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On the double Laplace-typed integral transform for solving partial differential equations via quantum calculus

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Abstract

In this work, we extend double Laplace-type integral transform (DLTT) and apply the concept of quantum calculus or q-calculus to establish the definitions of double q-Laplace-type integral transform (q-DLTT). We present some main properties and compute some functions of the q-DLTT. Also, we apply them to establish simple formulas for the general solution to solve the q-PDEs, such as q-wave equation, q-Klein-Gordon equation, q-Laplace equation, q-heat equation, and q-telegraph equation. By using the Mathematica program, the 3D plots, which represent for the q-PDEs, are provided. Finally, a convergence comparison between the exact solution and q-calculus of PDEs is also given in this research.

Keywords: q-Laplace-typed integral transform, q-derivative, q-integral, q-calculus, q-patial differential equations, q-convolution theorem.

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1. Introduction

Integral transform methods are among the essential techniques implemented in solving PDEs. Many phenomena in mathematical physics and scientific fields can be written in terms of PDEs, such as wave equation, Klein-Gordon equation, Laplace equation, heat equation, and telegraph equation. Researchers and scientists have worked very hard to find exact solutions of PDEs using integral transform. Several integral transforms, such as Laplace transform [38], Sumudu transform [42], σ-integral Laplace transform [33], Elzaki transform [20], Mohand transform [1], Sawi transform [2], Kamal transform [30], Laplace-type integral transform (or G-transform) [32], and Pourreza transform [7]. Recently, the double integral transform has been applied to solve PDEs with unknown functions of two variables. Furthermore, the double transform has been developed in the relevant literature, such as double Laplace transform [16, 18], double Sumudu transform [19, 41], double Shehu transform [3], and double Elzaki transform [24].

Quantum calculus (also called q-calculus) is the study of calculus without limits. It can be reduced to ordinary calculus if we take the limit q tends to be 1. Euler was the first mathematician to study q-calculus in the early eighteenth (1707-1783). In 1910, Jackson [25, 26] introduced the q-derivative and the q-integral. The q-calculus has a lot of applications in several mathematical areas, such as number theory,

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combinatorics, quantum theory, mechanics, basic hypergeometric functions, orthogonal polynomials, and theory of relativity, see for instance [4–6, 14, 17, 21, 27, 34, 39, 40] and their respective references. Also, the fundamental explanation of the q-calculus aspects can be found in the book by Kac and Cheung [29].

Around a decade ago, the topic of the q-integral transform interested many researchers, leading to various investigation forms. Many researchers study the properties of q-integral transform such as q-Laplace transform of two variables [23], q-Sumudu transform [9], q-Laplace transform [15], q-natural transform [11], q-Laplace-type integral transform [12], and q-Mellin transform [13].

We are mainly inspired by a series of the above-mentioned literature, and therefore, propose to extend the Laplace-type integral transform (or G-transform) to introduce double q-Laplace-type integral transform (double G_q-transform or q-DLTT). We present some main properties of the q-DLTT and compute it for some functions. The q-convolution theorem of two functions and some properties with proofs are introduced, and to apply those properties to solve q-PDEs. Finally, a convergence comparison between the exact solution and q-calculus of PDEs is also provided in this paper. Note that, we will frequently use some properties of paper [28] when p=1.

In this paper, we consider the nonhomogeneous linear PDE in q-calculus as follows:

$$A\frac{\partial_q^2 F}{\partial_q \xi^2}(\xi,\eta) + B\frac{\partial_q^2 F}{\partial_q \eta^2}(\xi,\eta) + C\frac{\partial_q F}{\partial_q \xi}(\xi,\eta) + D\frac{\partial_q F}{\partial_q \eta}(\xi,\eta) + EF(\xi,\eta) = \omega(\xi,\eta).$$

with the initial conditions

$$F(\xi,0) = F_1(\xi), \quad \frac{\partial_q g}{\partial_q \eta}(\xi,0) = F_2(\xi),$$

and the boundary conditions

$$F(0,\eta) = g_1(\eta), \quad \frac{\partial_q g}{\partial_q \xi}(0,\eta) = g_2(\eta),$$

where A, B, C, D, E are constants, $F(\xi,\eta)$ is the unknow function, and $\omega(\xi,\eta)$ is the source term.

2. Basic knowledge of q-calculus

In this section, some mathematical symbols used in the q-calculus are denoted to be of help in further study and can be found in [29, 32, 37]. In the entirety of this work, let $0 < q < 1$ be constants. If $q \to 1$, then we can reduce q of any forms to classical calculus. First, we present the following definition. The q-number for $n \in \mathbb{N}$ is defined by

$$[n]_q = 1 + q + \cdots + q^{n-1} = \begin{cases} \frac{1-q^n}{1-q}, & \text{if } q \neq 1, \\ n, & \text{if } q \to 1. \end{cases}$$

Also, we write the q-factorial for $n \in \mathbb{N}$ is defined as

$$[n]_q! = \begin{cases} \prod_{j=1}^n [j]_q = [1]_q[2]_q \cdots [n]_q, & \text{if } n \geqslant 1, \\ 1, & \text{if } n = 0. \end{cases}$$

The q-derivatives of a function $F : [0,\infty) \to \mathbb{R}$ is defined by

$$D_q F(\xi) = \frac{F(\xi) - F(q\xi)}{(1-q)\xi}, \quad \xi \neq 0, \quad D_q F(0) = \lim_{\xi \to 0} D_q F(\xi).$$

The q-binomial coefficients are defined by

$$\binom{n}{j}_q = \frac{[n]_q!}{[j]_q[n-j]_q!}, \quad 0 \leqslant j \leqslant n.$$

The q-addition are defined by

$$(x+y)_q^n = \sum_{j=0}^{n} \binom{n}{j}_q x^j y^{n-j}. \tag{2.1}$$

The q-derivatives of higher order are given by

$$\left(D_q^0 \mathcal{F}\right)(\xi) = \mathcal{F}(\xi) \quad \text{and} \quad \left(D_q^j \mathcal{F}\right)(\xi) = D_q\left(D_q^{j-1}\mathcal{F}\right)(\xi), \quad j \in \mathbb{N}.$$

The q-derivatives of the product and quotient rules of two functions are as follows:

$$D_q(\mathcal{F}(\xi)g(\xi)) = \mathcal{F}(\xi)D_q g(\xi) + g(q\xi)D_q\mathcal{F}(\xi),$$

and

$$D_q\left(\frac{\mathcal{F}(\xi)}{g(\xi)}\right) = \frac{g(q\xi)D_q\mathcal{F}(\xi) - \mathcal{F}(q\xi)D_q g(\xi)}{g(\xi)g(q\xi)}, \quad g(\xi) \neq 0.$$

The q-integral from 0 to c and from 0 to ∞ of $\mathcal{F}$ are defined as

$$\int_0^c \mathcal{F}(\xi)d_q\xi = (1-q)c\sum_{k=0}^{\infty} q^k \mathcal{F}\left(q^k c\right)$$

and

$$\int_0^{\infty} \mathcal{F}(\xi)d_q\xi = (1-q)\sum_{k=-\infty}^{\infty} q^k \mathcal{F}\left(q^k\right),$$

respectively. The q-integral in an interval $[a, b]$ of $D_q\mathcal{F}$ is given by

$$\int_a^b D_q\mathcal{F}(\xi)d_q\xi = \mathcal{F}(b) - \mathcal{F}(a).$$

The q-integration by parts is represented as

$$\int_a^b g(q\xi)D_q\mathcal{F}(\xi)d_q\xi = \mathcal{F}(b)g(b) - \mathcal{F}(a)g(a) - \int_a^b \mathcal{F}(\xi)(D_q g(\xi))d_q\xi. \tag{2.2}$$

The two types of q-exponential functions are defined as follows:

$$e_q(\xi) = \sum_{n=0}^{\infty} \frac{1}{[n]_q!}\xi^n, \tag{2.3}$$

$$E_q(\xi) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_q!}\xi^n.$$

In addition, the derivative of the q-exponential functions are given as

$$D_q e_q(n\xi) = n e_q(n\xi), \quad D_q E_q(n\xi) = nE_q(nq\xi).$$

The q-trigonometric functions cosine and sine are as follows:

$$\cos_q(\xi) = \frac{e_q(i\xi) + e_q(-i\xi)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_q!}\xi^{2n}, \tag{2.4}$$

$$\mathrm{Cos}_q(\xi) = \frac{E_q(i\xi) + E_q(-i\xi)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n}{2}}}{[2n]_q!}\xi^{2n},$$

$$\sin_q(\xi) = \frac{e_q(i\xi) - e_q(-i\xi)}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_q!}\xi^{2n+1}, \tag{2.5}$$

$$\mathrm{Sin}_q(\xi) = \frac{E_q(i\xi) - E_q(-i\xi)}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n+1}{2}}}{[2n+1]_q!}\xi^{2n+1}.$$

The function of the hyperbolic q-cosine and the hyperbolic q-sine of ξ are as follows:

$$\cosh_q(\xi) = \frac{e_q(\xi) + e_q(-\xi)}{2} = \sum_{n=0}^{\infty} \frac{\xi^{2n}}{[2n]_q!}, \tag{2.6}$$

$$\mathrm{Cosh}_q(\xi) = \frac{E_q(\xi) + E_q(-\xi)}{2} = \sum_{n=0}^{\infty} \frac{q^{\binom{2n}{3}}}{[2n]_q!}\xi^{2n},$$

$$\sinh_q(\xi) = \frac{e_q(\xi) - e_q(-\xi)}{2} = \sum_{n=0}^{\infty} \frac{\xi^{2n+1}}{[2n+1]_q!}, \tag{2.7}$$

$$\mathrm{Sinh}_q(\xi) = \frac{E_q(\xi) - E_q(-\xi)}{2} = \sum_{n=0}^{\infty} \frac{q^{\binom{2n+1}{2}}}{[2n+1]_q!}\xi^{2n+1}.$$

The q-gamma function is defined as follows:

$$\Gamma_q(n) = \int_0^{\infty} \xi^{n-1} E_q(-q\xi) d_q\xi.$$

It satisfies the following properties:

$$\Gamma_q(n+1) = [n]_q \Gamma_q(n) \quad \text{and} \quad \Gamma_q(n+1) = [n]_q!.$$

The q-integral on $(0, \infty)$ for $\beta \in \mathbb{R} \setminus \{0\}$ is given by

$$\int_0^{\infty} F(\beta\xi) d_q\xi = \frac{1}{\beta} \int_0^{\infty} F(\xi) d_q\xi. \tag{2.8}$$

The q-beta function is defined as

$$B_q(l, s) = \int_0^1 \xi^{l-1}(1 - q\xi)_q^{s-1} d_q\xi. \tag{2.9}$$

The relation between the q-gamma function and the q-beta function is

$$B_q(l, s) = \frac{\Gamma_q(l)\Gamma_q(s)}{\Gamma_q(l+s)}. \tag{2.10}$$

3. Some properties of double q-Laplace-typed integral transform

In this section, we define q-Laplace-type integral transform and investigate some of their properties and relations. First, we present the following definition.

Definition 3.1. The q-Laplace-type integral transform of function $F(\xi, \eta)$ is a continuous function of two variables $\xi > 0$ and $\eta > 0$. Then the q-DLTT of the function $F(\xi, \eta)$ is given by

$$G_{2,q}[F(\xi, \eta)](u, v) = u^{\alpha} v^{\alpha} \int_0^{\infty} \int_0^{\infty} F(\xi, \eta) E_q\left(-\frac{q\xi}{u}\right) E_q\left(-\frac{q\eta}{v}\right) d_q\xi d_q\eta, \quad u, v > 0,$$

provided the integral exists.

Clearly, The q-DLTT is a linear integral transformation for any $\tau,\beta \in \mathbb{R}$ and $G_{2,q}[F(\xi,\eta)](u,v)$, $G_{2,q}[g(\xi,\eta)](u,v)$ exist, we have

$$\begin{aligned} G_{2,q}[\tau F(\xi,\eta)+\beta g(\xi,\eta)](u,v) &= u^{\alpha}v^{\alpha}\int_0^{\infty}\int_0^{\infty}[\tau F(\xi,\eta)+\beta g(\xi,\eta)]E_q\left(-\frac{q\xi}{u}\right)E_q\left(-\frac{q\eta}{v}\right)d_q\xi d_q\eta \\ &= \tau u^{\alpha}v^{\alpha}\int_0^{\infty}\int_0^{\infty}F(\xi,\eta)E_q\left(-\frac{q\xi}{u}\right)E_q\left(-\frac{q\eta}{v}\right)d_q\xi d_q\eta \\ &\quad+\beta u^{\alpha}v^{\alpha}\int_0^{\infty}\int_0^{\infty}g(\xi,\eta)E_q\left(-\frac{q\xi}{u}\right)E_q\left(-\frac{q\eta}{v}\right)d_q\xi d_q\eta \\ &= \tau G_{2,q}[F(\xi,\eta)](u,v)+\beta G_{2,q}[g(\xi,\eta)](u,v). \end{aligned}$$

Now, we apply q-DLTT on some special functions as follows.

1. Let $F(\xi,\eta)=1$ for $\xi,\eta>0$, then

$$\begin{aligned} G_{2,q}[1](u,v) &= u^{\alpha}v^{\alpha}\int_0^{\infty}\int_0^{\infty}E_q\left(-\frac{q\xi}{u}\right)E_q\left(-\frac{q\eta}{v}\right)d_q\xi d_q\eta \\ &= u^{\alpha+1}v^{\alpha+1}\int_0^{\infty}D_qE_q\left(-\frac{\xi}{u}\right)\int_0^{\infty}D_qE_q\left(-\frac{\eta}{v}\right)d_q\xi d_q\eta = u^{\alpha+1}v^{\alpha+1}. \end{aligned}$$

2. Let $F(\xi,\eta)=\xi^m\eta^n$ for $\xi>0,\eta>0$ and m,n are constants, then

$$G_{2,q}[\xi^m\eta^n](u,v) = u^{\alpha}v^{\alpha}\int_0^{\infty}\int_0^{\infty}\xi^m\eta^nE_q\left(-\frac{q\xi}{u}\right)E_q\left(-\frac{q\eta}{v}\right)d_q\xi d_q\eta.$$

Consider

$$\begin{aligned} \int_0^{\infty}\xi^mE_q\left(-\frac{q\xi}{u}\right)d_q\xi &= -u\int_0^{\infty}\xi^mD_qE_q\left(-\frac{\xi}{u}\right)d_q\xi \\ &= u[m]_q\int_0^{\infty}\xi^{m-1}E_q\left(-\frac{q\xi}{u}\right)d_q\xi \text{ (by (2.2))} \\ &= u^2[m]_q[m-1]_q\int_0^{\infty}\xi^{m-2}E_q\left(-\frac{q\xi}{u}\right)d_q\xi \\ &= u^m[m]_q[m-1]_q\cdots[1]_q\int_0^{\infty}E_q\left(-\frac{q\xi}{u}\right)d_q\xi \\ &= -u^{m+1}[m]_q!\int_0^{\infty}D_qE_q\left(-\frac{\xi}{u}\right)d_q\xi = u^{m+1}[m]_q!. \end{aligned}$$

In the same way $\int_0^{\infty}\eta^nE_q\left(-\frac{q\eta}{v}\right)=v^{n+1}[n]_q!$, then we have

$$G_{2,q}[\xi^m\eta^n](u,v) = u^{m+\alpha+1}v^{n+\alpha+1}[m]_q![n]_q!. \tag{3.1}$$

In the particular case, when $c,d>-1$, then

$$G_{2,q}[\xi^c\eta^d](u,v) = u^{c+\alpha+1}v^{d+\alpha+1}\Gamma_q(c+1)\Gamma_q(d+1).$$

3. For $\xi>0,\eta>0$, and m,n being constants, then we have

$$\begin{aligned} G_{2,q}[F(a\xi,b\eta)](u,v) &= u^{\alpha}v^{\alpha}\int_0^{\infty}\int_0^{\infty}F(a\xi,b\eta)E_q\left(-\frac{q\xi}{u}\right)E_q\left(-\frac{q\eta}{v}\right)d_q\xi d_q\eta \\ &= u^{\alpha}v^{\alpha}\int_0^{\infty}E_q\left(-\frac{q\eta}{v}\right)\int_0^{\infty}F(a\xi,b\eta)E_q\left(-\frac{q\xi}{u}\right)d_q\xi d_q\eta \end{aligned}$$

$$
\begin{aligned}
&=\frac{u^{\alpha}v^{\alpha}}{a}\int_0^{\infty}E_q\left(-\frac{q\eta}{v}\right)\int_0^{\infty}\digamma(\xi,b\eta)E_q\left(-\frac{q\xi}{au}\right)d_q\xi d_q\eta \text{ (by (2.8))}\\
&=\frac{u^{\alpha}v^{\alpha}}{a}\int_0^{\infty}E_q\left(-\frac{q\xi}{au}\right)\int_0^{\infty}\digamma(\xi,b\eta)E_q\left(-\frac{q\eta}{v}\right)d_q\eta d_a\xi\\
&=\frac{u^{\alpha}v^{\alpha}}{ab}\int_0^{\infty}\int_0^{\infty}\digamma(\xi,\eta)E_q\left(-\frac{q\xi}{au}\right)E_q\left(-\frac{q\eta}{bv}\right)d_q\xi d_q\eta\\
&=\frac{1}{a^{\alpha+1}b^{\alpha+1}}G_{2,q}[\digamma(\xi,\eta)](au,bv).
\end{aligned}
$$

4. Let $\digamma(\xi,\eta)=(a\xi+b\eta)_q^n$ for $\xi>0,\eta>0$ and $n\in\mathbb{N}$, then

$$
\begin{aligned}
G_{2,q}[(a\xi+b\eta)_q^n](u,v)&=\sum_{k=0}^{n}\binom{n}{k}_q G_{2,q}[(a\xi)^k(b\eta)^{n-k}] \text{ (by (2.1))}\\
&=\frac{1}{a^{\alpha+1}b^{\alpha+1}}\sum_{k=0}^{n}\binom{n}{k}_q G_{2,q}[\xi^k\eta^{n-k}]\\
&=\frac{1}{a^{\alpha+1}b^{\alpha+1}}\sum_{k=0}^{n}\binom{n}{k}_q (au)^{k+\alpha+1}(uv)^{n-k+\alpha+1}[k]_q![n-k]_q! \text{ (by (3.1))}\\
&=b^n[n]_q!u^{\alpha+1}v^{n+\alpha+1}\sum_{k=0}^{n}\left(\frac{au}{bv}\right)^k\\
&=\frac{[n]_q!u^{\alpha+1}v^{\alpha+1}}{au-bv}[(au)^{n+1}-(bv)^{n+1}],\quad \frac{v}{u}<\frac{a}{b}.
\end{aligned}
\tag{3.2}
$$

5. Let $\digamma(\xi,\eta)=e_q(a\xi+b\eta)$ for $\xi>0,\eta>0$ and a,b are constants, then

$$
\begin{aligned}
G_{2,q}[e_q(a\xi+b\eta)](u,v)&=\sum_{n=0}^{\infty}G_{2,q}\left[\frac{(a\xi+b\eta)_q^n}{[n]_q!}\right] \text{ (by (2.3))}\\
&=\frac{u^{\alpha+1}v^{\alpha+1}}{au-bv}\sum_{n=0}^{\infty}\left[(au)^{n+1}-(bv)^{n+1}\right] \text{ (by (3.2))}\\
&=\frac{u^{\alpha+1}v^{\alpha+1}}{au-bv}\left[\frac{au}{1-au}-\frac{bv}{1-bv}\right]=\frac{u^{\alpha+1}v^{\alpha+1}}{(1-au)(1-bv)},\quad u<\frac{1}{a},v<\frac{1}{b}.
\end{aligned}
$$

Using q-Euler's formulas (2.4), (2.5), (2.6), and (2.7), we obtain

$$
\begin{aligned}
G_{2,q}[\sin_q(a\xi+b\eta)](u,v)&=\frac{1}{2i}\left[\frac{u^{\alpha}v^{\alpha}}{(1-aui)(1-bvi)}-\frac{u^{\alpha+1}v^{\alpha+1}}{(1+aui)(1+bvi)}\right]=\frac{u^{\alpha+1}v^{\alpha+1}(bv-au)}{(1+a^2u^2)(1+b^2v^2)},\\
G_{2,q}[\cos_q(a\xi+b\eta)](u,v)&=\frac{1}{2}\left[\frac{u^{\alpha}v^{\alpha}}{(1-aui)(1-bvi)}+\frac{u^{\alpha+1}v^{\alpha+1}}{(1+aui)(1+bvi)}\right]=\frac{u^{\alpha+1}v^{\alpha+1}(1-abuv)}{(1+a^2u^2)(1+b^2v^2)},\\
G_{2,q}[\sinh_q(a\xi+b\eta)](u,v)&=\frac{1}{2}\left[\frac{u^{\alpha}v^{\alpha}}{(1-au)(1-bv)}-\frac{u^{\alpha+1}v^{\alpha+1}}{(1+au)(1+bv)}\right]=\frac{u^{\alpha+1}v^{\alpha+1}(bv+au)}{(1-a^2u^2)(1-b^2v^2)},\\
G_{2,q}[\cosh_q(a\xi+b\eta)](u,v)&=\frac{1}{2}\left[\frac{u^{\alpha}v^{\alpha}}{(1-au)(1-bv)}+\frac{u^{\alpha+1}v^{\alpha+1}}{(1+au)(1+bv)}\right]=\frac{u^{\alpha+1}v^{\alpha+1}(1+abuv)}{(1-a^2u^2)(1-b^2v^2)}.
\end{aligned}
$$

Theorem 3.2. *Let $\digamma(\xi)$ be a continuous function and assume that $\digamma(\xi)$ has the q-Taylor expansion*

$$
\digamma(\xi)=\sum_{k=0}^{\infty}a_n\frac{\xi^n}{[n]_q!}.
\tag{3.3}
$$

Therefore,

$$G_{2,q}[F(\kappa\xi+\lambda\eta)](u,v)=\frac{1}{\kappa u-\lambda v}\left(\kappa u v^{\alpha+1}G_q[F(\kappa\xi)](u)-\lambda v u^{\alpha+1}G_q[F(\lambda\xi)](v)\right),$$

when κ *and* λ *are constants.*

Proof. We have the following

$$F(\kappa\xi+\lambda\eta)_q \overset{(3.3)}{=} \sum_{n=0}^{\infty}\frac{a_n(\kappa\xi+\lambda\eta)_q^n}{[n]_q!} \overset{(2.1)}{=} \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_q(\kappa\xi)^k(\lambda\eta)^{n-k}\right)\frac{a_n}{[n]_q!}.$$

Hence, we can write that

$$\begin{aligned}G_{2,q}[F(\kappa\xi+\lambda\eta)_q](u,v)&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\binom{n}{k}_q G_{2,q}[(\kappa\xi)^k(\lambda\eta)^{n-k}](u,v)\right)\frac{a_n}{[n]_q!}\\&=\frac{u^{\alpha+1}v^{\alpha+1}}{\kappa u-\lambda v}\left[\sum_{n=0}^{\infty}a_n(\kappa u)^{n+1}-\sum_{n=0}^{\infty}a_n(\lambda v)^{n+1}\right]\ (\text{by } (3.2))\\&=\frac{u^{\alpha+1}v^{\alpha+1}}{\kappa u-\lambda v}\left[\kappa u\sum_{n=0}^{\infty}a_n(\kappa u)^{n}-\lambda v\sum_{n=0}^{\infty}a_n(\lambda v)^{n}\right]\\&=\frac{1}{\kappa u-\lambda v}(\kappa u v^{\alpha+1}G_q[F(\kappa\xi)](u)-\lambda v u^{\alpha+1}G_q[F(\lambda\xi)](v)).\end{aligned}$$

Therefore, the proof is completed. □

Theorem 3.3 (Convolution theorem)**.** *If* $F(\xi,\eta)=(\xi\eta)^{\gamma}$ *and* $g(\xi,\eta)=(\xi\eta)^{\beta-1},\gamma,\beta>0$, *then*

$$G_{2,q}[(F**g)_q(\xi,\eta)](u,v)=\frac{1}{u^{\alpha}v^{\alpha}}G_{2,q}[F(\xi,\eta)](u,v)G_{2,q}[g(\xi,\eta)](u,v),$$

where

$$(F*g)_q(\xi,\eta)=\int_0^{\xi}\int_0^{\eta}F(x,y)g(\xi-qx,\eta-qy)d_qxd_qy. \tag{3.4}$$

Proof. Using (2.9) and (3.4), we get

$$(F*g)_q(\xi,\eta)=\int_0^{\xi}\int_0^{\eta}x^{\gamma}y^{\gamma}(\xi-qx)_q^{\beta-1}(\eta-qy)_q^{\beta-1}d_qxd_qy.$$

Changing the variables in the equation above by $x=r\xi$, $y=h\eta$ and using (2.9), results in the following form:

$$(F*g)_q(\xi,\eta)=\xi^{\gamma+\beta}\int_0^1 r^{\gamma}(1-qr)_q^{\beta-1}d_qr\times\eta^{\gamma+\beta}\int_0^1 h^{r}(1-qh)_q^{\beta-1}d_qh=\xi^{\gamma+\beta}\eta^{\gamma+\beta}\left[B_q(\gamma+1,\beta)\right]^2.$$

We then take q-DLTT in the equation above and using (2.10), we can write that

$$\begin{aligned}G_{2,q}[(F*g)_q(\xi,\eta)](u,v)&=[B_q(\gamma+1,\beta)]^2\int_0^{\infty}\int_0^{\infty}\xi^{\gamma+\beta}\eta^{\gamma+\beta}E_q\left(-\frac{q\xi}{u}\right)E_q\left(-\frac{q\eta}{v}\right)d_q\xi d_q\eta\\&=[B_q(\gamma+1,\beta)]^2\,[[\gamma+\beta]_q!]^2u^{\gamma+\beta+\alpha+1}v^{\gamma+\beta+\alpha+1}\\&=[\Gamma_q(\gamma+1)]^2\,[\Gamma_q(\beta)]^2\,u^{\gamma+\beta+\alpha+1}v^{\gamma+\beta+\alpha+1}\\&=\frac{1}{u^{\alpha}v^{\alpha}}G_{2,q}[F(\xi,\eta)](u,v)G_{2,q}[g(\xi,\eta)](u,v).\end{aligned}$$

This completes the proof of the theorem. □

Theorem 3.4 (Partial q-derivatives order one and two)**.** *Let $F(\xi,\eta)$ be a continuous function, then we get the following identities hold:*

(i) $G_{2,q}\left[\frac{\partial_q F}{\partial_q \xi}(\xi,\eta)\right](u,v) = -u^\alpha G_q[F(0,\eta)](v) + \frac{1}{u}G_{2,q}[F(\xi,\eta)](u,v)$;

(ii) $G_{2,q}\left[\frac{\partial_q F}{\partial_q \eta}(\xi,\eta)\right](u,v) = -v^\alpha G_q[F(\xi,0)](u) + \frac{1}{v}G_{2,q}[F(\xi,\eta)](u,v)$;

(iii) $G_{2,q}\left[\frac{\partial_q^2 F}{\partial_q \xi^2}(\xi,\eta)\right](u,v) = -u^{\alpha-1} G_q[F(0,\eta)](v) - u^\alpha G_q\left[\frac{\partial_q F}{\partial_q \xi}(0,\eta)\right](v) + \frac{1}{u^2}G_{2,q}[F(\xi,\eta)](u,v)$;

(iv) $G_{2,q}\left[\frac{\partial_q^2 F}{\partial_q \eta^2}(\xi,\eta)\right](u,v) = -v^{\alpha-1} G_q[F(\xi,0)](u) - v^\alpha G_q\left[\frac{\partial_q F}{\partial_q \eta}(\xi,0)\right](u) + \frac{1}{v^2}G_{2,q}[F(\xi,\eta)](u,v)$;

(v) $G_{2,q}\left[\frac{\partial_q^2 F}{\partial_q \xi \partial_q \eta}(\xi,\eta)\right](u,v) = u^\alpha v^\alpha F(0,0) - \frac{v^\alpha}{u}G_q[F(\xi,0)](u) - \frac{u^\alpha}{v}G_q[F(0,\eta)](v) + \frac{1}{uv}G_{2,q}[F(\xi,\eta)](u,v)$.

Proof.

(i) From Definition 3.1 and using q-integrating by parts (2.2), we obtain

$$\begin{aligned}
G_{2,q}\left[\frac{\partial_q F}{\partial_q \xi}(\xi,\eta)\right](u,v) &= u^\alpha v^\alpha \int_0^\infty \int_0^\infty \frac{\partial_q F}{\partial_q \xi}(\xi,\eta) E_q\left(-\frac{q\xi}{u}\right) E_q\left(-\frac{q\eta}{v}\right) d_q\xi d_q\eta \\
&= u^\alpha v^\alpha \int_0^\infty \left[-F(0,\eta) + \frac{1}{u}\int_0^\infty F(\xi,\eta) E_q\left(-\frac{q\xi}{u}\right) d_q\xi\right] E_q\left(-\frac{q\eta}{v}\right) d_q\eta \\
&= -u^\alpha G_q[F(0,\eta)](v) + \frac{1}{u}G_{2,q}[F(\xi,\eta)](u,v).
\end{aligned}$$

(ii) From Definition 3.1 and using q-integrating by parts (2.2), we obtain

$$\begin{aligned}
G_{2,q}\left[\frac{\partial_q F}{\partial_q \eta}(\xi,\eta)\right](u,v) &= u^\alpha v^\alpha \int_0^\infty \int_0^\infty \frac{\partial_q F}{\partial_q \eta}(\xi,\eta) E_q\left(-\frac{q\eta}{v}\right) E_q\left(-\frac{q\xi}{u}\right) d_q\xi d_q\eta \\
&= u^\alpha v^\alpha \int_0^\infty \left[-F(\xi,0) + \frac{1}{v}\int_0^\infty f(\xi,\eta) E_q\left(-\frac{q\eta}{u}\right) d_q\eta\right] E_q\left(-\frac{q\xi}{u}\right) d_q\xi \\
&= -v^\alpha G_q[F(\xi,0)](u) + \frac{1}{v}G_{2,q}[F(\xi,\eta)](u,v).
\end{aligned}$$

(iii) From Definition 3.1 and using q-integrating by parts (2.2), we obtain

$$\begin{aligned}
G_{2,q}\left[\frac{\partial_q^2 F}{\partial_q \xi^2}(\xi,\eta)\right](u,v) &= u^\alpha v^\alpha \int_0^\infty \int_0^\infty \frac{\partial_q^2 F}{\partial_q \xi^2}(\xi,\eta) E_q\left(-\frac{q\xi}{u}\right) E_q\left(-\frac{q\eta}{v}\right) d_q\xi d_q\eta \\
&= u^\alpha v^\alpha \int_0^\infty \left[-\frac{\partial_q F}{\partial_q \xi}(\xi,\eta) + \frac{1}{u}\int_0^\infty \frac{\partial_q F}{\partial_q \xi}(\xi,\eta) E_q\left(-\frac{q\xi}{u}\right) d_q\xi\right] E_q\left(-\frac{q\eta}{v}\right) d_q\eta \\
&= -u^{\alpha-1} G_q[F(0,\eta)](v) - u^\alpha G_q\left[\frac{\partial_q F}{\partial_q \xi}(0,\eta)\right](v) + \frac{1}{u^2}G_{2,q}[F(\xi,\eta)](u,v).
\end{aligned}$$

(iv) From Definition 3.1 and using q-integrating by parts (2.2), we obtain

$$\begin{aligned}
G_{2,q}\left[\frac{\partial^2 F}{\partial_q \eta^2}(\xi,\eta)\right](u,v) &= u^\alpha v^\alpha \int_0^\infty \int_0^\infty \frac{\partial_q^2 F}{\partial_q \eta^2}(\xi,\eta) E_q\left(-\frac{q\xi}{u}\right) E_q\left(-\frac{q\eta}{v}\right) d_q\xi d_q\eta \\
&= u^\alpha v^\alpha \int_0^\infty \left[-\frac{\partial_q F}{\partial_q \eta}(\xi,0) + \frac{1}{v}\int_0^\infty \frac{\partial_q F}{\partial_q \eta}(\xi,\eta) E_q\left(-\frac{q\eta}{v}\right) d_q\xi\right] E_q\left(-\frac{q\xi}{u}\right) d_q\xi \\
&= -v^{\alpha-1} G_q[F(\xi,0)](u) - v^\alpha G_q\left[\frac{\partial_q F}{\partial_q \eta}(\xi,0)\right](u) + \frac{1}{v^2}G_{2,q}[F(\xi,\eta)](u,v).
\end{aligned}$$

(v) From Definition 3.1, the result of (i), and using q-integrating by parts (2.2), we obtain

$$G_{2,q}\left[\frac{\partial_q^2 F}{\partial_q \xi \partial_q \eta}(\xi,\eta)\right](u,v) = u^\alpha v^\alpha \int_0^\infty \int_0^\infty \frac{\partial_q^2 F}{\partial_q \xi \partial_q \eta}(\xi,\eta) E_q\left(-\frac{q\xi}{u}\right) E_q\left(-\frac{q\eta}{v}\right) d_q\xi d_q\eta$$

$$= u^\alpha v^\alpha \int_0^\infty \left[-\frac{\partial_q F}{\partial_q \xi}(\xi,0) + \frac{1}{v}\int_0^\infty \frac{\partial_q F}{\partial_q \xi}(\xi,\eta) E_q\left(-\frac{q\eta}{v}\right) d_q\eta \right] E_q\left(-\frac{q\xi}{u}\right) d_q\xi$$
$$= u^\alpha v^\alpha F(0,0) - \frac{v^\alpha}{u} G_q[F(\xi,0)](u) - \frac{u^\alpha}{v} G_q[F(0,\eta)](v) + \frac{1}{uv} G_{2,q}[F(\xi,\eta)](u,v).$$

Therefore, the proof is completed. □

Theorem 3.5 (Partial q-derivatives of order n)**.** *Let $F(\xi,\eta)$ be a continuous function, then we get the following identities hold:*

(i) $G_{2,q}\left[\frac{\partial_q^n F}{\partial_q \xi^n}(\xi,\eta)\right](u,v) = \frac{1}{u^n} G_{2,q}[F(\xi,\eta)](u,v) - \sum_{k=0}^{n-1} u^{\alpha-n+k+1} G_q\left[\frac{\partial_q^k F}{\partial_q \xi^k}(0,\eta)\right](v)$;

(ii) $G_{2,q}\left[\frac{\partial_q^n F}{\partial_q \eta^n}(\xi,\eta)\right](u,v) = \frac{1}{v^n} G_{2,q}[F(\xi,\eta)](u,v) - \sum_{k=0}^{n-1} v^{\alpha-n+k+1} G_q\left[\frac{\partial_q^k F}{\partial_q \xi^k}(\xi,0)\right](u)$.

Proof. We prove Theorem 3.5 by mathematical induction and therefore is omitted. □

4. Application of double q-Laplace-typed integral transform

In this section, we apply the q-DLTT to a family of PDEs and obtain a simple formula for the general solution. We consider a second order linear partial differential equation in two independent variables ξ and η of the form

$$A\frac{\partial_q^2 F}{\partial_q \xi^2}(\xi,\eta) + B\frac{\partial_q^2 F}{\partial_q \eta^2}(\xi,\eta) + C\frac{\partial_q F}{\partial_q \xi}(\xi,\eta) + D\frac{\partial_q F}{\partial_q \eta}(\xi,\eta) + EF(\xi,\eta) = \omega(\xi,\eta), \tag{4.1}$$

with the initial conditions

$$F(\xi,0) = F_1(\xi), \quad \frac{\partial_q g}{\partial_q \eta}(\xi,0) = F_2(\xi), \tag{4.2}$$

and the boundary conditions

$$F(0,\eta) = g_1(\eta), \quad \frac{\partial_q g}{\partial_q \xi}(0,\eta) = g_2(\eta), \tag{4.3}$$

where A, B, C, D, E are constants, $F(\xi,\eta)$ is the unknown function, and $\omega(\xi,\eta)$ is the source term. The main idea of this technique is to apply the q-DLTT to (4.1), q-Laplace-typed integral transform to the initial conditions in (4.2),

$$G_q[F(\xi,0)](u) = G_q[F_1(\xi)](u) = F_1(u), \quad G_q\left[\frac{\partial_q g}{\partial_q \eta}(\xi,0)\right](u) = G_q[F_2(\xi)](u) = F_2(u),$$

q-Laplace-typed integral transform to the boundary conditions in (4.3),

$$G_q[F(0,\eta)](v) = G_q[g_1(\eta)](v) = H_1(v), \quad G_q\left[\frac{\partial_q g}{\partial_q \xi}(0,\eta)\right](v) = G_q[g_2(\eta)](v) = H_2(v),$$

and $G_{2,q}[\omega(\xi,\eta)](u,v) = \Omega(u,v)$. Now, applying the q-DLTT to (4.1), we have

$$AG_{2,q}\left[\frac{\partial_q^2 F}{\partial_q \xi^2}(\xi,\eta)\right](u,v) + BG_{2,q}\left[\frac{\partial_q^2 F}{\partial_q \eta^2}(\xi,\eta)\right](u,v) + CG_{2,q}\left[\frac{\partial_q F}{\partial_q \xi}(\xi,\eta)\right](u,v)$$
$$+ DG_{2,q}\left[\frac{\partial_q F}{\partial_q \eta}(\xi,\eta)\right](u,v) + EG_{2,q}[F(\xi,\eta)](u,v) = G_{2,q}[\omega(\xi,\eta)](u,v).$$

Using the derivatives of the q-DLTT and above conditions, we get

$$\begin{aligned}
&A\left(-u^{\alpha-1}H_1(v)-u^{\alpha}H_2(v)+\frac{1}{u^2}G_{2,q}[\mathcal{F}(\xi,\eta)](u,v)\right)\\
&+B\left(-v^{\alpha-1}F_1(u)-v^{\alpha}F_2(u)+\frac{1}{v^2}G_{2,q}[\mathcal{F}(\xi,\eta)](u,v)\right)+C\left(-u^{\alpha}H_1(v)+\frac{1}{u}G_{2,q}[\mathcal{F}(\xi,\eta)](u,v)\right)\\
&+D\left(-v^{\alpha}F_1(u)+\frac{1}{v}G_{2,q}[\mathcal{F}(\xi,\eta)](u,v)\right)+EG_{2,q}[\mathcal{F}(\xi,\eta)](u,v)=\Omega(u,v).
\end{aligned}\tag{4.4}$$

After simple computations, (4.4) can be written as

$$\begin{aligned}
&G_{2,q}[\mathcal{F}(\xi,\eta)](u,v)\\
&=\frac{\Omega(u,v)+A(u^{\alpha-1}H_1(v)+u^{\alpha}H_2(v))+B(v^{\alpha-1}F_1(u)+v^{\alpha}F_2(u))+Cu^{\alpha}H_1(v)+Dv^{\alpha}F_1(u)}{\frac{A}{u^2}+\frac{B}{v^2}+\frac{C}{u}+\frac{D}{v}+E}.
\end{aligned}\tag{4.5}$$

Taking the inverse q-DLTT to both sides of (4.5), we have

$$\mathcal{F}(\xi,\eta)=G^{-1}_{2,q}\left[\frac{\Omega(u,v)+A(u^{\alpha-1}H_1(v)+u^{\alpha}H_2(v))+B(v^{\alpha-1}F_1(u)+v^{\alpha}F_2(u))+Cu^{\alpha}H_1(v)+Dv^{\alpha}F_1(u)}{\frac{A}{u^2}+\frac{B}{v^2}+\frac{C}{u}+\frac{D}{v}+E}\right].$$

Now, the examples of some familiar PDEs are solved by applying the q-DLTT such as q-wave equation, q-Klein-Gordon equation, q-Laplace equation, q-heat equation, and q-telegraph equation. All the following figures of the selected examples were obtained using MATLAB software.

Example 4.1. Find the solution of the q-wave equation

$$\frac{\partial_q^2\mathcal{F}}{\partial_q\xi^2}(\xi,\eta)=\frac{\partial_q^2\mathcal{F}}{\partial_q\eta^2}(\xi,\eta),\quad \xi,\eta\geqslant 0,\tag{4.6}$$

with the initial conditions are $\mathcal{F}(\xi,0)=\sin_q(\xi)$ and $\frac{\partial_q\mathcal{F}}{\partial_q\eta}(\xi,0)=3$, and boundary conditions are $\mathcal{F}(0,\eta)=3\eta$ and $\frac{\partial_q\mathcal{F}}{\partial_q\xi}(0,\eta)=\cos_q(\eta)$. Applying q-Laplace-typed integral transform to initial and boundary conditions, we obtain

$$F_1(u)=\frac{u^{\alpha+2}}{1+u^2},\quad F_2(u)=3u^{\alpha+1},\quad H_1(v)=3v^{\alpha+2},\quad\text{and}\quad H_2(v)=\frac{v^{\alpha+1}}{1+v^2}.$$

Substituting the values of functions $F_1,F_2,H_1,H_2,A=1,B=-1$, and $C=D=E=\Omega=0$ in (4.5) we obtain

$$G_{2,q}[\mathcal{F}(\xi,\eta)](u,v)=\frac{3u^{\alpha-1}v^{\alpha+2}+\frac{u^{\alpha}v^{\alpha+1}}{1+v^2}-\frac{u^{\alpha+2}v^{\alpha-1}}{1+u^2}-3u^{\alpha+1}v^{\alpha}}{\frac{1}{u^2}-\frac{1}{v^2}}=3u^{\alpha+1}v^{\alpha+2}+\frac{u^{\alpha+2}v^{\alpha+1}}{(u^2+1)(v^2+1)}.\tag{4.7}$$

Taking the inverse q-DLTT in (4.7), then the solution of (4.6) is

$$\mathcal{F}(\xi,\eta)=G^{-1}_{2,q}\left[3u^{\alpha+1}v^{\alpha+2}+\frac{u^{\alpha+2}v^{\alpha+1}}{(u^2+1)(v^2+1)}\right]=3\eta+\sin_q(\xi)\cos_q(\eta).$$

Figure 1 shows the solution of q-wave equation with conditions when we set $q=0.8045$, which is shown in the left side of Figure 1 and we set $q\to 1$, which is shown in the right side of Figure 1.

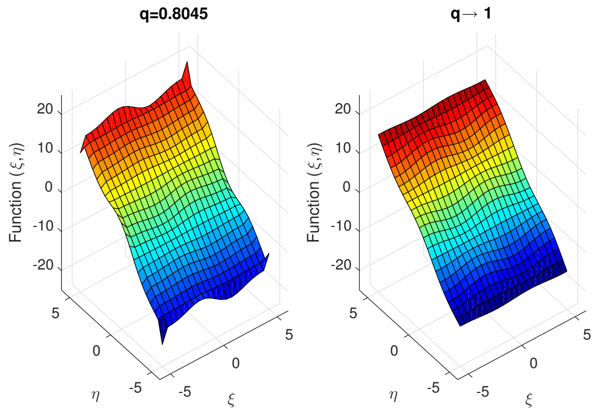

Figure 1: Plot illustration for Example 4.1.

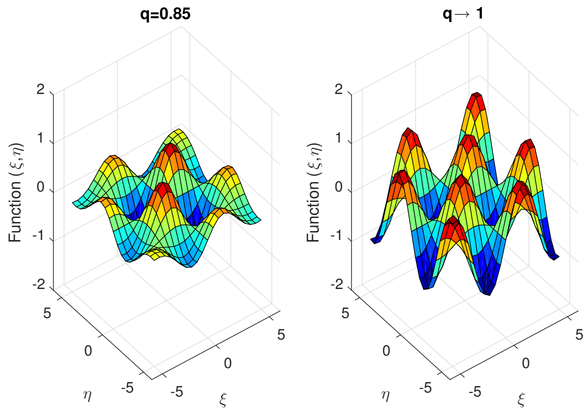

Figure 2: Plot illustration for Example 4.2.

Example 4.2. Find the solution of the q-Klein-Gordon equation

$$\frac{\partial_q^2 \mathcal{F}}{\partial_q \eta^2}(\xi,\eta) - \frac{\partial_q^2 \mathcal{F}}{\partial_q \xi^2}(\xi,\eta) = 2\mathcal{F}(\xi,\eta) - 2\sin_q(\xi)\sin_q(\eta). \tag{4.8}$$

with the initial conditions $\mathcal{F}(\xi,0) = 0$, $\frac{\partial_q \mathcal{F}}{\partial_q \eta}(\xi,0) = \sin_q(\xi)$ and boundary conditions $\mathcal{F}(0,\eta) = 0$, $\frac{\partial_q \mathcal{F}}{\partial_q \xi}(0,\eta) = \sin_q(\eta)$.

Applying q-Laplace-typed integral transform to initial and boundary conditions, we obtain

$$F_1(u) = 0, \quad F_2(u) = \frac{u^{\alpha+2}}{1+u^2}, \quad H_1(v) = 0, \quad \text{and} \quad H_2(v) = \frac{v^{\alpha+2}}{1+v^2}.$$

Substituting the values of functions $F_1, F_2, H_1, H_2, A = -1, B = 1, C = D = 0, E = -2$, and $\Omega = -\frac{2u^{\alpha+2}v^{\alpha+2}}{(1+u^2)(1+v^2)}$ in (4.5), we obtain

$$G_{2,q}[\mathcal{F}(\xi,\eta)](u,v) = \frac{-\frac{2u^{\alpha+2}v^{\alpha+2}}{(1+u^2)(1+v^2)} - \frac{u^{\alpha}v^{\alpha+2}}{1+v^2} + \frac{u^{\alpha+2}v^{\alpha}}{1+u^2}}{\frac{1}{v^2} - \frac{1}{u^2} - 2} = \frac{u^{\alpha+2}v^{\alpha+2}}{(u^2+1)(v^2+1)}. \tag{4.9}$$

Taking the inverse q-DLTT in (4.9), then the solution of (4.8) is

$$\mathcal{F}(\xi,\eta) = G_{2,q}^{-1}\left[\frac{u^{\alpha+2}v^{\alpha+2}}{(u^2+1)(v^2+1)}\right] = \sin_q(\xi)\sin_q(\eta).$$

Figure 2 shows the solution of q-Klein-Gordon equation with conditions when we set $q = 0.85$, which is shown in the left side of Figure 2 and we set $q \to 1$, which is shown in the right side of Figure 2.

Example 4.3. Find the solution of the q-Laplace equation

$$\frac{\partial_q^2 \mathcal{F}}{\partial_q \eta^2}(\xi,\eta) + \frac{\partial_q^2 \mathcal{F}}{\partial_q \xi^2}(\xi,\eta) = 0, \quad \xi,\eta > 0, \tag{4.10}$$

with the initial conditions $\mathcal{F}(\xi,0) = 0$, $\frac{\partial_q \mathcal{F}}{\partial_q \eta}(\xi,0) = \cos_q(\xi)$ and boundary conditions $\mathcal{F}(0,\eta) = \sinh_q(\eta)$, $\frac{\partial_q \mathcal{F}}{\partial_q \xi}(0,\eta) = 0$.

Applying q-Laplace-typed integral transform to initial and boundary conditions, we obtain

$$F_1(u) = 0, \quad F_2(u) = \frac{u^{\alpha+1}}{1+u^2}, \quad H_1(v) = \frac{v^{\alpha+2}}{1-v^2}, \quad \text{and} \quad H_2(v) = 0.$$

Substituting the values of functions $F_1, F_2, H_1, H_2, A = 1, B = 1$, and $C = D = E = \Omega = 0$ in (4.5), we obtain

$$G_{2,q}[\mathcal{F}(\xi,\eta)](u,v) = \frac{\frac{u^{\alpha-1}v^{\alpha+2}}{1-v^2} + \frac{u^{\alpha+1}v^{\alpha}}{1+u^2}}{\frac{1}{u^2} + \frac{1}{v^2}} = \frac{u^{\alpha+1}v^{\alpha+2}}{(u^2+1)(1-v^2)}. \tag{4.11}$$

Taking the inverse q-DLTT in (4.11), then the solution of (4.10) is

$$\mathcal{F}(\xi,\eta) = G_{2,q}^{-1}\left[\frac{u^{\alpha+1}v^{\alpha+2}}{(u^2+1)(1-v^2)}\right] = \cos_q(\xi)\sinh_q(\eta).$$

Figure 3 shows the solution of q-Klein-Gordon equation with conditions when we set $q = 0.85$, which is shown in the left side of Figure 3 and we set $q \to 1$, which is shown in the right side of Figure 3.

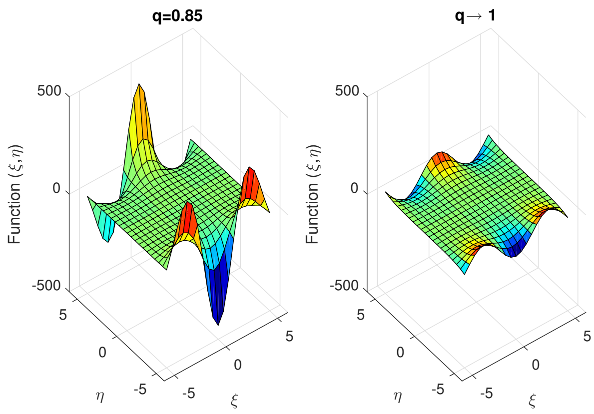

Figure 3: Plot illustration for Example 4.3.

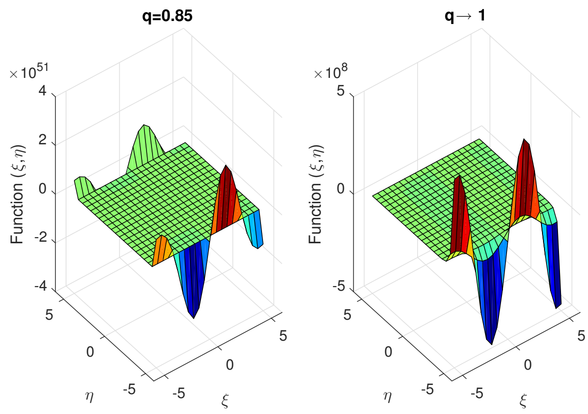

Figure 4: Plot illustration for Example 4.4.

Example 4.4. Find the solution of the q-heat equation

$$\frac{\partial_q^2 \mathcal{F}}{\partial_q \xi^2}(\xi,\eta) - \frac{\partial_q \mathcal{F}}{\partial_q \eta}(\xi,\eta) - 3\mathcal{F}(\xi,\eta) = -3, \quad \xi,\eta \geqslant 0, \tag{4.12}$$

with the initial conditions $\mathcal{F}(\xi,0) = 1 + \sin_q(\xi)$ and boundary conditions $\mathcal{F}(0,\eta) = 1$, $\frac{\partial_q \mathcal{F}}{\partial_q \xi}(0,\eta) = e_q(-4\eta)$.

Applying q-Laplace-typed integral transform to initial and boundary conditions, we obtain

$$F_1(u) = u^{\alpha+1} + \frac{u^{\alpha+2}}{1+u^2}, \quad H_1(v) = v^{\alpha+1}, \quad \text{and} \quad H_2(v) = \frac{v^{\alpha+1}}{1+4v}.$$

Substituting the values of functions $F_1, H_1, H_2, A = 1, B = C = 0, D = -1, E = -3$, and $\Omega = 3u^{\alpha+1}v^{\alpha+1}$ in (4.5), we obtain

$$\begin{aligned} G_{2,q}[\mathcal{F}(\xi,\eta)](u,v) &= \frac{3u^{\alpha+1}v^{\alpha+1} + u^{\alpha-1}v^{\alpha+1} + \frac{u^{\alpha}v^{\alpha+1}}{1+4v} - u^{\alpha+1}v^{\alpha} - \frac{u^{\alpha+2}v^{\alpha}}{1+u^2}}{\frac{1}{u^2} - \frac{1}{v} - 3} \\ &= u^{\alpha+1}v^{\alpha+1} + \frac{u^{\alpha+2}v^{\alpha+1}}{(u^2+1)(4v+1)}. \end{aligned} \tag{4.13}$$

Taking the inverse q-DLTT in (4.13), then the solution of (4.12) is

$$\mathcal{F}(\xi,\eta) = G_{2,q}^{-1}\left[u^{\alpha+1}v^{\alpha+1} + \frac{u^{\alpha+2}v^{\alpha+1}}{(u^2+1)(4v+1)}\right] = 1 + \sin_q(\xi)e_q(-4\eta).$$

Figure 4 shows the solution of q-Klein-Gordon equation with conditions when we set $q = 0.85$, which is shown in the left side of Figure 4 and we set $q \to 1$, which is shown in the right side of Figure 4.

Example 4.5. Find the solution of the q-telegraph equation

$$\frac{\partial_q^2 F}{\partial_q \xi^2}(\xi,\eta) - \frac{\partial_q^2 F}{\partial_q \eta^2}(\xi,\eta) - \frac{\partial_q F}{\partial_q \eta}(\xi,\eta) + F(\xi,\eta) = 0, \quad \xi,\eta \geqslant 0, \tag{4.14}$$

with the initial conditions $F(\xi,0) = e_q(\xi)$, $\frac{\partial_q F}{\partial_q \eta}(\xi,0) = -2e_q(\xi)$ and boundary conditions $F(0,\eta) = e_q{-2\eta}$, $\frac{\partial_q F}{\partial_q \xi}(0,\eta) = e_q{-2\eta}$.

Applying q-Laplace-typed integral transform to initial and boundary conditions, we obtain

$$F_1(u) = \frac{u^{\alpha+1}}{1-u}, \quad F_2(u) = -\frac{2u^{\alpha+1}}{1-u}, \quad H_1(v) = \frac{v^{\alpha+1}}{1+2v}, \quad \text{and} \quad H_2(v) = \frac{v^{\alpha+1}}{1+2v}.$$

Substituting the values of functions $F_1, H_1, H_2, A = 1, B = -1, C = 0, D = -1, E = 1$, and $\Omega = 0$ in (4.5), we obtain

$$G_{2,q}[F(\xi,\eta)](u,v) = \frac{\frac{u^{\alpha-1}v^{\alpha+1}}{1+2v} + \frac{u^{\alpha}v^{\alpha+1}}{1+2v} - \frac{u^{\alpha+1}v^{\alpha-1}}{1-u} - \frac{u^{\alpha+1}v^{\alpha}}{1-u} + \frac{2u^{\alpha+1}v^{\alpha}}{1-u}}{\frac{1}{u^2} - \frac{1}{v^2} - \frac{1}{v} + 1} = \frac{u^{\alpha+1}v^{\alpha+1}}{(1-u)(1+2v)}. \tag{4.15}$$

Taking the inverse q-DLTT in (4.15), then the solution of (4.14) is

$$F(\xi,\eta) = G_{2,q}^{-1}\left[\frac{u^{\alpha+1}v^{\alpha+1}}{(1-u)(1+2v)}\right] = e_q(\xi)e_q(-2\eta).$$

Figure 5 shows the solution of q-Klein-Gordon equation with conditions when we set $q = 0.85$, which is shown in the left side of Figure 5 and we set $q \to 1$, which is shown in the right side of Figure 5.

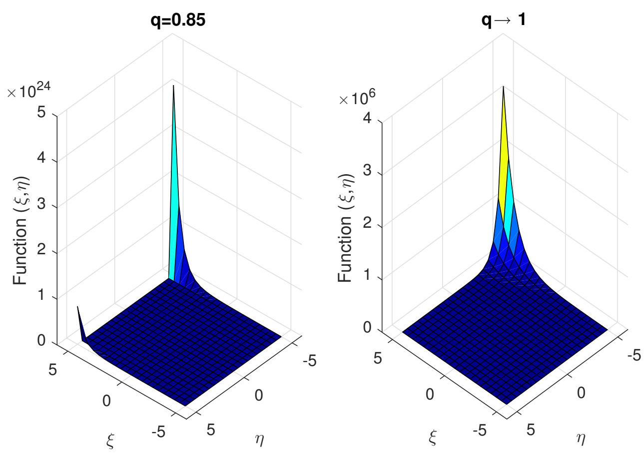

Figure 5: Plot illustration for Example 4.5.

In Table 1, we compare exact solution and q-calculus of PDEs by using infinity norm error (L_∞) with the value of N from 10, 20, and 100, which is a parameter of the convergence of the proposed scheme. When $q = 0.8045, q = 0.95, q = 0.99999$, and $q \to 1$ the errors are decreasing when N is a higher. In Table 2, when $q = 0.85, q = 0.95, q = 0.99999$, and $q \to 1$ the errors are decreasing when N is a higher. In Table 3, when $q = 0.85, q = 0.95, q = 0.99999$, and $q \to 1$ the errors are decreasing when N is a higher. In Table 4, when $q = 0.85$ and $q = 0.95$ the errors are increasing when N is greater but $q = 0.99999$ and $q \to 1$ the errors are decreasing when N is a higher. In Table 5, when $q = 0.85$ and $q = 0.95$ the errors are increasing when N is a higher but $q = 0.99999$ and $q \to 1$ the errors are decreasing when N is a higher.

Table 1: Comparison between the exact solution and quantum calculus of PDE regarding Example 4.1.

Number of iteration (N)	Infinity norm error (L_∞)			
	$q = 0.8045$	$q = 0.95$	$q = 0.99999$	$q \to 1$
10	2.7095×10^4	3.1671	8.66814×10^{-4}	5.1558×10^{-4}
20	1.1230×10^4	3.1639	7.3939×10^{-4}	1.4211×10^{-14}
100	29.4669	3.1693	7.3939×10^{-4}	1.4211×10^{-14}

Table 2: Comparison between the exact solution and q-calculus of PDE regarding Example 4.2.

Number of iteration (N)	Infinity norm error (L_∞)			
	$q = 0.85$	$q = 0.95$	$q = 0.99999$	$q \to 1$
10	98.3967	4.6863	0.0010	1.4377×10^{-4}
20	9.3509	4.6038	0.0009	4.5574×10^{-14}
100	9.3304	4.6038	0.0009	4.5574×10^{-14}

Table 3: Comparison between the exact solution and q-calculus of PDE regarding Example 4.3.

Number of iteration (N)	Infinity norm error (L_∞)			
	$q = 0.85$	$q = 0.95$	$q = 0.99999$	$q \to 1$
10	9.0382×10^3	2.9289×10^2	0.0459	0.0063
20	3.3283×10^3	2.9211×10^2	0.0398	3.9080×10^{-13}
100	3.3070	2.9211×10^2	0.0398	3.9080×10^{-13}

Table 4: Comparison between the exact solution and q-calculus of PDE regarding Example 4.4.

Number of iteration (N)	Infinity norm error (L_∞)			
	$q = 0.85$	$q = 0.95$	$q = 0.99999$	$q \to 1$
10	4.9657×10^9	6.6311×10^9	6.7297×10^9	6.7297×10^9
20	1.6299×10^{14}	1.2514×10^{11}	2.9971×10^9	2.9996×10^9
100	3.0587×10^{52}	3.1791×10^{15}	6.6324×10^6	5.0366×10^{-6}

Table 5: Comparison between the exact solution and q-calculus of PDE regarding Example 4.5.

Number of iteration (N)	Infinity norm error (L_∞)			
	$q = 0.85$	$q = 0.95$	$q = 0.99999$	$q \to 1$
10	1.4560×10^8	4.9930×10^6	3.5141×10^6	3.5150×10^6
20	4.1138×10^{10}	4.6663×10^7	1.0741×10^4	1.3195×10^4
100	7.1047×10^{24}	5.1731×10^7	2.4693×10^3	3.3182×10^{-9}

5. Conclusion

In this study, we have introduced new extensions of the double Laplace-type integral transform (DLTT) by utilizing quantum calculus, which we called the double q-Laplace-type integral transform (q-DLLT). Then, we have derived several basic properties and theorems related to q-Taylor expansion, q-convolution, partial q-derivatives of order one and two, and partial q-derivatives of order n. After that, we established the formula that can solve q-PDEs, such as q-wave equation, q-Klein-Gordon equation, q-Laplace equation, q-heat equation, and q-telegraph equation; and plot 3D using q-DLLT. Also, after we made the convergence comparison, we discussed changing the value of N; the obtained L_∞ error was reduced up to a high and acceptable order of convergence when $q = 0.99999$ and $q \to 1$.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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