheta_{2\varphi+1}) \leqslant \tau \max\{&\mathcal{S}(\mathcal{H}^2\vartheta_{2\varphi}, \mathcal{H}^2\vartheta_{2\varphi}, \mathcal{K}\vartheta_{2\varphi+1}), \mathcal{S}(\Phi\mathcal{H}\vartheta_{2\varphi}, \Phi\mathcal{H}\vartheta_{2\varphi}, \mathcal{H}^2\vartheta_{2\varphi}),\\ &\mathcal{S}(\Psi\vartheta_{2\varphi+1}, \Psi\vartheta_{2\varphi+1}, \mathcal{K}\vartheta_{2\varphi+1}), \mathcal{S}(\Phi\mathcal{H}\vartheta_{2\varphi}, \Phi\mathcal{H}\vartheta_{2\varphi}, \Psi\vartheta_{2\varphi+1})\}.\end{aligned} \tag{3.7}$$

Taking the upper limit when $\varphi \to \infty$ in equation (3.7), we conclude that

$$\begin{aligned}\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w) &= \lim_{\varphi\to\infty} \mathcal{S}(\Phi\mathcal{H}\vartheta_{2\varphi}, \Phi\mathcal{H}\vartheta_{2\varphi}, \mathcal{K}\vartheta_{2\varphi+1})\\ &\leqslant \tau \max\{\lim_{\varphi\to\infty} \mathcal{S}(\mathcal{H}^2\vartheta_{2\varphi}, \mathcal{H}^2\vartheta_{2\varphi}, \mathcal{K}\vartheta_{2\varphi+1}), \lim_{\varphi\to\infty} \mathcal{S}(\Phi\mathcal{H}\vartheta_{2\varphi}, \Phi\mathcal{H}\vartheta_{2\varphi}, \mathcal{H}^2\vartheta_{2\varphi}),\\ &\qquad \lim_{\varphi\to\infty} \mathcal{S}(\Psi\vartheta_{2\varphi+1}, \Psi\vartheta_{2\varphi+1}, \mathcal{K}\vartheta_{2\varphi+1}), \lim_{\varphi\to\infty} \mathcal{S}(\Phi\mathcal{H}\vartheta_{2\varphi}, \Phi\mathcal{H}\vartheta_{2\varphi}, \Psi\vartheta_{2\varphi+1})\}\\ &\leqslant \tau \max\{\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w), 0, 0, \mathcal{S}(\mathcal{H}w, \mathcal{H}w, w)\} = \tau\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w).\end{aligned}$$

Consequently, $\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w) \leqslant \tau\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w)$, as $0 < \tau < 1$, it follows that $\mathcal{H}w = w$. Similarly, since $\mathcal{K}$ is continuous, we have

$$\lim_{\varphi\to\infty} \mathcal{K}^2\vartheta_{2\varphi+1} = \mathcal{K}w, \quad \lim_{\varphi\to\infty} \mathcal{K}\Psi\vartheta_{2\varphi+1} = \mathcal{K}w,$$

and since Ψ and $\mathcal{K}$ are compatible, then

$$\lim_{\varphi\to\infty} \mathcal{S}(\Psi\mathcal{K}\vartheta_{2\varphi+1}, \Psi\mathcal{K}\vartheta_{2\varphi+1}, \mathcal{K}\Psi\vartheta_{2\varphi+1}) = 0.$$

Hence, by Lemma 3.1, we deduce that $\lim_{\varphi\to\infty} \Psi\mathcal{K}\vartheta_{2\varphi+1} = \mathcal{K}w$. Now, by putting $\vartheta = w = \vartheta_{2\varphi}$ and $\eta = \mathcal{K}\vartheta_{2\varphi+1}$ in equation (3.1), we obtain

$$\begin{aligned}\mathcal{S}(\Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi}, \Psi k\vartheta_{2\varphi+1}) \leqslant \tau \max\{&\mathcal{S}(\mathcal{H}\vartheta_{2\varphi}, \mathcal{H}\vartheta_{2\varphi}, \mathcal{K}^2\vartheta_{2\varphi+1}), \mathcal{S}(\Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi}, \mathcal{H}\vartheta_{2\varphi}),\\ &\mathcal{S}(\Psi\mathcal{K}\vartheta_{2\varphi+1}, \Psi\mathcal{K}\vartheta_{2\varphi+1}, \mathcal{K}^2\vartheta_{2\varphi+1}), \mathcal{S}(\Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi}, \Psi\mathcal{K}\vartheta_{2\varphi+1})\}.\end{aligned} \tag{3.8}$$

Taking the upper limit when $\varphi \to \infty$ in equation (3.8), we get

$$\mathcal{S}(w, w, \mathcal{K}w) = \lim_{\varphi\to\infty} \mathcal{S}(\Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi}, \Psi\mathcal{K}\vartheta_{2\varphi+1}) \leqslant \tau \max\{\mathcal{S}(w, w, \mathcal{K}w), 0, 0, \mathcal{S}(w, w, \mathcal{K}w)\},$$

this means $\mathcal{K}w = w$. Also, applying equation (3.1) we obtain

$$\mathcal{S}(\Phi w, \Phi w, \Psi\mathcal{K}\vartheta_{2\varphi+1}) \leqslant \tau \max\{\mathcal{S}(\mathcal{H}w, \mathcal{H}w, \mathcal{K}\vartheta_{2\varphi+1}), \mathcal{S}(\Phi w, \Phi w, \mathcal{H}w), \mathcal{S}(\Psi\vartheta_{2\varphi+1}, \Psi\vartheta_{2\varphi+1}, \mathcal{K}\vartheta_{2\varphi+1}), \mathcal{S}(\Phi w, \Phi w, \Psi\vartheta_{2\varphi+1})\}.$$

Taking the upper limit when $\varphi \to \infty$ as $\mathcal{H}w = \mathcal{K}w = w$, we have

$$\mathcal{S}(\Phi w, \Phi w, w) \leqslant \tau \max\{\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w), \mathcal{S}(\Phi w, \Phi w, w), \mathcal{S}(w, w, w), \mathcal{S}(\Phi w, \Phi w, w)\} = \tau\mathcal{S}(\Phi w, \Phi w, w).$$

Since $0 < \tau < 1$, we get $\mathcal{S}(\Phi w, \Phi w, w) = 0$, and $\Phi w = w$. Now, using equation (3.1), and as $\mathcal{H}w = \mathcal{K}w = \Phi w = w$, we obtain

$$\begin{aligned}\mathcal{S}(w, w, \Psi w) &= \mathcal{S}(\Phi w, \Phi w, \Psi w)\\ &\leqslant \tau \max\{\mathcal{S}(\mathcal{H}w, \mathcal{H}w, \mathcal{K}w), \mathcal{S}(\Phi w, \Phi w, \mathcal{H}w), \mathcal{S}(\Psi w, \Psi w, \mathcal{K}w), \mathcal{S}(\Phi w, \Phi w, \Psi w)\} = \tau\mathcal{S}(w, w, \Psi w),\end{aligned}$$

which implies that $\mathcal{S}(w, w, \Psi w) = 0$, and $\Psi w = w$. Thus, by all the above conclusions, we proved that $\mathcal{H}w = \mathcal{K}w = \Phi w = \Psi w = w$. Next, if $\exists$ another CFP $w^\star$ in Σ of all $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$, then

$$\begin{aligned}\mathcal{S}(w^\star, w^\star, w) &= \mathcal{S}(\Phi w^\star, \Phi w^\star, \Psi w)\\ &\leqslant \tau \max\{\mathcal{S}(\mathcal{H}w^\star, \mathcal{H}w^\star, \mathcal{K}w), \mathcal{S}(\Phi w^\star, \Phi w^\star, \mathcal{H}w^\star), \mathcal{S}(\Psi w, \Psi w, \mathcal{K}w), \mathcal{S}(\Phi w^\star, \Phi w^\star, \Psi w)\}\\ &\leqslant \tau \max\{\mathcal{S}(w^\star, w^\star, w), \mathcal{S}(w^\star, w^\star, w^\star), \mathcal{S}(w, w, w), \mathcal{S}(w^\star, w^\star, w)\} = \tau\mathcal{S}(w^\star, w^\star, w),\end{aligned}$$

which yields that $\mathcal{S}(w^\star, w^\star, w) = 0$, and $w^\star = w$. Hence, w is a unique CFP of $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$. □

Example 3.3. Let $\Sigma = [0, 1]$, and a mapping $\mathcal{S} : \Sigma^3 \to \mathbb{R}^+$, is an usual $\mathcal{S}$-metric on $\mathbb{R}$, and defined as $\mathcal{S}(\vartheta, w, \eta) = |\vartheta - \eta| + |w - \eta|$, $\forall \vartheta, w, \eta \in \Sigma$, and a $br_\perp$ on Σ is $\vartheta \perp w$, if $\vartheta w \leqslant 2\vartheta$, this implies that $(\Sigma, \mathcal{S}, \perp)$ is a complete OSMS. Now, define $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ on Σ by

$$\Phi\vartheta = (\frac{\vartheta}{5})^8, \quad \Psi\vartheta = (\frac{\vartheta}{5})^4, \quad \mathcal{H}\vartheta = (\frac{\vartheta}{5})^2, \quad \mathcal{K}\vartheta = \frac{\vartheta}{5},$$

since $\Phi\vartheta \subseteq \mathcal{K}\vartheta$ and $\Psi\vartheta \subseteq \mathcal{H}\vartheta$. Therefore, the pairs $\{\Phi, \mathcal{H}\}$ and $\{\Psi, \mathcal{K}\}$ are compatible mappings. Also,

$$\Phi\vartheta.\Phi w = (\frac{\vartheta}{5})^8.(\frac{w}{5})^8 \leqslant 2\Phi\vartheta, \qquad \Psi\vartheta.\Psi w = (\frac{\vartheta}{5})^4.(\frac{w}{5})^4 \leqslant 2\Psi\vartheta,$$
$$\mathcal{H}\vartheta.\mathcal{H}w = (\frac{\vartheta}{5})^2.(\frac{w}{5})^2 \leqslant 2\mathcal{H}\vartheta, \qquad \mathcal{K}\vartheta.\mathcal{K}w = (\frac{\vartheta}{5}).(\frac{w}{5}) \leqslant 2\mathcal{K}\vartheta.$$

Hence, $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $\perp_{\mathcal{P}}$. Moreover, for each $\vartheta, w, \eta \in \Sigma$, we have

$$\begin{aligned}\mathcal{S}(\Phi\vartheta, \Phi w, \Psi\eta) &= |\Phi\vartheta - \Psi\eta| + |\Phi w - \Psi\eta|\\ &= \left|(\frac{\vartheta}{5})^8 - (\frac{\eta}{5})^4\right| + \left|(\frac{w}{5})^8 - (\frac{\eta}{5})^4\right|\\ &= \left|(\frac{\vartheta}{5})^4 - (\frac{\eta}{5})^2\right| \left|(\frac{\vartheta}{5})^4 + (\frac{\eta}{5})^2\right| + \left|(\frac{w}{5})^4 - (\frac{\eta}{5})^2\right| \left|(\frac{w}{5})^4 + (\frac{\eta}{5})^2\right|\\ &\leqslant \frac{26}{625}\left|(\frac{\vartheta}{5})^2 - \frac{\eta}{5}\right| \left|(\frac{\vartheta}{5})^2 + \frac{\eta}{5}\right| + \frac{26}{625}\left|(\frac{w}{5})^2 - \frac{\eta}{5}\right| \left|(\frac{w}{5})^2 + \frac{\eta}{5}\right|\\ &\leqslant \frac{156}{15,625}\left|(\frac{\vartheta}{5})^2 - \frac{\eta}{5}\right| + \frac{156}{15,625}\left|(\frac{w}{5})^2 - \frac{\eta}{5}\right|\\ &\leqslant \frac{156}{15,625}\left|\mathcal{H}\vartheta - \mathcal{K}\eta\right| + \frac{156}{15,625}\left|\mathcal{H}w - \mathcal{K}\eta\right|\\ &= \frac{156}{15,625}\Big(\left|\mathcal{H}\vartheta - \mathcal{K}\eta\right| + \left|\mathcal{H}w - \mathcal{K}\eta\right|\Big)\end{aligned}$$

$$\begin{aligned}
&= \frac{156}{15,625}\mathcal{S}(\mathcal{H}\vartheta,\mathcal{H}w,\mathcal{K}\eta)\\
&\leqslant \frac{156}{15,625}\max\{\mathcal{S}(\mathcal{H}\vartheta,\mathcal{H}w,\mathcal{K}\eta),\mathcal{S}(\Phi\vartheta,\Phi\vartheta,\mathcal{H}\vartheta),\mathcal{S}(\Psi\eta,\Psi\eta,\mathcal{K}\eta),\mathcal{S}(\Phi w,\Phi w,\Psi\eta)\}\\
&\leqslant \tau\max\{\mathcal{S}(\mathcal{H}\vartheta,\mathcal{H}w,\mathcal{K}\eta),\mathcal{S}(\Phi\vartheta,\Phi\vartheta,\mathcal{H}\vartheta),\mathcal{S}(\Psi\eta,\Psi\eta,\mathcal{K}\eta),\mathcal{S}(\Phi w,\Phi w,\Psi\eta)\},
\end{aligned}$$

where $\frac{156}{15,625} \leqslant \tau < 1$. Thus, $\Phi,\Psi,\mathcal{H}$, and $\mathcal{K}$ satisfies the hypothesis of the Theorem 3.2, and 0 is the unique CFP of $\Phi,\Psi,\mathcal{H}$, and $\mathcal{K}$.

In this situation, we state and prove our second result.

Theorem 3.4. *Let $\Phi,\Psi,\mathcal{H},\mathcal{K}:\Sigma\to\Sigma$ are $\perp_{\mathcal{P}}$, $\perp_{\mathcal{C}}$ mappings of a complete OSMS $(\Sigma,\mathcal{S},\perp)$. Also, let the pairs $\{\Phi,\mathcal{H}\}$ and $\{\Psi,\mathcal{K}\}$ be compatible self mappings on a complete OSMS, and $\forall\vartheta,w,\eta\in\Sigma$ with $\vartheta\perp w\perp\eta$, satisfying*

$$\begin{aligned}
\mathcal{S}(\Phi\vartheta,\Phi w,\Psi\eta) \leqslant{}& \alpha_1\mathcal{S}(\mathcal{H}\vartheta,\mathcal{H}w,\mathcal{K}\eta)+\alpha_2\mathcal{S}(\Phi\vartheta,\Phi\vartheta,\mathcal{K}\eta)+\alpha_3\mathcal{S}(\mathcal{H}\vartheta,\mathcal{H}w,\Psi\eta)\\
&+\alpha_4\mathcal{S}(\Phi w,\Phi w,\mathcal{K}\eta)+\alpha_5\mathcal{S}(\Psi\eta,\Psi\eta,\mathcal{K}\eta),
\end{aligned} \tag{3.9}$$

where $\alpha_j \geqslant 0$, $j=1,2,3,4,5$, are real constants with $\alpha_1+3\alpha_2+3\alpha_3+3\alpha_4+\alpha_5<1$. If $\Phi(\Sigma)\subseteq\mathcal{T}(\Sigma)$, $\Psi(\Sigma)\subseteq\mathcal{H}(\Sigma)$, $\mathcal{H}$, and $\mathcal{K}$ are continuous, then the four maps $\Phi,\Psi,\mathcal{H}$, and $\mathcal{K}$ have a unique CFP.

Proof. The orthogonality of a non-void set implies that $\exists\vartheta_0\in\Sigma$, fulfilling

$$(\forall w\in\Sigma,\vartheta_0\perp w)\quad\text{or}\quad(\forall w\in\Sigma,w\perp\vartheta_0).$$

It follows that $\vartheta_0\perp\Phi\vartheta_0$ or $\Phi\vartheta_0\perp\vartheta_0$. Since $\Phi(\Sigma)\subseteq\mathcal{K}(\Sigma)$, $\exists\ \vartheta_1\in\Sigma$, such that $\mathcal{K}\vartheta_1=\Phi\vartheta_0$, and also $\Psi\vartheta_1\in\mathcal{H}(\vartheta)$. Now, let $\vartheta_2\in\Sigma$ be such that $\mathcal{H}\vartheta_2=\Psi\vartheta_1$. In general, $\vartheta_{2\varphi+1}\in\Sigma$ is chosen such that $\mathcal{K}\vartheta_{2\varphi+1}=\Phi\vartheta_{2\varphi}$, and $\vartheta_{2\varphi+2}\in\Sigma$, such that $\mathcal{H}\vartheta_{2\varphi+2}=\Psi\vartheta_{2\varphi+1}$; $\varphi=0,1,2,\ldots$. We set

$$w_{2\varphi}=\mathcal{K}\vartheta_{2\varphi+1}=\Phi\vartheta_{2\varphi},\quad w_{2\varphi+1}=\mathcal{H}\vartheta_{2\varphi+2}=\Psi\vartheta_{2\varphi+1},\quad\varphi\geqslant 0.$$

Next, we show that $\{w_\varphi\}$ is a Cauchy $\perp$-seq. For end this, we have

$$\begin{aligned}
&\mathcal{S}(w_{2\varphi},w_{2\varphi},w_{2\varphi+1})\\
&\ = \mathcal{S}(\Phi\vartheta_{2\varphi},\Phi\vartheta_{2\varphi},\Psi\vartheta_{2\varphi+1})\\
&\ \leqslant \alpha_1\mathcal{S}(\mathcal{H}\vartheta_{2\varphi},\mathcal{H}\vartheta_{2\varphi},\mathcal{K}\vartheta_{2\varphi+1})+\alpha_2\mathcal{S}(\Phi\vartheta_{2\varphi},\Phi\vartheta_{2\varphi},\mathcal{K}\vartheta_{2\varphi+1})+\alpha_3\mathcal{S}(\mathcal{H}\vartheta_{2\varphi},\mathcal{H}\vartheta_{2\varphi},\Psi\vartheta_{2\varphi+1})\\
&\qquad+\alpha_4\mathcal{S}(\Phi\vartheta_{2\varphi},\Phi\vartheta_{2\varphi},\mathcal{K}\vartheta_{2\varphi+1})+\alpha_5\mathcal{S}(\Psi\vartheta_{2\varphi+1},\Psi\vartheta_{2\varphi+1},\mathcal{K}\vartheta_{2\varphi+1})\\
&\ = \alpha_1\mathcal{S}(w_{2\varphi-1},w_{2\varphi-1},w_{2\varphi})+\alpha_2\mathcal{S}(w_{2\varphi},w_{2\varphi},w_{2\varphi})\\
&\qquad+\alpha_3\mathcal{S}(w_{2\varphi-1},w_{2\varphi-1},w_{2\varphi+1})+\alpha_4\mathcal{S}(w_{2\varphi},w_{2\varphi},w_{2\varphi})+\alpha_5\mathcal{S}(w_{2\varphi+1},w_{2\varphi+1},w_{2\varphi})\\
&\ \leqslant \alpha_1\mathcal{S}(w_{2\varphi-1},w_{2\varphi-1},w_{2\varphi})+\alpha_3\big[2\mathcal{S}(w_{2\varphi-1},w_{2\varphi-1},w_{2\varphi})\\
&\qquad+\mathcal{S}(w_{2\varphi+1},w_{2\varphi+1},w_{2\varphi})\big]+\alpha_5\mathcal{S}(w_{2\varphi},w_{2\varphi},w_{2\varphi+1}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{S}(w_{2\varphi},w_{2\varphi},w_{2\varphi+1}) \leqslant{}& \alpha_1\mathcal{S}(w_{2\varphi-1},w_{2\varphi-1},w_{2\varphi})+2\alpha_3\mathcal{S}(w_{2\varphi-1},w_{2\varphi-1},w_{2\varphi})\\
&+(\alpha_3+\alpha_5)\mathcal{S}(w_{2\varphi},w_{2\varphi},w_{2\varphi+1}).
\end{aligned} \tag{3.10}$$

Now, we prove that $\mathcal{S}(w_{2\varphi},w_{2\varphi},w_{2\varphi+1})\leqslant\mathcal{S}(w_{2\varphi-1},w_{2\varphi-1},w_{2\varphi})$, for each $\varphi\in\mathbb{N}$. If $\mathcal{S}(w_{2\varphi-1},w_{2\varphi-1},w_{2\varphi}) < \mathcal{S}(w_{2\varphi},w_{2\varphi},w_{2\varphi+1})$, for some $\varphi\in\mathbb{N}$, then from equation (3.10), we have

$$\begin{aligned}
\mathcal{S}(w_{2\varphi},w_{2\varphi},w_{2\varphi+1}) &< \alpha_1\mathcal{S}(w_{2\varphi},w_{2\varphi},w_{2\varphi+1})+2\alpha_3\mathcal{S}(w_{2\varphi},w_{2\varphi},w_{2\varphi+1})+(\alpha_3+\alpha_5)\mathcal{S}(w_{2\varphi},w_{2\varphi},w_{2\varphi+1})\\
&= (\alpha_1+3\alpha_3+\alpha_5)\mathcal{S}(w_{2\varphi},w_{2\varphi},w_{2\varphi+1}) < \mathcal{S}(w_{2\varphi},w_{2\varphi},w_{2\varphi+1}),
\end{aligned}$$

which is a contradiction. So, we get $\mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi+1}) \leqslant \mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi})$ for each $\varphi \in \mathbb{N}$, and from equation (3.10), we find

$$\mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi+1}) \leqslant (\alpha_1 + 3\alpha_3 + \alpha_5)\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi}). \tag{3.11}$$

Further, we have

$$\begin{aligned}
&\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi})\\
&\quad = \mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-1})\\
&\quad = \mathcal{S}(\Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi}, \Psi\vartheta_{2\varphi-1})\\
&\quad \leqslant \alpha_1\mathcal{S}(\mathcal{H}\vartheta_{2\varphi}, \mathcal{H}\vartheta_{2\varphi}, \mathcal{K}\vartheta_{2\varphi-1}) + \alpha_2\mathcal{S}(\Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi}, \mathcal{K}\vartheta_{2\varphi-1}) + \alpha_3\mathcal{S}(\mathcal{H}\vartheta_{2\varphi}, \mathcal{H}\vartheta_{2\varphi}, \Psi\vartheta_{2\varphi-1})\\
&\qquad + \alpha_4\mathcal{S}(\Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi}, \mathcal{K}\vartheta_{2\varphi-1}) + \alpha_5\mathcal{S}(\Psi\vartheta_{2\varphi-1}, \Psi\vartheta_{2\varphi-1}, \mathcal{K}\vartheta_{2\varphi-1})\\
&\quad = \alpha_1\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi-2}) + \alpha_2\mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-2})\\
&\qquad + \alpha_3\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi-1}) + \alpha_4\mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-2}) + \alpha_5\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi-2})\\
&\quad \leqslant \alpha_1\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi-2}) + (2\alpha_2 + 2\alpha_4)\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi})\\
&\qquad + (\alpha_2 + \alpha_4)\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi-2}) + \alpha_5\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi-2}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-1}) \leqslant \alpha_1\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi-2}) &+ (2\alpha_2 + 2\alpha_4)\mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-1})\\
&+ (\alpha_2 + \alpha_4 + \alpha_5)\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi-2}).
\end{aligned} \tag{3.12}$$

Similarly, if $\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi-2}) < \mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-1})$, for some $\varphi \in \mathbb{N}$, then by equation (3.12), we obtain

$$\mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-1}) \leqslant (\alpha_1 + 3\alpha_2 + 3\alpha_4 + \alpha_5)\mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-1}) < \mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-1}),$$

which is a contradiction. So, we have $\mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-1}) \leqslant \mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi-2})$, for each $\varphi \in \mathbb{N}$, and from equation (3.12), we get

$$\mathcal{S}(w_{2\varphi}, w_{2\varphi}, w_{2\varphi-1}) \leqslant (\alpha_1 + 3\alpha_2 + 3\alpha_4 + \alpha_5)\mathcal{S}(w_{2\varphi-1}, w_{2\varphi-1}, w_{2\varphi-2}). \tag{3.13}$$

Now, in view of equations (3.11) and (3.13), we have

$$\mathcal{S}(w_{\varphi}, w_{\varphi}, w_{\varphi-1}) \leqslant \lambda\mathcal{S}(w_{\varphi-1}, w_{\varphi-1}, w_{\varphi-2}), \quad \varphi \geqslant 2,$$

where $\lambda = \min\{\alpha_1 + 3\alpha_3 + \alpha_5, \alpha_1 + 3\alpha_2 + 3\alpha_4 + \alpha_5\}$, since $\lambda \in (0, 1)$, Hence, for $\varphi \geqslant 2$, one has

$$\mathcal{S}(w_{\varphi}, w_{\varphi}, w_{\varphi-1}) \leqslant \cdots \leqslant \lambda^{\varphi-1}\mathcal{S}(w_1, w_1, w_0). \tag{3.14}$$

By the triangle inequality in OSMS, for $\varphi > \ell$, one finds

$$\mathcal{S}(w_{\varphi}, w_{\varphi}, w_{\ell}) \leqslant 2\mathcal{S}(w_{\ell}, w_{\ell}, w_{\ell+1}) + 2\mathcal{S}(w_{\ell+1}, w_{\ell+1}, w_{\ell+2}) + \cdots + 2\mathcal{S}(w_{\varphi-1}, w_{\varphi-1}, w_{\varphi}).$$

Hence, by equation (3.14), and as $\lambda < 1$, we have

$$\begin{aligned}
\mathcal{S}(w_{\varphi}, w_{\varphi}, w_{\ell}) &\leqslant 2(\lambda^{\ell} + \lambda^{\ell+1} + \cdots + \lambda^{\varphi-1})\mathcal{S}(w_1, w_1, w_0)\\
&\leqslant 2\lambda^{\ell}[1 + \lambda + (\lambda)^2 + \cdots]\mathcal{S}(w_1, w_1, w_0)\\
&\leqslant \frac{2\lambda^{\ell}}{1-\lambda}\mathcal{S}(w_1, w_1, w_0) = \frac{2\lambda^{\ell}}{1-\lambda}\mathcal{S}(w_1, w_1, w_0) \to 0 \quad \text{as} \quad \ell \to \infty.
\end{aligned}$$

Therefore $\{w_\varphi\}$ is a Cauchy $\perp$-seq. Let $w \in \Sigma$ be such that

$$\lim_{\varphi\to\infty} \Phi\vartheta_{2\varphi} = \lim_{\varphi\to\infty} \mathcal{K}\vartheta_{2\varphi+1} = \lim_{\varphi\to\infty} \Psi\vartheta_{2\varphi+1} = \lim_{\varphi\to\infty} \mathcal{H}\vartheta_{2\varphi+2} = w.$$

Since $\mathcal{H}$ is continuous, we get

$$\lim_{\varphi\to\infty} \mathcal{H}^2\vartheta_{2\varphi+2} = \mathcal{H}w, \quad \lim_{\varphi\to\infty} \mathcal{H}\vartheta_{2\varphi} = \mathcal{H}w.$$

Also, since Φ and $\mathcal{H}$ are compatible, then $\lim_{\varphi\to\infty} \mathcal{S}(\Phi\mathcal{H}\vartheta_{2\varphi}, \Phi\mathcal{H}\vartheta_{2\varphi}, \mathcal{H}\Phi\vartheta_{2\varphi}) = 0$. So, by Lemma 3.1, $\lim_{\varphi\to\infty} \Phi\mathcal{H}\vartheta_{2\varphi} = \mathcal{H}w$, and from equation (3.9), we obtain

$$\begin{aligned}\mathcal{S}(\Phi\mathcal{H}\vartheta_{2\varphi}, \Phi\mathcal{H}\vartheta_{2\varphi}, \Psi\vartheta_{2\varphi+1}) \leqslant{}& \alpha_1\mathcal{S}(\mathcal{H}^2\vartheta_{2\varphi}, \mathcal{H}^2\vartheta_{2\varphi}, \mathcal{K}\vartheta_{2\varphi+1}) + \alpha_2\mathcal{S}(\Phi\mathcal{H}\vartheta_{2\varphi}, \Phi\mathcal{H}\vartheta_{2\varphi}, \mathcal{K}\vartheta_{2\varphi+1})\\ &+ \alpha_3\mathcal{S}(\mathcal{H}^2\vartheta_{2\varphi}, \mathcal{H}^2\vartheta_{2\varphi}, \Psi\vartheta_{2\varphi+1}) + \alpha_4\mathcal{S}(\Phi\mathcal{H}\vartheta_{2\varphi}, \Phi\mathcal{H}\vartheta_{2\varphi}, \mathcal{K}\vartheta_{2\varphi+1})\\ &+ \alpha_5\mathcal{S}(\Psi\vartheta_{2\varphi+1}, \Psi\vartheta_{2\varphi+1}, \mathcal{K}\vartheta_{2\varphi+1}).\end{aligned}$$

Taking the upper limit as $\varphi \to \infty$, one gets

$$\begin{aligned}\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w) &\leqslant a_1\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w) + \alpha_2\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w) + \alpha_3\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w) + \alpha_4\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w) + \alpha_5\mathcal{S}(w, w, w)\\ &\leqslant a_1\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w) + \alpha_2\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w) + \alpha_3\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w) + \alpha_4\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w)\\ &= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w) \leqslant (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w).\end{aligned}$$

Therefore, $\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w) \leqslant (\alpha_1 + 3\alpha_2 + 3\alpha_3 + 3\alpha_4 + \alpha_5)\mathcal{S}(\mathcal{H}w, \mathcal{H}w, w)$ as $\alpha_1 + 3\alpha_2 + 3\alpha_3 + 3\alpha_4 + \alpha_5 < 1$, it implies that $\mathcal{H}w = w$. Similarly, since $\mathcal{K}$ is continuous, we have

$$\lim_{\varphi\to\infty} \mathcal{K}^2\vartheta_{2\varphi+1} = \mathcal{K}w, \quad \lim_{\varphi\to\infty} \mathcal{K}\Psi\vartheta_{2\varphi+1} = \mathcal{K}w.$$

Since Ψ and $\mathcal{K}$ are compatible, then $\lim_{\varphi\to\infty} \mathcal{S}(\Psi\mathcal{K}\vartheta_{2\varphi+1}, \Psi\mathcal{K}\vartheta_{2\varphi+1}, \mathcal{K}\Psi\vartheta_{2\varphi+1}) = 0$. So, by Lemma 3.1, $\lim_{\varphi\to\infty} \Psi\mathcal{K}\vartheta_{2\varphi+1} = \mathcal{K}w$, and by equation (3.9), one finds

$$\begin{aligned}\mathcal{S}(\Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi}, \Psi\mathcal{K}\vartheta_{2\varphi+1}) \leqslant{}& \alpha_1\mathcal{S}(\mathcal{H}\vartheta_{2\varphi}, \mathcal{H}\vartheta_{2\varphi}, \mathcal{K}^2\vartheta_{2\varphi+1}) + \alpha_2\mathcal{S}(\Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi}, \mathcal{K}^2\vartheta_{2\varphi+1})\\ &+ \alpha_3\mathcal{S}(\mathcal{H}\vartheta_{2\varphi}, \mathcal{H}\vartheta_{2\varphi}, \Psi\mathcal{K}\vartheta_{2\varphi+1}) + \alpha_4\mathcal{S}(\Phi\vartheta_{2\varphi}, \Phi\vartheta_{2\varphi}, \mathcal{K}^2\vartheta_{2\varphi+1})\\ &+ \alpha_5\mathcal{S}(\Psi\mathcal{K}\vartheta_{2\varphi+1}, \Psi\mathcal{K}\vartheta_{2\varphi+1}, \mathcal{K}^2\vartheta_{2\varphi+1}).\end{aligned}$$

Taking the upper limit as $\varphi \to \infty$, one has

$$\begin{aligned}\mathcal{S}(w, w, \mathcal{K}w) &\leqslant a_1\mathcal{S}(w, w, \mathcal{K}w) + \alpha_2\mathcal{S}(w, w, \mathcal{K}w) + \alpha_3\mathcal{S}(w, w, \mathcal{K}w) + \alpha_4\mathcal{S}(w, w, \mathcal{K}w) + \alpha_5\mathcal{S}(\mathcal{K}w, \mathcal{K}w, \mathcal{K}w)\\ &= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\mathcal{S}(w, w, \mathcal{K}w) \leqslant (\alpha_1 + 3\alpha_2 + 3\alpha_3 + 3\alpha_4 + \alpha_5)\mathcal{S}(w, w, \mathcal{K}w).\end{aligned}$$

That is, $\mathcal{S}(w, w, \mathcal{K}w) \leqslant (\alpha_1 + 3\alpha_2 + 3\alpha_3 + 3\alpha_4 + \alpha_5)\mathcal{S}(w, w, \mathcal{K}w)$. Therefore, by $\alpha_1 + 3\alpha_2 + 3\alpha_3 + 3\alpha_4 + \alpha_5 < 1$, we have $\mathcal{K}w = w$. Again, from equation (3.9), it follows

$$\begin{aligned}\mathcal{S}(\Phi w, \Phi w, \Psi\vartheta_{2\varphi+1}) &\leqslant \alpha_1\mathcal{S}(\mathcal{H}w, \mathcal{H}w, \mathcal{K}\vartheta_{2\varphi+1}) + \alpha_2\mathcal{S}(\Phi w, \Phi w, \mathcal{K}\vartheta_{2\varphi+1}) + \alpha_3\mathcal{S}(\mathcal{H}w, \mathcal{H}w, \Psi\vartheta_{2\varphi+1})\\ &\quad + \alpha_4\mathcal{S}(\Phi w, \Phi w, \mathcal{K}\vartheta_{2\varphi+1}) + \alpha_5\mathcal{S}(\Psi\vartheta_{2\varphi+1}, \Psi\vartheta_{2\varphi+1}, \mathcal{K}\vartheta_{2\varphi+1})\\ &\leqslant (\alpha_2 + \alpha_4)\mathcal{S}(\Phi w, \Phi w, w).\end{aligned}$$

Taking the upper limit when $\varphi \to \infty$ as $\mathcal{H}w = w$ and $\mathcal{K}w = w$, we have

$$\begin{aligned}\mathcal{S}(\Phi w, \Phi w, w) &\leqslant \alpha_1\mathcal{S}(w, w, w) + \alpha_2\mathcal{S}(\Phi w, \Phi w, w) + \alpha_3\mathcal{S}(w, w, w) + \alpha_4\mathcal{S}(\Phi w, \Phi w, w) + \alpha_5\mathcal{S}(w, w, w)\\ &\leqslant (\alpha_2 + \alpha_4)\mathcal{S}(\Phi w, \Phi w, w).\end{aligned}$$

Therefore, $\mathcal{S}(\Phi w, \Phi w, w) \leqslant (\alpha_1 + 3\alpha_2 + 3\alpha_3 + 3\alpha_4 + \alpha_5)\mathcal{S}(\Phi w, \Phi w, w)$, and by $\alpha_1 + 3\alpha_2 + 3\alpha_3 + 3\alpha_4 + \alpha_5 < 1$, one has $\Phi w = w$. Once, from equation (3.9), we have $\mathcal{S}(\Phi w, \Phi w, \Psi w) = 0$, and hence $\Phi w = \Psi w$. Then, we prove that $\Phi w = \Psi w = \mathcal{H} w = \mathcal{K} w = w$. Next, if $\exists$ another CFP $w^\star$ in Σ of four maps $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$, then

$$\begin{aligned}
\mathcal{S}(w^\star, w^\star, w) &= \mathcal{S}(\Phi w^\star, \Phi w^\star, \Psi w) \\
&\leqslant \alpha_1 \mathcal{S}(\mathcal{H} w^\star, \mathcal{H} w^\star, \mathcal{K} w) + \alpha_2 \mathcal{S}(\Phi w^\star, \Phi w^\star, \mathcal{K} w) + \alpha_3 \mathcal{S}(\mathcal{H} w^\star, \mathcal{H} w^\star, \Psi w) \\
&\quad + \alpha_4 \mathcal{S}(\Phi w^\star, \Phi w^\star, \mathcal{K} w) + \alpha_5 \mathcal{S}(\Psi w, \Psi w, \mathcal{K} w) \\
&= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\mathcal{S}(w^\star, w^\star, w) \leqslant (\alpha_1 + 3\alpha_2 + 3\alpha_3 + 3\alpha_4 + \alpha_5)\mathcal{S}(w^\star, w^\star, w),
\end{aligned}$$

which follows

$$\mathcal{S}(w^\star, w^\star, w) \leqslant (\alpha_1 + 3\alpha_2 + 3\alpha_3 + 3\alpha_4 + \alpha_5)\mathcal{S}(w^\star, w^\star, w).$$

Since $\alpha_1 + 3\alpha_2 + 3\alpha_3 + 3\alpha_4 + \alpha_5 < 1$, we have $\mathcal{S}(w^\star, w^\star, w) = 0$, i.e., $w^\star = w$. Hence, we conclude that w is a unique CFP of four maps $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$. □

4. Applications

4.1. Application on the Volterra-Type integral system

Let $\mathcal{I} = \mathcal{C}([c, d], \mathbb{R})$ be the space of real valued continuous function defined on $[c, d]$. Also, let the function $\mathcal{S} : \mathcal{I} \times \mathcal{I} \times \mathcal{I} \to [0, \infty)$ be defined by

$$\mathcal{S}(\vartheta, w, \eta) = \sup_{\theta \in [c,d]} |\vartheta(\theta) - w(\theta)| + \sup_{\theta \in [c,d]} |\vartheta(\theta) - \eta(\theta)| + \sup_{\theta \in [c,d]} |w(\theta) - \eta(\theta)|, \ \forall \vartheta, w, \eta \in \mathcal{C}([c, d], \mathbb{R}).$$

Define a $\mathfrak{br}_\perp$ in $\mathcal{I}$; $\vartheta \perp w$ if $\vartheta(\theta) w(\theta) \geqslant w(\theta)$, $\forall \theta \in [c, d]$. Obviously, $(\Sigma, \mathcal{S}, \perp)$ is a complete OSMS. Consider the Volterra-type integral system given by

$$\begin{cases} \vartheta(\theta) = \mathfrak{p}(\theta) + \int_c^\theta \mathcal{L}(\theta, \mathfrak{r}, \mathcal{H}(\vartheta(\theta)))d\mathfrak{r}, \\ \vartheta(\theta) = \mathfrak{p}(\theta) + \int_c^\theta \mathcal{M}(\theta, \mathfrak{r}, \mathcal{K}(\vartheta(\theta)))d\mathfrak{r}, \ \forall \theta \in [c, d], \end{cases} \tag{4.1}$$

where $\mathfrak{p} : \mathcal{I} \to \mathbb{R}$ is a continuous mapping, and $\mathcal{L}, \mathcal{M} : [c, d] \times [c, d] \times \mathcal{I} \to \mathbb{R}$.

Now, we demonstrate the following theorem to verify the existence of a solution of the system of integral equations (4.1).

Theorem 4.1. *Let $\mathcal{I} = \mathcal{C}([c, d], \mathbb{R})$, and define the mappings $\Phi, \Psi, \mathcal{H}, \mathcal{K} : (\mathcal{I}, \mathcal{S}, \perp) \to (\mathcal{I}, \mathcal{S}, \perp)$ by*

$$\Phi\vartheta(\theta) = \mathfrak{p}(\theta) + \int_c^\theta \mathcal{L}(\theta, \mathfrak{r}, \mathcal{H}(\vartheta(\theta)))d\mathfrak{r} \quad \text{and} \quad \Psi\vartheta(\theta) = \mathfrak{p}(\theta) + \int_c^\theta \mathcal{M}(\theta, \mathfrak{r}, \mathcal{K}(\vartheta(\theta)))d\mathfrak{r},$$

where $\mathfrak{p} : \mathcal{I} \to \mathbb{R}$, and $\mathcal{H}, \mathcal{K} : [c, d] \times [c, d] \times \mathcal{I} \to \mathbb{R}$ are continuous. Assume that the following axioms are fulfilled.

1. *$\exists$ a continuous function $\mathcal{G} : \mathcal{I} \to [0, \infty)$, such that*
$$|\mathcal{L}(\theta, \mathfrak{r}, \mathcal{H}(\vartheta(\theta))) - \mathcal{M}(\theta, \mathfrak{r}, \mathcal{K}(w(\theta)))| \leqslant \mathcal{G}(\theta, \mathfrak{r})|\vartheta(\theta) - w(\theta)|,$$
$\forall \vartheta, w \in \mathbb{R}, \vartheta \perp w$, $\forall \theta, \mathfrak{r} \in [c, d]$, *and* $\mathcal{H}, \mathcal{K} \in \mathcal{I}$.
2. *$\exists \sup_{\theta \in [c,d]} \int_c^\theta \mathcal{G}(\theta, \mathfrak{r})d\mathfrak{r} < \tau$, for each $\tau \leqslant 1$.*
3. *$\exists$ an $\perp$-seq $\{\mathfrak{r}_\varphi\}$ in $\mathcal{I}$, such that*
$$\lim_{\varphi \to \infty} \mathcal{S}(\Phi\mathcal{H}(\mathfrak{r}_\varphi), \Phi\mathcal{H}(\mathfrak{r}_\varphi), \mathcal{H}\Phi(\mathfrak{r}_\varphi)) = 0 \quad \text{and} \quad \lim_{\varphi \to \infty} \mathcal{S}(\Psi\mathcal{K}(\mathfrak{r}_\varphi), \Psi\mathcal{K}(\mathfrak{r}_\varphi), \mathcal{K}\Psi(\mathfrak{r}_\varphi)) = 0,$$
whenever,
$$\lim_{\varphi \to \infty} \Phi(\mathfrak{r}_\varphi) = \lim_{\varphi \to \infty} \mathcal{H}(\mathfrak{r}_\varphi) = \theta \quad \text{and} \quad \lim_{\varphi \to \infty} \Psi(\mathfrak{r}_\varphi) = \lim_{\varphi \to \infty} \mathcal{K}(\mathfrak{r}_\varphi) = \theta, \quad \text{for some} \quad \theta \in \mathcal{I}.$$

Then, the Volterra-type integral system (4.1) has a unique common solution.

Proof. First, we claim that for every $\vartheta \in \mathcal{J}$, $\Phi\vartheta \in \mathcal{J}$, and $\forall\ \theta \in [c, d]$, we have

$$\Phi\vartheta(\theta) = \mathfrak{p}(\theta) + \int_c^\theta \mathcal{L}(\theta, \mathfrak{r}, \mathcal{H}(\vartheta(\theta)))d\mathfrak{r} \geqslant 1 \quad \text{and} \quad \Psi w(\theta) = \mathfrak{p}(\theta) + \int_c^\theta \mathcal{M}(\theta, \mathfrak{r}, \mathcal{K}(w(\theta)))d\mathfrak{r} \geqslant 1.$$

We conclude that $\Phi\vartheta(\theta) > 1$, and we have $\Phi\vartheta \in \mathcal{J}$. Now, we aim to show that

1. $\exists\ \vartheta_0 \in \mathcal{J}$ such that $\vartheta_0 \perp \Phi\vartheta$, $\forall\vartheta \in \mathcal{J}$, and $\vartheta_0 \perp \Psi\vartheta$, $\forall\vartheta \in \mathcal{J}$;
2. $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $\perp$-preserving;
3. $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $(\mathcal{S}, \perp)$-contraction;
4. $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $\perp$-continuous.

Here, we are going to prove the above issues as following.

1. Putting $\vartheta_0 = \mathfrak{a}$ (the constant function ϑ_0), we have $\mathfrak{a} \perp \Phi\vartheta$, $\mathfrak{a} \perp \Psi\vartheta$, $\mathfrak{a} \perp \mathcal{H}\vartheta$ and $\mathfrak{a} \perp \mathcal{K}\vartheta\ \forall\vartheta \in \mathcal{J}$.
2. Φ is $\perp_{\mathcal{P}}$ if for every $\vartheta, w \in \mathcal{J}$, $\vartheta \perp w$, we have $\Phi\vartheta \perp \Phi w$. Also, we get $\Phi\vartheta(\theta) > 1$, $\forall\theta \in [c, d]$ $\Rightarrow \Phi\vartheta(\theta)\Phi w(\theta) \geqslant \Phi w(\theta)$, $\forall\theta \in [c, d]$, therefore $\Phi\vartheta \perp \Phi w$. Similarly, Ψ is $\perp_{\mathcal{P}}$ if for every $\vartheta, w \in \mathcal{J}$, $\vartheta \perp w$, we have $\Psi\vartheta \perp \Psi w$. Further, we find $\Psi\vartheta(\theta) > 1$, $\forall\theta \in [c, d] \Rightarrow \Psi\vartheta(\theta)\Psi w(\theta) \geqslant \Psi w(\theta)$, $\forall\theta \in [c, d]$, therefore $\Psi\vartheta \perp \Psi w$. Since the pairs $\{\Phi, \mathcal{H}\}$ and $\{\Psi, \mathcal{K}\}$ are compatible. Hence, $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $\perp_{\mathcal{P}}$.
3. Let $\vartheta, w \in \mathcal{J}$, $\vartheta \perp w$, and $\theta \in [c, d]$, we have

$$\begin{aligned} |\Phi\vartheta(\theta) - \Psi w(\theta)| &= \left|\mathfrak{p}(\theta) + \int_c^\theta \mathcal{L}(\theta, \mathfrak{r}, \mathcal{H}(\vartheta(\theta)))d\mathfrak{r} - \mathfrak{p}(\theta) - \int_c^\theta \mathcal{M}(\theta, \mathfrak{r}, \mathcal{K}(w(\theta)))d\mathfrak{r}\right| \\ &= \left|\int_c^\theta \mathcal{L}(\theta, \mathfrak{r}, \mathcal{H}(\vartheta(\theta)))d\mathfrak{r} - \int_c^\theta \mathcal{M}(\theta, \mathfrak{r}, \mathcal{K}(w(\theta)))d\mathfrak{r}\right| \\ &\leqslant \int_c^\theta \left|\mathcal{L}(\theta, \mathfrak{r}, \mathcal{H}(\vartheta(\theta))) - \mathcal{M}(\theta, \mathfrak{r}, \mathcal{K}(w(\theta)))\right| d\mathfrak{r} \leqslant \int_c^\theta \mathcal{G}(\theta, \mathfrak{r})|\vartheta(\theta) - w(\theta)|d\mathfrak{r}. \end{aligned}$$

   Hence,

$$\sup_{\theta\in[c,d]} |\Phi\vartheta(\theta) - \Psi w(\theta)| \leqslant \sup_{\theta\in[c,d]} |\vartheta(\theta) - w(\theta)| \sup_{\theta\in[c,d]} \int_c^\theta \mathcal{G}(\theta, \mathfrak{r})d\mathfrak{r} \leqslant \tau \sup_{\theta\in[c,d]} |\vartheta(\theta) - w(\theta)|.$$

   Also, we have

$$\mathcal{S}(\Phi\vartheta, \Phi\vartheta, \Psi w) = 2 \sup_{\theta\in[c,d]} |\Phi\vartheta(\theta) - \Psi w(\theta)| \leqslant 2\tau \sup_{\theta\in[c,d]} |\vartheta(\theta) - w(\theta)| = \tau\mathcal{S}(\vartheta, \vartheta, w).$$

   Therefore, $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $(\mathcal{S}, \perp)$-contraction with $\tau < 1$.
4. Let $\{\vartheta_\varphi\}$ be an $\perp$-seq in $\mathcal{J}$, such that $\{\vartheta_\varphi\}$ converges to some $\vartheta \in \mathcal{J}$. Since $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $\perp_{\mathcal{P}}$, and $\{\Phi\vartheta_\varphi\}$ is an $\perp$-seq. Then, for each $\varphi \in \mathbb{N}$, we have

$$|\Phi\vartheta_\varphi - \Psi\vartheta| < \tau|\vartheta_\varphi - \vartheta|, \quad 0 < \tau < 1.$$

   As $\varphi \to \infty$, we conclude that $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $\perp_{\mathcal{C}}$.

Hence, the mappings $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ satisfy the hypotheses of the Theorem 3.2. Therefore, the Volterra-type integral system (4.1) has a unique common solution. □

4.2. Application to fractional differential equations

Consider the Caputo fractional derivative using fractional differential equation.

$$\begin{cases} \mathbb{D}^{\beta}_{0+}\vartheta(z) + r_1(z, \mathcal{H}(\vartheta(z))) = 0, & 0 < z < 1, \\ \mathbb{D}^{\beta}_{0+}\vartheta(z) + r_2(z, \mathcal{K}(\vartheta(z))) = 0, & 0 < z < 1, \end{cases} \tag{4.2}$$

where, $1 < \beta \leqslant 2, \vartheta(0) + \vartheta'(0) = 0, \vartheta(1) + \vartheta'(1) = 0$ are the boundary conditions with $r_1, r_2 \colon [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. Let $\mathcal{Q} = \mathcal{C}([0,1], \mathbb{R})$. Define

$$\mathcal{S}(\vartheta, w, \eta) = \sup_{z\in[0,1]} |\vartheta(z) - w(z)| + \sup_{z\in[0,1]} |\vartheta(z) - \eta(z)| + \sup_{z\in[0,1]} |w(z) - \eta(z)|, \ \forall \vartheta, w, \eta \in \mathcal{Q},$$

with $\perp$ given by

$$\vartheta \perp \eta \iff \vartheta(z)\eta(z) \geqslant \vartheta(z) \ \text{ or } \ \vartheta(z)\eta(z) \geqslant \eta(z),$$

for all $z \in [0,1]$. Then $(\mathcal{Q}, \mathcal{S}, \perp)$ is a complete $\mathcal{OSMS}$. Note that $\vartheta \in \mathcal{Q}$ solves (4.2) whenever $\vartheta \in \mathcal{Q}$ is the solution of

$$\begin{cases} \vartheta(z) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1}(1-z) r_1(t, \mathcal{H}(\vartheta(t)))dt + \frac{1}{\Gamma(\beta-1)} \int_0^1 (1-t)^{\beta-2}(1-z) r_1(t, \mathcal{H}(\vartheta(t)))dt \\ \qquad + \frac{1}{\Gamma(\beta)} \int_0^z (z-t)^{\beta-1} r_1(t, \mathcal{H}(\vartheta(t)))dt, \\ \vartheta(z) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1}(1-z) r_2(t, \mathcal{K}(\vartheta(t)))dt + \frac{1}{\Gamma(\beta-1)} \int_0^1 (1-t)^{\beta-2}(1-z) r_2(t, \mathcal{K}(\vartheta(t)))dt \\ \qquad + \frac{1}{\Gamma(\beta)} \int_0^z (z-t)^{\beta-1} r_2(t, \mathcal{K}(\vartheta(t)))dt, \ \forall z \in [0,1]. \end{cases}$$

Theorem 4.2. *Let the mappings $\Phi, \Psi, \mathcal{H}, \mathcal{K} : (\mathcal{Q}, \mathcal{S}, \perp) \to (\mathcal{Q}, \mathcal{S}, \perp)$ as, $\forall z \in [0,1]$,*

$$\begin{cases} \Phi(\vartheta(z)) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1}(1-z) r_1(t, \mathcal{H}(\vartheta(t)))dt \\ \qquad + \frac{1}{\Gamma(\beta-1)} \int_0^1 (1-t)^{\beta-2}(1-z) r_1(t, \mathcal{H}(\vartheta(t)))dt + \frac{1}{\Gamma(\beta)} \int_0^z (z-t)^{\beta-1} r_1(t, \mathcal{H}(\vartheta(t)))dt, \\ \Psi(\vartheta(z)) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1}(1-z) r_2(t, \mathcal{K}(\vartheta(t)))dt \\ \qquad + \frac{1}{\Gamma(\beta-1)} \int_0^1 (1-t)^{\beta-2}(1-z) r_2(t, \mathcal{K}(\vartheta(t)))dt + \frac{1}{\Gamma(\beta)} \int_0^z (z-t)^{\beta-1} r_2(t, \mathcal{K}(\vartheta(t)))dt. \end{cases}$$

Suppose the conditions

1. *for all $\vartheta, \eta \in \mathcal{Q}, r_1, r_2 \colon [0,1] \times [0,\infty) \to [0,\infty)$, and $\wp > 0$ satisfies*
$$|r_1(t, \mathcal{H}(\vartheta(t))) - r_2(t, \mathcal{K}(\eta(t)))| \leqslant \wp |\vartheta(t) - \eta(t)|,$$
2. $\sup_{z\in[0,1]} \left| \frac{1-z}{\Gamma(\beta+1)} + \frac{1-z}{\Gamma(\beta)} + \frac{z^{\beta}}{\Gamma(\beta+1)} \right| = \xi < 1,$

hold. Then, equation (4.2) has a unique solution.

Proof. First, we claim that for every $\vartheta \in \mathcal{Q}$, $\Phi\vartheta \in \mathcal{Q}$, and $\forall \ z \in [0,1]$, we have

$$\begin{aligned} \Phi\vartheta(z) &= \frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1}(1-z) r_1(t, \mathcal{H}(\vartheta(t)))dt + \frac{1}{\Gamma(\beta-1)} \int_0^1 (1-t)^{\beta-2}(1-z) r_1(t, \mathcal{H}(\vartheta(t)))dt \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^z (z-t)^{\beta-1} r_1(t, \mathcal{H}(\vartheta(t)))dt \geqslant 1, \\ \Psi\eta(z) &= \frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1}(1-z) r_2(t, \mathcal{K}(\eta(t)))dt + \frac{1}{\Gamma(\beta-1)} \int_0^1 (1-t)^{\beta-2}(1-z) r_2(t, \mathcal{K}(\eta(t)))dt \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^z (z-t)^{\beta-1} r_2(t, \mathcal{K}(\eta(t)))dt \geqslant 1. \end{aligned}$$

We conclude that $\Phi\vartheta(z) > 1$, and we have $\Phi\vartheta \in \mathcal{Q}$. Also, $\Psi\eta(z) > 1$, and we have $\Psi\eta \in \mathcal{Q}$. Now, our aim is to show that

1. $\exists\, \vartheta_0 \in \mathcal{Q}$ such that $\vartheta_0 \perp \Phi\vartheta$, $\forall \vartheta \in \mathcal{Q}$, and $\vartheta_0 \perp \Psi\vartheta$, $\forall \vartheta \in \mathcal{Q}$;
2. $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $\perp$-preserving;
3. $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $(\mathcal{S}, \perp)$-contraction;
4. $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $\perp$-continuous.

Here, we are going to prove the above issues as following.

1. Putting $\vartheta_0 = a$ (the constant function ϑ_0), we have $a \perp \Phi\vartheta, a \perp \Psi\vartheta, a \perp \mathcal{H}\vartheta$ and $a \perp \mathcal{K}\vartheta\ \forall \vartheta \in \mathcal{Q}$.
2. Φ is $\perp_{\mathcal{P}}$ if for every $\vartheta, w \in \mathcal{Q}$, $\vartheta \perp w$, we have $\Phi\vartheta \perp \Phi w$. Also, we get $\Phi\vartheta(z) > 1$, $\forall z \in [0,1]$ $\Rightarrow \Phi\vartheta(z)\Phi w(z) \geqslant \Phi w(z)$, $\forall z \in [0,1]$, therefore $\Phi\vartheta \perp \Phi w$. Similarly, Ψ is $\perp_{\mathcal{P}}$ if for every $\vartheta, w \in \mathcal{J}$, $\vartheta \perp w$, we have $\Psi\vartheta \perp \Psi w$. Further, we find $\Psi\vartheta(z) > 1$, $\forall z \in [0, \mathfrak{f}] \Rightarrow \Psi\vartheta(z)\Psi w(z) \geqslant \Psi w(z)$, $\forall z \in [0, \mathfrak{f}]$, therefore $\Psi\vartheta \perp \Psi w$. Since the pairs $\{\Phi, \mathcal{H}\}$, and $\{\Psi, \mathcal{K}\}$ are compatible. Hence, $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $\perp_{\mathcal{P}}$.
3. Let $\vartheta, \eta \in \mathcal{Q}$ and consider

$$\begin{aligned}
|\Phi\vartheta(z) - \Psi\eta(z)| &= \Bigg| \frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1}(1-z)(r_1(t, \mathcal{H}(\vartheta(t))) - r_2(t, \mathcal{K}(\eta(t))))dt \\
&\quad + \frac{1}{\Gamma(\beta-1)} \int_0^1 (1-t)^{\beta-2}(1-z)(r_1(t, \mathcal{H}(\vartheta(t))) - r_2(t, \mathcal{K}(\eta(t))))dt \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^z (z-t)^{\beta-1}(r_1(t, \mathcal{H}(\vartheta(t))) - r_2(t, \mathcal{K}(\eta(t))))dt \Bigg| \\
&\leqslant \frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1}(1-z) \Big| (r_1(t, \mathcal{H}(\vartheta(t))) - r_2(t, \mathcal{K}(\eta(t)))) \Big| dt \\
&\quad + \frac{1}{\Gamma(\beta-1)} \int_0^1 (1-t)^{\beta-2}(1-z) \Big| (r_1(t, \mathcal{H}(\vartheta(t))) - r_2(t, \mathcal{K}(\eta(t)))) \Big| dt \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^z (z-t)^{\beta-1} \Big| (r_1(t, \mathcal{H}(\vartheta(t))) - r_2(t, \mathcal{K}(\eta(t)))) \Big| dt \\
&\leqslant \frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1}(1-z)\wp|\vartheta(t) - \eta(t)|dt \\
&\quad + \frac{1}{\Gamma(\beta-1)} \int_0^1 (1-t)^{\beta-2}(1-z)\wp|\vartheta(t) - \eta(t)|dt \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^z (z-t)^{\beta-1}\wp|\vartheta(t) - \eta(t)|dt \\
&= \wp|\vartheta(z) - \eta(z)| \Bigg(\frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1}(1-z)dt \\
&\quad + \frac{1}{\Gamma(\beta-1)} \int_0^1 (1-t)^{\beta-2}(1-z)dt + \frac{1}{\Gamma(\beta)} \int_0^z (z-t)^{\beta-1}dt \Bigg) \\
&= \wp|\vartheta(z) - \eta(z)| \left(\frac{1-z}{\Gamma(\beta+1)} + \frac{1-z}{\Gamma(\beta)} + \frac{z^\beta}{\Gamma(\beta+1)} \right) \\
&\leqslant \wp|\vartheta(z) - \eta(z)| \sup_{z \in [0,1]} \left(\frac{1-z}{\Gamma(\beta+1)} + \frac{1-z}{\Gamma(\beta)} + \frac{z^\beta}{\Gamma(\beta+1)} \right),
\end{aligned}$$

so, we have

$$\Big| \Phi\vartheta(z) - \Psi\eta(z) \Big| = \xi\wp|\vartheta(z) - \eta(z)| \leqslant \wp|\vartheta(z) - \eta(z)|,$$

i.e.,

$$\sup_{z \in [0,1]} \Big| \Phi\vartheta(z) - \Psi\eta(z) \Big| \leqslant \wp \sup_{z \in [0,1]} |\vartheta(z) - \eta(z)|,$$

thus, we have

$$\mathcal{S}(\Phi\vartheta,\Phi\vartheta,\Psi\eta)=2\sup_{z\in[0,1]}|\Phi\vartheta(z)-\Psi\eta(z)|\leqslant 2\wp\sup_{z\in[0,\mathfrak{f}]}|\vartheta(z)-\eta(z)|=\wp\mathcal{S}(\vartheta,\vartheta,\eta).$$

Therefore, $\Phi,\Psi,\mathcal{H}$, and $\mathcal{K}$ are $(\mathcal{S},\perp)$-contraction with $0<\wp<1$.

4. Let $\{\vartheta_\varphi\}$ be an $\perp$-seq in $\mathcal{Q}$, such that $\{\vartheta_\varphi\}$ converges to some $\vartheta\in\mathcal{Q}$. Since $\Phi,\Psi,\mathcal{H}$, and $\mathcal{K}$ are $\perp_{\mathcal{P}}$; and $\{\Phi\vartheta_\varphi\}$ is an $\perp$-seq. Then, for each $\varphi\in\mathbb{N}$, we have

$$|\Phi\vartheta_\varphi-\Psi\vartheta|<\wp|\vartheta_\varphi-\vartheta|,\quad 0<\wp<1.$$

As $\varphi\to\infty$, we conclude that $\Phi,\Psi,\mathcal{H}$, and $\mathcal{K}$ are $\perp_{\mathcal{C}}$.

Hence, the mappings $\Phi,\Psi,\mathcal{H}$, and $\mathcal{K}$ satisfy the hypotheses of the Theorem 3.2. Therefore, the fractional differential equation (4.2) has a unique common solution. □

5. Application in production-consumption equilibrium

Our results are applied to the dynamic market equilibrium problem, an important economics topic, where we solve an initial value problem and develop a mathematical model. Daily pricing trends and prices show an important effect on markets for both production $\mathcal{B}_\wp$ and consumption $\mathcal{B}_\mathfrak{c}$, despite price movements. Consequently, the economist is interested in knowing the current price $\vartheta(z)$. Let us consider

$$\mathcal{B}_\wp=t_1+\varpi_1\vartheta(z)+\xi_1\frac{d\vartheta(z)}{dz}+\mu_1\frac{d^2\vartheta(z)}{dz^2},\qquad \mathcal{B}_\mathfrak{c}=t_2+\varpi_2\vartheta(z)+\xi_2\frac{d\vartheta(z)}{dz}+\mu_2\frac{d^2\vartheta(z)}{dz^2},$$

initially $\vartheta(0)=0,\frac{d\vartheta}{dz}(0)=0$, where $t_1,t_2,\varpi_1,\varpi_2,\xi_1,\xi_2,\mu_1$, and μ_2 are constants. A state of dynamic economic equilibrium occurs when market forces are in balance, meaning that the current gap between production and consumption stabilizes, that is, $\mathcal{B}_\wp=\mathcal{B}_\mathfrak{c}$. Thus,

$$\begin{aligned}
t_1+\varpi_1\vartheta(z)+\xi_1\frac{d\vartheta(z)}{dz}+\mu_1\frac{d^2\vartheta(z)}{dz^2}&=t_2+\varpi_1\vartheta(z)+\xi_2\frac{d\vartheta(z)}{dz}+\mu_2\frac{d^2\vartheta(z)}{dz^2},\\
(t_1-t_2)+(\varpi_1-\varpi_2)\vartheta(z)+(\xi_1-\xi_2)\frac{d\vartheta(z)}{dz}+(\mu_1-\mu_2)\frac{d^2\vartheta(z)}{dz^2}&=0,\\
\mu\frac{d^2\vartheta(z)}{dz^2}+\xi\frac{d\vartheta(z)}{dz}+\varpi\vartheta(z)&=-t,\\
\frac{d^2\vartheta(z)}{dz^2}+\frac{\xi}{\mu}\frac{d\vartheta(z)}{dz}+\frac{\varpi}{\mu}\vartheta(z)&=-\frac{t}{\mu},
\end{aligned}$$

where $t=t_1-t_2$, $\varpi=\varpi_1-\varpi_2$, $\xi=\xi_1-\xi_2$, and $\mu=\mu_1-\mu_2$. Our initial value problem is now represented as follows:

$$\begin{cases}\vartheta''(z)+\frac{\xi}{\mu}\vartheta'(z)+\frac{\varpi}{\mu}\mathcal{H}(\vartheta(z))=-\frac{t}{\mu},\\ \vartheta''(z)+\frac{\xi}{\mu}\vartheta'(z)+\frac{\varpi}{\mu}\mathcal{K}(\vartheta(z))=-\frac{t}{\mu},\ \text{with } \vartheta(0)=0 \text{ and } \vartheta'(0)=0.\end{cases}\tag{5.1}$$

Now, we study the production and consumption duration time $\mathfrak{f}$, problem (5.1) is equivalent to the system of equations:

$$\begin{cases}\vartheta(z)=\int_0^\mathfrak{f}\mathcal{F}(z,z^*)\mathcal{A}_1(z^*,z,\mathcal{H}(\vartheta(z)))dz,\\ \vartheta(z)=\int_0^\mathfrak{f}\mathcal{F}(z,z^*)\mathcal{A}_2(z^*,z,\mathcal{K}(\vartheta(z)))dz,\ \forall z\in[0,\mathfrak{f}],\end{cases}$$

where Green function $\mathcal{F}(z,z^*)$ is

$$\mathcal{F}(z,z^*)=\begin{cases}z\hbar^{\frac{\varpi}{2\xi}}(z^*-z), & 0\leqslant z\leqslant\mathfrak{e}\leqslant\mathfrak{f},\\ \mathfrak{e}\hbar^{\frac{\varpi}{2\xi}}(z-z^*), & 0\leqslant\mathfrak{e}\leqslant z\leqslant\mathfrak{f}.\end{cases}$$

Let $\mathcal{J} = \mathcal{C}([0,\mathfrak{f}],\mathbb{R})$ and $\mathcal{A}_1, \mathcal{A}_2 \colon [0,\mathfrak{f}] \times \mathcal{J} \times \mathcal{J} \to \mathbb{R}$ are continuous functions. Define

$$\mathcal{S}(\vartheta, w, \eta) = \sup_{z\in[0,\mathfrak{f}]} |\vartheta(z) - w(z)| + \sup_{z\in[0,\mathfrak{f}]} |\vartheta(z) - \eta(z)| + \sup_{z\in[0,\mathfrak{f}]} |w(z) - \eta(z)|, \ \forall \vartheta, w, \eta \in \mathcal{C}([0,\mathfrak{f}],\mathbb{R}),$$
$$\mathcal{S}(\vartheta,\vartheta,\eta) = 2 \sup_{z\in[0,\mathfrak{f}]} |\vartheta(z) - \eta(z)|,$$

$\forall \vartheta, \eta \in \mathcal{J}$ with $\perp$ given by

$$\vartheta \perp \eta \iff \vartheta(z)\eta(z) \geqslant \vartheta(z) \ \text{ or } \ \vartheta(z)\eta(z) \geqslant \eta(z),$$

$\forall z \in [0,\mathfrak{f}]$. Then $(\mathcal{J}, \mathcal{S}, \perp)$ is a complete OSMS. Also $\Phi, \Psi, \mathcal{H}, \mathcal{K} : (\mathcal{J}, \mathcal{S}, \perp) \to (\mathcal{J}, \mathcal{S}, \perp)$ is defined by

$$\begin{cases} \Phi(\vartheta(z)) = \int_0^{\mathfrak{f}} \mathcal{F}(z, z^*)\mathcal{A}_1(z^*, z, \mathcal{H}(\vartheta(z)))dz, \\ \Psi(\vartheta(z)) = \int_0^{\mathfrak{f}} \mathcal{F}(z, z^*)\mathcal{A}_2(z^*, z, \mathcal{K}(\vartheta(z)))dz, \ \forall z \in [0,\mathfrak{f}]. \end{cases} \tag{5.2}$$

Let us consider, the solution to the dynamic market equilibrium problem, which is represented as (5.1), is a common fixed point of $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ (5.2). Now, the current price $\vartheta(z)$ is given by (5.1).

Theorem 5.1. *Consider the operators $\Phi, \Psi, \mathcal{H}, \mathcal{K} : (\mathcal{J}, \mathcal{S}, \perp) \to (\mathcal{J}, \mathcal{S}, \perp)$ be complete $\mathcal{OSMS}$, satisfying*

1. *we can find $z \in [0,\mathfrak{f}]$ and $0 < \wp < 1$ such that $|\mathcal{A}_1(z^*, z, \mathcal{H}(\vartheta(z))) - \mathcal{A}_2(z^*, z, \mathcal{K}(\vartheta(z)))| \leqslant \frac{\wp}{\mathfrak{f}} |\vartheta(z) - w(z)|$;*
2. *a continuous function $\mathcal{F} \colon [0,\mathfrak{f}] \times [0,\mathfrak{f}] \to \mathbb{R}$ that satisfies $\sup_{z\in[0,\mathfrak{f}]} \int_0^{\mathfrak{f}} \mathcal{F}(z, z^*)dz \leqslant 1$.*

Then, the dynamic market equilibrium problem (5.1) has exactly one solution.

Proof. First, we claim that for every $\vartheta \in \mathcal{J}$, $\Phi\vartheta \in \mathcal{J}$ and $\forall$ $z \in [0,\mathfrak{f}]$, we have

$$\Phi(\vartheta(z)) = \int_0^{\mathfrak{f}} \mathcal{F}(z, z^*)\mathcal{A}_1(z^*, z, \mathcal{H}(\vartheta(z)))dz \geqslant 1 \quad \text{and} \quad \Psi(w(z)) = \int_0^{\mathfrak{f}} \mathcal{F}(z, z^*)\mathcal{A}_2(z^*, z, \mathcal{K}(w(z)))dz \geqslant 1.$$

We conclude that $\Phi\vartheta(\theta) > 1$, and we have $\Phi\vartheta \in \mathcal{J}$. Also, $\Psi w(z) > 1$, and we have $\Psi w \in \mathcal{J}$. Now, our aim is to show that

1. $\exists\, \vartheta_0 \in \mathcal{J}$ such that $\vartheta_0 \perp \Phi\vartheta, \forall \vartheta \in \mathcal{J}$, and $\vartheta_0 \perp \Psi\vartheta, \forall \vartheta \in \mathcal{J}$;
2. $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $\perp$-preserving;
3. $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $(\mathcal{S}, \perp)$-contraction;
4. $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $\perp$-continuous.

Here, we are going to prove the above issues as following. Putting $\vartheta_0 = a$ (the constant function ϑ_0), we have $a \perp \Phi\vartheta, a \perp \Psi\vartheta, a \perp \mathcal{H}\vartheta$ and $a \perp \mathcal{K}\vartheta$ $\forall \vartheta \in \mathcal{J}$. Clearly, $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $\perp$-preserving and $\perp$-continuous. Let $\vartheta, w \in \mathcal{J}$, $\vartheta \perp w$, and $z \in [0,\mathfrak{f}]$, we have

$$\begin{aligned} |\Phi\vartheta(z) - \Psi w(z)| &= \left| \int_0^{\mathfrak{f}} \mathcal{F}(z, z^*)\mathcal{A}_1(z^*, z, \mathcal{H}(\vartheta(z)))dz - \int_0^{\mathfrak{f}} \mathcal{F}(z, z^*)\mathcal{A}_2(z^*, z, \mathcal{K}(w(z)))dz \right| \\ &\leqslant \int_0^{\mathfrak{f}} \mathcal{F}(z, z^*) \left|\mathcal{A}_1(z^*, z, \mathcal{H}(\vartheta(z))) - \mathcal{A}_2(z^*, z, \mathcal{K}(w(z)))\right| dz \\ &\leqslant \int_0^{\mathfrak{f}} \mathcal{F}(z, z^*)dz \int_0^{\mathfrak{f}} \left|\mathcal{A}_1(z^*, z, \mathcal{H}(\vartheta(z))) - \mathcal{A}_2(z^*, z, \mathcal{K}(w(z)))\right| dz \\ &\leqslant \frac{\wp}{\mathfrak{f}} |\vartheta(z) - w(z)|(\mathfrak{f}) \int_0^{\mathfrak{f}} \mathcal{F}(z, z^*)dz \leqslant \wp|\vartheta(z) - w(z)| \int_0^{\mathfrak{f}} \mathcal{F}(z, z^*)dz. \end{aligned}$$

Now,

$$\sup_{z\in[0,\mathfrak{f}]} |\Phi\vartheta(z) - \Psi w(z)| \leqslant \wp \sup_{z\in[0,\mathfrak{f}]} |\vartheta(z) - w(z)| \sup_{z\in[0,\mathfrak{f}]} \int_0^{\mathfrak{f}} \mathcal{F}(z, z^*)dz \leqslant \wp \sup_{z\in[0,\mathfrak{f}]} |\vartheta(z) - w(z)|.$$

Also, we have

$$\mathcal{S}(\Phi\vartheta, \Phi\vartheta, \Psi w) = 2 \sup_{z\in[0,\mathfrak{f}]} |\Phi\vartheta(z) - \Psi w(z)| \leqslant 2\wp \sup_{z\in[0,\mathfrak{f}]} |\vartheta(z) - w(z)| = \wp\mathcal{S}(\vartheta, \vartheta, w).$$

Therefore, $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $(\mathcal{S}, \perp)$-contraction with $0 < \wp < 1$. Let $\{\vartheta_\varphi\}$ be an $\perp$-seq in $\mathcal{J}$, such that $\{\vartheta_\varphi\}$ converges to some $\vartheta \in \mathcal{J}$. Since $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $\perp_{\mathcal{P}}$; and $\{\Phi\vartheta_\varphi\}$ is an $\perp$-seq. Then, for each $\varphi \in \mathbb{N}$, we have

$$|\Phi\vartheta_\varphi - \Psi\vartheta| < \wp|\vartheta_\varphi - \vartheta|, \quad 0 < \wp < 1.$$

As $\varphi \to \infty$, we conclude that $\Phi, \Psi, \mathcal{H}$, and $\mathcal{K}$ are $\perp_{\mathcal{C}}$. Hence, the dynamic market equilibrium problem (5.1) has exactly one solution. □

6. Conclusion

In this article, we have established CFP theorems of four mappings on OSMS. Moreover, the practical significance of our theoretical result is demonstrated through a concrete example, showcasing its applicability in real-world scenarios. Finally, the existence and uniqueness result of Volterra-type integral system, production-consumption equilibrium, and fractional differential equations are enhanced to illustrate the applications of our finding. Future research could explore applying these results to other integral and differential equations, potentially unlocking new directions for theoretical and practical progress in mathematical analysis. Additionally, extending our CFP theorems to other types of metric spaces, such as probabilistic metric spaces, fuzzy metric spaces, or those with different ordering structures, could further broaden the scope of their applicability.

Funding

"La derivada fraccional generalizada, nuevos resultados y aplicaciones a desigualdades integrales" Cod UIO-077-2024. This study is supported via funding from Prince Sattam bin Abdulaziz University project number PSAU/2024/R/1446.

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