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Exponential stability of gene regulatory networks with distributed delays

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Abstract

This paper explores the global exponential stability (GES) for a class of genetic regulatory networks (GRNs) with distributed time-varying delays. Firstly, we establish a new class of GRNs with recent-history distributed time-varying delays, which can be useful for better understanding of GRNs dynamics more precisely. Then, to simplify the complexity of the main results, a suitable Lyapunov functional is constructed with the help of Lyapunov stability theory, and some new delay-dependent exponential stability conditions for the considered GRNs are derived. Different from other papers, this paper presents novel results that are easy to solve and new insights into GES of GRNs. Finally, two standard numerical examples along with their simulation analysis are presented to illustrate the effectiveness of theoretical results.

Keywords: Gene regulatory networks, global exponential stability, distributed delays, Lyapunov functional.

2020 MSC: 92B20, 93D23, 93D05.

1. Introduction

The concept of GRNs is a collection of molecules that interact with one another and with other molecules within the cell to regulate gene expression levels, which was first proposed in the 1960s, and it has received significant attention from numerous researchers [1–8]. The study of GRNs is multidisciplinary, involving biology, control science, computing, electronics, and theoretical physics [9–13]. From a mathematical perspective, there are two classes of GRNs: differential equation model which is a continuous model [14, 16], and Boolean model which is a discrete model [2, 15]. In the continuous model, the variables describing the concentrations of both mRNA and proteins are continuous values, which can provide a detailed explanation of the nonlinear dynamical behavior that is exhibited by GRNs [17–20]. On the other hand, stability is an important aspect of designing or controlling GRNs; it is of great importance to study the stability of GRNs. To date, several results on the stability of the GRNs have been published in the literature by employing Lyapunov functions and inequality techniques [24, 25, 28–32, 34, 35].

In real-life situations, time delay is unavoidable in the process of propagating signals of biological networks in the context of very large scale integration circuits since the transmission speed of information

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is finite in such circuits [3, 18, 25, 29, 30]. There is also a special nature of NNs, since they consist of many parallel pathways of different axon lengths and sizes, which is why they can be modeled with distributed delays because they consist of many parallel pathways [9, 15, 21, 24, 25]. In reality, it is actually more practical to use a GRN model with distributed delays when dealing with individual molecules, since movement of mRNA from transcription sites to translation sites is an active process with a wide range of transport times. The existence of time-delayed signals in networks the dynamics leads to instability in the networks, resulting in poor performance and a loss of efficiency. As a result of the slow process of translation, transcription, and translocation in GRNs, there can be a time delay in signal, and in practice, this delayed signal is a time-varying signal, and it is important to consider the time delays when designing GRNs. As a results, several studies have been published in the literature which examine the dynamics of GRNs by considering different time delays [16, 21, 22, 28–30].

In recent years, there are several researchers have conducted studies to investigate the various dynamical behaviors of GRNs including, synchronization, asymptotic stability and exponential stability [21–24, 26, 27, 33]. Unlike asymptotic stability, exponential stability is particularly advantageous because it ensures that the system's state converges to equilibrium at a rate proportional to an exponential function, which is generally faster and more predictable than other forms of stability [10, 15, 21]. Consequently, much effort has been spent on studies of the GES of GRNs, and a wide range of scientific results have been published in the literature regarding the GES of GRNs [2, 10, 15, 21]. For example, in [21], the authors studied the decentralized event-triggered exponential stability for uncertain delayed GRNs with Markov jump parameters and distributed delays. In [2], the GES problem of discrete-time GRNs with time delays has been discussed, and several conditions have been established using the Lyapunov method. In [15], the authors have studied the GES problem of discrete-time GRNs with time-varying discrete delays and unbounded distributed delays, and also several conditions have been established. However, there have been very few studies conducted on the GES of GRNs with distributed time-varying delays. In addition, there is still a lot of room to explore further in the GES of GRNs with distributed time-varying delays. This motivates us to investigate the present study.

Motivated by the above discussions, this study mainly focuses on the GES problem of GRNs with with distributed time-varying delays. This study includes the following significant merites. (i) To represent more realistic dynamical behaviors of GRNs, we considered a general form of GRNs with distributed time-varying delays. (ii) By constructing suitable Lyapunov functions, several delay-dependent sufficient conditions have been derived to ensure the GES of the considered NNs. In addition, the obtained GES criteria are directly expressed by the system parameters, which is more straightforward and can be easily verified in practice. (iii) To verify the effectiveness and the correctness of theoretical results, two standard numerical examples along with their simulation analysis are presented.

This study is structured in the following manner. Section 2 delivers the problem model and preliminaries. Section 3 delivers the GES results; Theorem 3.1, Corollary 3.3, and Corollary 3.5 provide several suitable conditions for GES. Section 4 delivers two numerical examples and Section 5 delivers the conclusions of this study.

2. Problem model and preliminaries

2.1. Notations

Throughout this paper, the symbol $\mathbb{R}^n$ represents the n-dimensional Euclidean space. $\mathbb{R}^{n\times m}$ denotes the set of all real matrices with n rows and m columns. The matrix $A > 0 (A < 0)$ indicates that positive (negative) definite matrix. The transpose of matrix A is denoted as A^T, I represents the identity matrix and $\|\cdot\|_1$ denotes the 1-norm, $\|\alpha\|_1 = \sum_{i=1}^{n} |\alpha_i|$, $\|A\|_1 = \sup_{\alpha\neq 0} \dfrac{\|A\alpha\|_1}{\|\alpha\|_1}$.

2.2. Problem model

This paper considers a class of GRNs with distributed delays as follows:

$$\begin{cases} \dfrac{dm_i(t)}{dt} = -d_i m_i(t) + \sum_{j=1}^{n} a_{ij} g_j(p_j(t-\tau(t))) + \sum_{j=1}^{n} b_{ij} \int_0^{\tau(t)} K_{ij}(s) g_j(p_j(t-s)) ds + I_i, \\ \dfrac{dp_i(t)}{dt} = -c_i p_i(t) + e_i \int_0^{\tau(t)} K_i(s) m_i(t-s) ds, \end{cases} \tag{2.1}$$

where $i = 1, 2, \ldots, n$; $m_i(t)$, $p_i(t)$ are the mRNA and protein concentrations of the i^{th} node at time t; d_i and c_i represent the degradation rates of mRNA and protein, respectively; and e_i is the translation rate; $\tau(t)$ is the transcriptional and translational delays, which satisfies $0 \leqslant \tau(t) \leqslant \tau^*$, $\dot{\tau}(t) \leqslant \delta < 1$; $g_j(p_j(\cdot)) = \frac{(p_j(\cdot)/\nu_j)^{H_j}}{1+(p_j(\cdot)/\nu_j)^{H_j}}$ is the feedback regulation of the protein in the transcription process, where H_j is the Hill coefficient and $\nu_j > 0$ is a constant; $K_{ij}(s)$ is the delay kernel; $I_i = \sum_{\kappa_i} \vartheta_{ij}$ is the basal rate, where κ_i is the set of all the j which is a repressor of the gene i and ϑ_{ij} is the dimensionless transcriptional rate of the transcription factor j to the gene i; a_{ij}, b_{ij} are coupling matrix of the gene network and denoted by

$$a_{ij} = \begin{cases} \vartheta_{ij}, & \text{transcription factor activator of gene } i, \\ 0, & \text{no connection between } j \text{ and } i, \\ -\vartheta_{ij}, & \text{transcription factor repressor of gene } i, \end{cases}$$

$$b_{ij} = \begin{cases} \vartheta_{ij}, & \text{transcription factor activator of gene } i, \\ 0, & \text{no connection between } j \text{ and } i, \\ -\vartheta_{ij}, & \text{transcription factor repressor of gene } i. \end{cases}$$

The initial condition of GRNs (2.1) is given by

$$m_i(t) = \gamma_i(t),\ t \in [-\tau,\ 0],\ i = 1, 2, \ldots n, \qquad p_i(t) = \zeta_i(t),\ t \in [-\tau,\ 0],\ i = 1, 2, \ldots n,$$

where $\gamma_i(t)$ and $\zeta_j(t)$ are continuous function defined on $[-\tau, 0]$.

The following assumption and definitions are crucial to derive the theoretical results presented in this study.

2.3. Preliminaries

(A1): The activation functions g_j $(j = 1, 2, \ldots n)$ are Lipschitz continuous on $\mathbb{R}^n$ with Lipschitz constants L_j^g, that is $|g_j(\alpha) - g_j(\beta)| \leqslant L_j^g |\alpha - \beta|$, $\forall \alpha, \beta \in \mathbb{R}$. Also we denote $\tilde{L}_g = \max_{1 \leqslant j \leqslant n} L_j^g$ and $K^* = \max_{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n} \left\{ \max_{s \in [\tau^*, 0]} K_{ji}(s) \right\}$.

(A2): Suppose that **(A1)** holds, GRNs (2.1) has at least one nonnegative equilibrium point $(m^*, p^*)^T$.

In order to shifting the equilibrium point into the origin, let $\alpha(\cdot) = m(\cdot) - m^*$, $\beta(\cdot) = p(\cdot) - p^*$, then the GRNs (2.1) becomes

$$\begin{cases} \dfrac{d\alpha_i(t)}{dt} = -d_i \alpha_i(t) + \sum_{j=1}^{n} a_{ij} g_j(\beta_j(t-\tau(t))) + \sum_{j=1}^{n} b_{ij} \int_0^{\tau(t)} K_{ij}(s) g_j(\beta_j(t-s)) ds, \\ \dfrac{d\beta_i(t)}{dt} = -c_i \beta_i(t) + e_i \int_0^{\tau(t)} K_i(s) \alpha_i(t-s) ds. \end{cases} \tag{2.2}$$

The vector matrix form of GRNs (2.2) can also be written as:

$$\begin{cases} \dfrac{d\alpha(t)}{dt} = -D\alpha(t) + Ag(\beta(t-\tau(t))) + B \int_0^{\tau(t)} K(s) g(\beta(t-s)) ds, \\ \dfrac{d\beta(t)}{dt} = -C\beta(t) + E \int_0^{\tau(t)} K(s) \alpha(t-s) ds, \end{cases} \tag{2.3}$$

where

$$\begin{aligned}
\alpha_i(t)) &= m_i(t) - m_i^*, & \beta_i(t)) &= p_i(t) - p_i^*,\\
\alpha(t) &= [\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t)]^T, & \beta(t) &= [\beta_1(t), \beta_2(t), \dots, \beta_n(t)]^T,\\
g_i(\beta_j(\cdot)) &= g_i(m_j(\cdot) + m_j^* + I_i) - g_i(m_j^* + I_i), & D &= \mathrm{diag}\{d_1, d_2, \dots, d_n\} \in \mathbb{R}^{n\times n},\\
C &= \mathrm{diag}\{c_1, c_2, \dots, c_n\} \in \mathbb{R}^{n\times n}, & E &= \mathrm{diag}\{e_1, e_2, \dots, e_n\} \in \mathbb{R}^{n\times n},\\
A &= (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}, & B &= (b_{ij})_{n\times n} \in \mathbb{R}^{n\times n}.
\end{aligned}$$

Definition 2.1. The zero equilibrium point $Z^* = [m^*, p^*]^T$ of GRNs (2.1) is said to be GES, if there exist constants $\theta > 0$ and $\epsilon > 0$ such that for any solution $Z(t) = [m(t), p(t)]^T$ with initial conditions $\psi(t) = [\gamma(t), \zeta(t)]^T \in \mathcal{C}([-\tau, 0], R^n)$ satisfies

$$\|Z(t) - Z^*\|_{1\Delta} \leqslant \theta e^{-\epsilon t}\|\psi(t) - z^*\|_{1\Delta},\ \forall t \geqslant 0,$$

where $\|\psi(t) - Z^*\|_{1\Delta} = \sum_{i=1}^n \sup_{t\in[-\tau^*,0]} |\gamma_i(t) - m_i^*| + \sum_{j=1}^n \sup_{t\in[-\tau^*,0]} |\zeta_j(t) - p_j^*|$.

3. Main results

This section explores the GES for GRNs (2.3) on the basis of Lyapunov stability theory.

Theorem 3.1. *Assume that* **(A1)** *holds, then the GRNs* (2.3) *is GES if the subsequent conditions are true:*

$$\frac{\|E\|_1}{1-\delta}K^*\tau^* < \|D\|_1, \quad \frac{\|A\|_1}{1-\delta}\tilde{L}_g + \frac{\|B\|_1}{1-\delta}K^*\tau^*\tilde{L}_g < \|C\|_1. \tag{3.1}$$

Proof. From (3.1), there exists $\epsilon > 0$, a small positive constant, such that

$$\|D\|_1 - \epsilon - \frac{\|E\|_1}{1-\delta}K^*\tau^*e^{\epsilon\tau^*} \geqslant 0, \quad \|C\|_1 - \epsilon - \frac{\|A\|_1}{1-\delta}\tilde{L}_g e^{\epsilon\tau^*} - \frac{\|B\|_1}{1-\delta}K^*\tau^*\tilde{L}_g e^{\epsilon\tau^*} \geqslant 0.$$

To obtain the result, we define the following Lyapunov functional $V(t)$ as follows

$$\begin{aligned}
V(t) =& \|\alpha(t)\|_1 e^{\epsilon t} + \|\beta(t)\|_1 e^{\epsilon t} + \frac{\|A\|_1}{1-\delta}\sum_{j=1}^n \int_{t-\tau(t)}^t |g_j(\beta_j(s))| e^{\epsilon(s+\tau^*)} ds\\
&+ \frac{\|B\|_1}{1-\delta}K^*\sum_{j=1}^n \int_{t-\tau(t)}^t (s - t + \tau(t))|g_j(\beta_j(s)| e^{\epsilon(s+\tau^*)} ds\\
&+ \frac{\|E\|_1}{1-\delta}K^*\sum_{i=1}^n \int_{t-\tau(t)}^t (s - t + \tau(t))\ \ |\alpha_i((s))| e^{\epsilon(s+\tau^*)} ds.
\end{aligned}$$

By employing the Dini right upper derivative of $V(t)$ of GRNs (2.3) along the trajectories, we have

$$\begin{aligned}
D^+V(t) =& \epsilon\|\alpha(t)\|_1 e^{\epsilon t} + D^+\|\alpha(t)\|_1 e^{\epsilon t} + \epsilon\|\beta(t)\|_1 e^{\epsilon t} + D^+\|\beta(t)\|_1 e^{\epsilon t} + \frac{\|A\|_1}{1-\delta}\sum_{j=1}^n |g_j(\beta_j(t))| e^{\epsilon(t+\tau^*)}\\
&- \frac{\|A\|_1}{1-\delta}\sum_{j=1}^n (1 - \dot{\tau}(t))|g_j(\beta_j(t - \tau(t)))| e^{\epsilon(t-\tau(t)+\tau^*)} + \frac{\|B\|_1}{1-\delta}K^*\tau(t)\sum_{j=1}^n |g_j(\beta_j(t))| e^{\epsilon(t+\tau^*)}\\
&- \frac{\|B\|_1}{1-\delta}K^*\sum_{j=1}^n \int_{t-\tau(t)}^t (1 - \dot{\tau}(t))|g_j(\beta_j(s))| e^{\epsilon(s+\tau^*)} ds + \frac{\|E\|_1}{1-\delta}K^*\tau(t)\sum_{i=1}^n |\alpha_i(t)| e^{\epsilon(s+\tau^*)}\\
&- \frac{\|E\|_1}{1-\delta}K^*\sum_{i=1}^n \int_{t-\tau(t)}^t (1 - \dot{\tau}(t))|\alpha_i(s)| e^{\epsilon(s+\tau^*)} ds
\end{aligned}$$

$$
\begin{aligned}
&\leqslant -(\|D\|-\epsilon)\|\alpha(t)\|_1 e^{\epsilon t} + \|A\|_1\|g(\beta(t-\tau(t)))\|_1 e^{\epsilon t} + \|B\|_1 K^* \int_0^{\tau(t)} |g(\beta(t-s))| ds e^{\epsilon t} \\
&\quad - (\|C\|-\epsilon)\|\beta(t)\|_1 e^{\epsilon t} + \|E\|_1 K^* \int_0^{\tau(t)} |\alpha(t-s))| ds e^{\epsilon t} + \frac{\|A\|_1}{1-\delta}\tilde{L}_g e^{\epsilon\tau^*}\|\beta(t)\|_1 e^{\epsilon t} \\
&\quad - \|A\|_1\|g(\beta(t-\tau(t)))\|_1 e^{\epsilon t} + \frac{\|B|_1}{1-\delta}K^*\tilde{L}_g e^{\epsilon\tau^*}\|\beta(t)\|_1 e^{\epsilon t} - \|B\|_1 K^* \int_0^{\tau(t)} g(\beta(t-s))|_1 ds e^{\epsilon t} \\
&\quad + \frac{\|E\|_1}{1-\delta}K^*\tau^* e^{\epsilon\tau^*}\|\alpha(t)\|_1 e^{\epsilon t} - \|E\|_1 K^* \int_0^{\tau(t)} |\alpha((t-s))| ds e^{\epsilon t} \\
&\leqslant -\left(\|D\|_1 - \epsilon - \frac{\|E\|_1}{1-\delta}e^{\epsilon\tau^*}\right)\|\alpha(t)\|_1 e^{\epsilon t} \\
&\quad - \left(\|C\|_1 - \epsilon - \frac{\|A\|_1}{1-\delta}\tilde{L}_g e^{\epsilon\tau^*} - \frac{\|B\|_1}{1-\delta}K^*\tau^*\tilde{L}_g e^{\epsilon\tau^*}\right)\|\beta(t)\|_1 e^{\epsilon t}
\end{aligned}
$$

$D^+V(t) \leqslant 0.$

Then, one can obtain that,

$$
\begin{aligned}
&\|\alpha(t)\|_1 e^{\epsilon t} + \|\beta(t)\|_1 e^{\epsilon t} \\
&\leqslant V(t) \\
&\leqslant V(0) \\
&= \|\alpha(0)\|_1 + \|\beta(0)\|_1 + \frac{\|A\|_1}{1-\delta}\sum_{j=1}^{n}\int_{-\tau(0)}^{0} |g_j(\beta_j(s)|e^{\epsilon(s+\tau^*)} ds \\
&\quad + \frac{\|B\|_1}{1-\delta}K^*\sum_{j=1}^{n}\int_{-\tau(0)}^{0}(s+\tau(0))|g_j(\beta_j(s)|e^{\epsilon(s+\tau^*)} ds + \frac{\|E\|_1}{1-\delta}K^*\sum_{i=1}^{n}\int_{-\tau(0)}^{0}(s+\tau(0))|\alpha_i(s)|e^{\epsilon(s+\tau^*)} ds \\
&\leqslant \|\alpha(0)\|_1 + \|\beta(0)\|_1 + \frac{\|A\|_1}{1-\delta}\tilde{L}_g\tau^* e^{\epsilon\tau^*}\sum_{j=1}^{n}\sup_{s\in[-\tau(0),0]}|\zeta_j(s)-p_j^*| \\
&\quad + \frac{\|B\|_1}{1-\delta}\tilde{L}_g K^*\tau^{*2}e^{\epsilon\tau^*}\sum_{j=1}^{n}\sup_{s\in[-\tau(0),0]}|\zeta_j(s)-p_j^*| + \frac{\|E\|_1}{1-\delta}K^*\tau^{*2}e^{\epsilon\tau^*}\sum_{i=1}^{n}\sup_{s\in[-\tau(0),0]}|\gamma_i(s)-m_i^*| \\
&\leqslant \left(1+\frac{\|E\|_1}{1-\delta}K^*\tau^{*2}e^{\epsilon\tau^*}\right)\|\gamma-m^*\|_{1\Delta} + \left(1+\frac{\|A\|_1}{1-\delta}\tilde{L}_g\tau^* e^{\epsilon\tau^*} + \frac{\|B\|_1}{1-\delta}K^*\tau^{*2}\tilde{L}_g e^{\epsilon\tau^*}\right)\|\zeta-p^*\|_{1\Delta} \\
&= \theta_1\|\gamma-m^*\|_{1\Delta} + \theta_2\|\zeta-p^*\|_{1\Delta} \leqslant \theta\Big(\|\gamma-m^*\|_{1\Delta} + \|\zeta-p^*\|_{1\Delta}\Big),
\end{aligned}
$$

where

$$
\theta_1 = 1+\frac{\|E\|_1}{1-\delta}\tau^{*2}e^{\epsilon\tau^*}, \quad \theta_2 = 1+\frac{\|A\|_1}{1-\delta}\tilde{L}_g\tau^* e^{\epsilon\tau^*} + \frac{\|B\|_1}{1-\delta}K^*\tau^{*2}\tilde{L}_g e^{\epsilon\tau^*}, \quad \theta = \max\Big\{\theta_1,\theta_2\Big\}.
$$

Hence, we obtain

$$
\|\alpha(t)\|_1 + \|\beta(t)\|_1 \leqslant \theta\Big(\|\gamma(t)\|_{1\Delta} + \|\zeta(t)\|_{1\Delta}\Big)e^{-\epsilon t}.
$$

By Definition 2.1, we obtained that the equilibrium point of (2.3) is GES. The proof is completed. □

Remark 3.2. When the time-varying delay $\tau(t) = \tau$, then the GRNs (2.3) becomes:

$$
\begin{cases}
\dfrac{d\alpha(t)}{dt} = -D\alpha(t) + Ag(\beta(t-\tau)) + B\int_0^{\tau} K(s)g(\beta(t-s))ds + I, \\
\dfrac{d\beta(t)}{dt} = -C\beta_i(t) + E\int_0^{\tau} K(s)\alpha(t-s)ds.
\end{cases} \tag{3.2}
$$

The following Corollary 3.3 gives the GES criterion for the GRNs.

Corollary 3.3. *Assume that* **(A1)** *holds, then the GRNs* (3.2) *is GES if the subsequent conditions are true:*

$$\|E\|_1 K^* \tau < \|D\|_1, \quad \|A\|_1 \tilde{L}_g + \|B\|_1 K^* \tau \tilde{L}_g < \|C\|_1.$$

Proof. Define the following Lyapunov functional,

$$V(t) = \|\alpha(t)\|_1 e^{\epsilon t} + \|\beta(t)\|_1 e^{\epsilon t} + \|A\|_1 \sum_{j=1}^{n} \int_{t-\tau}^{t} |g_j(\beta_j(s))| e^{\epsilon(s+\tau)} ds$$
$$+ \|B\|_1 K^* \sum_{j=1}^{n} \int_{t-\tau}^{t} (s-t+\tau) |g_j(\beta_j(s)| e^{\epsilon(s+\tau)} ds + \|E\|_1 K^* \sum_{i=1}^{n} \int_{t-\tau}^{t} (s-t+\tau) |\alpha_i((s))| e^{\epsilon(s+\tau)} ds.$$

By applying the similar procedure of Theorem 3.1, the remaining results can be obtained. □

Remark 3.4. When $A = 0$, then the GRNs (2.3) becomes:

$$\begin{cases} \frac{d\alpha(t)}{dt} = -D\alpha(t) + B \int_0^{\tau(t)} K(s) g(\beta(t-s)) ds, \\ \frac{d\beta(t)}{dt} = -C\beta(t) + E \int_0^{\tau(t)} K(s) \alpha(t-s) ds. \end{cases} \tag{3.3}$$

By applying the similar procedure of Theorem 3.1, the following Corollary 3.5 can be derived for the GRNs (3.3).

Corollary 3.5. *Assume that* **(A1)** *holds, then the GRNs* (3.3) *is GES if the subsequent conditions are true:*

$$\frac{\|E\|_1}{1-\delta} K^* \tau^* < \|D\|_1, \quad \frac{\|B\|_1}{1-\delta} K^* \tau^* \tilde{L}_g < \|C\|_1.$$

Remark 3.6. In recent years, there have been several notable studies have been conducted on the dynamical analysis of GRNs. For example, in [9], the authors examined GRNs with distributed delay and derived some sufficient conditions for robust stability by constructing suitable Lyapunov functional. In [13], the authors considered a class of GRNs with disturbance attenuation and discussed the stochastic stability analysis of GRNs. In [14], the authors examined Markovian switching uncertain stochastic GRNs with unbounded time-varying delays and also derived some sufficient conditions for robust μ-stability based on Lyapunov stability theory. In [16], the authors derived the multistability criteria for delayed GRNs with multivariable regulation function. It is important to note that there have been only few studies conducted on the GES of GRNs with distributed time-varying delays by Lyapunov stability theory. Therefore, this paper aims to fill such gap by investigating GES of GRNs with distributed time-varying delays.

4. Illustrative examples

This section presents two numerical case studies to describe the advantages and applicability of the theoretical results.

Example 4.1. Consider the following three-neuron GRNs with distributed delays as follows:

$$\frac{d\alpha_1(t)}{dt} = -d_1\alpha_1(t) + a_{11} g_1(\beta_1(t-\tau(t))) + a_{12} g_2(\beta_2(t-\tau(t))) + a_{13} g_3(\beta_3(t-\tau(t)))$$
$$+ b_{11} \int_0^{\tau(t)} K_1(s) g_1(\beta_1(t-s)) ds + b_{12} \int_0^{\tau(t)} K_1(s) g_2(\beta_2(t-s)) ds$$
$$+ b_{13} \int_0^{\tau(t)} K_1(s) g_3(\beta_3(t-s)) ds + I_1,$$

$$\begin{aligned}
\frac{d\alpha_2(t)}{dt} &= -d_2\alpha_2(t) + a_{21}g_1(\beta_1(t-\tau(t))) + a_{22}g_2(\beta_2(t-\tau(t)) + a_{23}g_3(\beta_3(t-\tau(t)) \\
&\quad + b_{21}\int_0^{\tau(t)} K_2(s)g_1(\beta_1(t-s))ds + b_{22}\int_0^{\tau(t)} K_2(s)g_2(\beta_2(t-s))ds \\
&\quad + b_{23}\int_0^{\tau(t)} K_2(s)g_3(\beta_3(t-s))ds + I_2, \qquad (4.1)\\
\frac{d\alpha_3(t)}{dt} &= -d_3\alpha_3(t) + a_{31}g_1(\beta_1(t-\tau(t))) + a_{32}g_2(\beta_2(t-\tau(t))) + a_{33}g_3(\beta_3(t-\tau(t))) \\
&\quad + b_{31}\int_0^{\tau(t)} K_3(s)g_1(\beta_1(t-s))ds + b_{32}\int_0^{\tau} K_3(s)g_2(\beta_2(t-s))ds \\
&\quad + b_{33}\int_0^{\tau(t)} K_3(s)g_3(\beta_3(t-s))ds + I_3, \\
\frac{d\beta_1(t)}{dt} &= -c_1\beta_1(t) + e_1\int_0^{\tau(t)} K_1(s)(\alpha_1(t-\tau(t)))ds, \\
\frac{d\beta_2(t)}{dt} &= -c_2\beta_2(t) + e_2\int_0^{\tau(t)} K_2(s)(\alpha_2(t-\tau(t))ds, \\
\frac{d\beta_3(t)}{dt} &= -c_3\beta_2(t) + e_3\int_0^{\tau(t)} K_3(s)(\alpha_3(t-\tau(t))ds,
\end{aligned}$$

where $d_1 = 1$, $d_2 = 1$, $d_3 = 1$, $a_{11} = -0.02$, $a_{12} = 0.01$, $a_{13} = 0.03$, $a_{21} = -0.01$, $a_{22} = -0.02$, $a_{23} = 0.07$, $a_{31} = 0.1$, $a_{32} = 0.03$, $a_{33} = -0.02$, $b_{11} = -0.03$, $b_{12} = 0.02$, $b_{13} = 0.01$, $b_{21} = -0.02$, $b_{22} = -0.04$, $b_{23} = 0.01$, $b_{31} = 0.2$, $b_{32} = -0.04$, $b_{33} = -0.01$, $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, $e_1 = 0.5$, $e_2 = 0.5$, $e_3 = 0.5$. The time-varying delay $\tau(t)$ is considered as $\tau(t) = 0.2 + 0.5\sin(t)$, which implies that $0 \leqslant \tau(t) \leqslant \tau^* = 0.7$. It is observable that $0 \leqslant \dot{\tau}(t) \leqslant \delta = 0 \leqslant 0.5\cos(t) \leqslant 0.3$. The nonlinear neuron activation function is taken as $g_1(\cdot) = 0.5\tanh(\cdot)$, $g_2(\cdot) = 0.5\tanh(\cdot)$, $g_3(\cdot) = 0.5\tanh(\cdot)$, it is easy to verify that the activation function is satisfied with $\tilde{L}_g = 1$.

Let $K_1(s) = K_2(s) = K_3(s) = 0.3(1-\sin(s))$, $K^* = 0.3$, according to Theorem 3.1, it is not hard to verify that

$$\begin{aligned}
\frac{\|E\|_1}{1-\delta}K^*\tau^* < \|D\|_1 &\Rightarrow \frac{0.5}{0.5} \times 0.3 \times 0.7 = 0.21 < 1, \\
\frac{\|A\|_1}{1-\delta}\tilde{L}_g + \frac{\|B\|_1}{1-\delta}K^*\tau^*\tilde{L}_g < \|C\|_1 &\Rightarrow \frac{0.13}{0.5} + \frac{0.25}{0.5} \times 0.3 \times 0.7 \times 1 = 0.365 < 1,
\end{aligned}$$

which implies that all the conditions of Theorem 3.1 is fulfilled with the above values. Therefore, according to Theorem 3.1 we can determine that GRNs (4.1) is GES.

Under the initial conditions $\zeta(t) = [0.4, -0.3, 0.2]^T$ and $\gamma(t) = [-0.3, 0.3, -0.6]^T$, the time response of mRNA and proteins are given in Figure 1. The phase plots of mRNA and proteins are shown in Figures 2 and 3. From simulation results in Figures 1-3, it is clear that the GRNs (4.1) is GES.

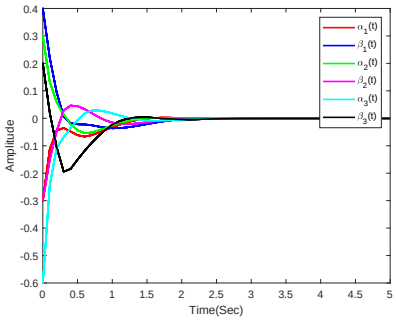

Figure 1: Time response of mRNA and protein concentrations $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$, $\beta_1(t)$, $\beta_2(t)$, $\beta_3(t)$ of GRNs (4.1).

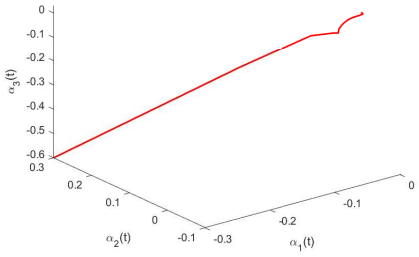

Figure 2: Phase plot of $\alpha_1(t)$, $\alpha_2(t)$ and $\alpha_3(t)$ of GRNs (4.1).

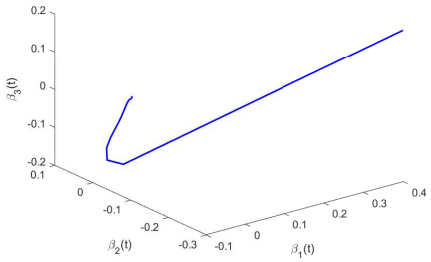

Figure 3: Phase plot of $\beta_1(t)$, $\beta_2(t)$, and $\beta_3(t)$ of GRNs (4.1).

Example 4.2. Consider the following two-neuron GRNs with distributed delays as follows:

$$\begin{aligned}
\frac{d\alpha_1(t)}{dt} &= -d_1\alpha_1(t) + a_{11}g_1(\beta_1(t-\tau)) + a_{12}g_2(\beta_2(t-\tau)) \\
&\quad + b_{11}\int_0^\tau K_1(s)g_1(\beta_1(t-s))ds + b_{12}\int_0^\tau K_1(s)g_2(\beta_2(t-s))ds + I_1, \\
\frac{d\alpha_2(t)}{dt} &= -d_2\alpha_2(t) + a_{21}g_1(\beta_1(t-\tau) + a_{22}g_2(\beta_2(t-\tau) \\
&\quad + b_{21}\int_0^\tau K_2(s)g_1(\beta_1(t-s))ds + b_{22}\int_0^\tau K_2(s)g_2(\beta_2(t-s))ds + I_2, \\
\frac{d\beta_1(t)}{dt} &= -c_1\beta_1(t) + e_1\int_0^\tau K_1(s)(\alpha_1(t-\tau))ds, \\
\frac{d\beta_2(t)}{dt} &= -c_2\beta_2(t) + e_2\int_0^\tau K_2(s)(\alpha_2(t-\tau)ds,
\end{aligned} \tag{4.2}$$

where $d_1 = 1$, $d_2 = 1$, $a_{11} = -0.03$, $a_{12} = 0.02$, $a_{21} = 0.05$, $a_{22} = 0.03$, $b_{11} = -0.04$, $b_{12} = 0.1$, $b_{21} = 0.02$, $b_{22} = 0.01$, $c_1 = 1$, $c_2 = 1$, $e_1 = 0.5$, $e_2 = 0.5$. The time delay τ is considered as $\tau = 0.5$, and the nonlinear activation function is taken as $g_1(\cdot) = 0.5\tanh(\cdot)$, $g_2(\cdot) = 0.5\tanh(\cdot)$. One can obtain that the activation function is satisfied with $\tilde{L}_g = 1$.

Let $K^* = 0.3$, according to Corollary 3.3, it is not hard to verify that

$$\begin{aligned}
\|E\|_1 K^*\tau < \|D\|_1 &\Rightarrow 0.5 \times 0.3 \times 0.5 = 0.075 < 1, \\
\|A\|_1\tilde{L}_g + \|B\|_1 K^*\tau\tilde{L}_g < \|C\|_1 &\Rightarrow 0.08 \times 1 + 0.11 \times 0.3 \times 0.5 \times 1 = 0.0965 < 1,
\end{aligned}$$

which implies that all the conditions of Corollary 3.3 is fulfilled with the above values. Therefore, according to Corollary 3.3 we can determine that GRNs 4.2 is GES.

Under randomly selected 10 initial conditions, the trajectories of mRNA and proteins are given Figures 4 and 5. The phase plots of mRNA and proteins are shown in Figures 6 and 7. From simulation results in Figures 4-7, it is clear that the GRNs (4.2) is GES.

5. Conclusions

This paper explored the GES problem for a class of GRNs with distributed time-varying delays. Firstly, we establish a new class of GRNs with recent-history distributed time-varying delays, which can be useful for better understanding of GRNs dynamics more precisely. Then, to simplify the complexity of the main results, a suitable Lyapunov functional is constructed with the help of Lyapunov stability theory, and some new delay-dependent exponential stability conditions for the considered GRNs are derived. Different from other papers, this paper presents novel results that are easy to solve and new insights into exponential stability of GRNs. Finally, two standard numerical examples along with their simulation analysis are presented to illustrate the effectiveness of theoretical results.

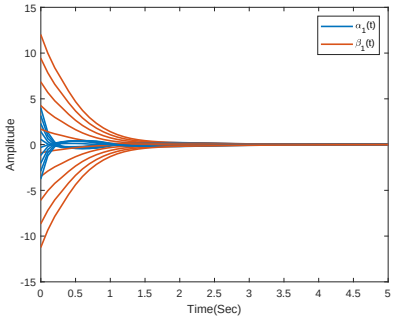

Figure 4: Time response of mRNA and protein concentrations $\alpha_1(t)$ and $\beta_1(t)$ of GRNs (4.2).

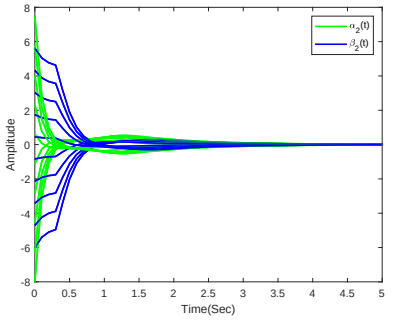

Figure 5: Time response of mRNA and protein concentrations $\alpha_2(t)$ and $\beta_2(t)$ of GRNs (4.2).

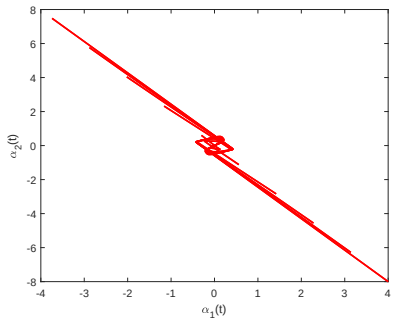

Figure 6: Phase plot of $\alpha_1(t)$ and $\alpha_2(t)$ of GRNs (4.2).

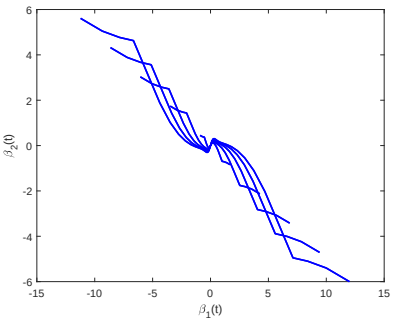

Figure 7: Phase plot of $\beta_1(t)$ and $\beta_2(t)$ of GRNs (4.2).

CRediT authorship contribution statement

S. Logeswari: conceptualization, methodology, investigation, writing-original draft, writing-review and editing, and software. R. Sriraman: supervision and validation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the paper.

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