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Numerical and visual analysis of Maclaurin type inequalities in the setting of generalized fractional calculus and applications

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Abstract

Yang local fractional calculus is very effective tools to investigate the non-differentiable functions. Moreover, local fractional calculus generalize the classical results and provide more general framework to investigate various problems. This study intends to construct new versions of Maclaurin's inequality for generalized fractional calculus. The study introduces a fresh equation for first-order local differentiable mappings. We develop new error estimates of Maclaurin's inequality incorporated with newly proposed identity, local fractional (L.F) variants of Hölder's type inequalities and generalized convexity. Additionally, we discuss some potential consequences of our primary findings to ensure the worth of our findings. We establish some interesting relations between generalized means, error boundaries of composite quadrature schemes within fractal space and probability distribution. We justify the accuracy of proposed results through visual analysis. The bounds obtained in our study are the better bounds as compared to previously established results. Also, for different values of $\sigma \in (0, 1]$, blend of inequalities can be obtained.

Keywords: Convex functions, Maclaurin inequality, Hölder's inequality, Newton-Cotes formula, local fractional.

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1. Introduction and preliminaries

In the literature, various methods are used to analyze errors in numerical integral techniques. One common challenge occurs when dealing with functions that are first- or second-order differentiable, making it difficult to precisely formulate the error terms for open and closed methods, like Simpson's schemes.

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Integral inequalities are used to establish error bounds for first-order differentiable convex functions. Another problem is how to find the reminder terms of quadrature rules for non-differentiable mapping. In the following scenarios, quantum calculus and Yang's L.F calculus are beneficial to tackle such problems. In addition to this, specific integral inequalities have been developed for non-convex functions with properties similar to convex functions to address these challenges. The notion of convex functions is defined as

$$\Psi((1-\ell)\mu+\ell y)\leqslant(1-\ell)\Psi(\mu)+\ell\Psi(y),\quad \forall \mu, y\in[\vartheta,\rho],$$

where $\ell \in [0,1]$. One of extensively investigated results pertaining to the convex functions in the theory of inequalities is Hermite-Hadamard's inequality. It reads as follows. Suppose $\Psi : I=[\vartheta,\rho]\subset\mathbb{R}\to\mathbb{R}$ is a convex function, then

$$\Psi\left(\frac{\vartheta+\rho}{2}\right)\leqslant\frac{1}{\rho-\vartheta}\int_{\vartheta}^{\rho}\Psi(\mu)d\mu\leqslant\frac{\Psi(\vartheta)+\Psi(\rho)}{2}.$$

It is worth mentioning here the analysis of Hermite-Hadamard's inequality from both the right and left sides provide us the bounds for the remainder terms of trapezoidal and midpoint rules used for numerical integration, see [3]. Simpson's rule is also one of the most commonly used three-point approximation integration rule. We now discuss Simpson's $\frac{1}{8}$-formula and its corresponding error inequality:

$$\int_{\vartheta}^{\rho}\Psi(\mu)d\mu=\frac{1}{6}\left[\Psi(\vartheta)+4\Psi\left(\frac{\vartheta+\rho}{2}\right)+\Psi(\rho)\right].$$

Theorem 1.1 ([2])**.** *Suppose* $\Psi : [\vartheta,\rho]\to\mathbb{R}$ *ia a fourth-order continuously differentiable function on* (ϑ,ρ) *and* $\|\Psi^{(4)}\|_{\infty}=\sup_{\mu\in(\vartheta,\rho)}|\Psi^{(4)}|<\infty$*, then*

$$\left|\frac{1}{6}\left[\Psi(\vartheta)+4\Psi\left(\frac{\vartheta+\rho}{2}\right)+\Psi(\rho)\right]-\frac{1}{\rho-\vartheta}\int_{\vartheta}^{\rho}\Psi(\mu)d\mu\right|\leqslant\frac{1}{2880}\|\Psi^{(4)}\|_{\infty}(\rho-\vartheta)^5.$$

The Maclaurin method is used to address the limitations of Simpson's approach as it does not incorporate with any boundary points in its quadrature method. We introduce the Maclaurin-type inequality, expressed as:

$$\left|\frac{1}{8}\left(3\Psi\left(\frac{5\vartheta+\rho}{6}\right)+2\Psi\left(\frac{\vartheta+\rho}{2}\right)+3\Psi\left(\frac{\vartheta+5\rho}{6}\right)\right)-\frac{1}{\rho-\vartheta}\int_{\vartheta}^{\rho}\Psi(\mu)d\mu\right|\leqslant\frac{7(\rho-\vartheta)^2}{51840}\|\Psi^{(4)}\|_{\infty}.$$

The concept of fractals has been utilized in various scientific fields for over a century, but its importance in mathematics has significantly increased in recent years. Mandelbrot authored several monographs on this subject and put forward the conception of a fractal set, defined as a set with Hausdorff dimensions greater than its topological dimensions. This has led to numerous studies on the subject. In [24], Yang developed certain σ-level sets based on the assertion that σ represents the fractal set dimension. The concept of fractional calculus emerged alongside classical concepts to calculate fractional order derivatives and integrals. It has gained significant attention for its utility in mathematical modelling. Continuity is typically required for functions to be differentiable. When functions are not differentiable, L.F calculus becomes essential. Mo et al. [10] came up with the theory of convexity over $\mathbb{R}^{\sigma}$ and analyzed aspects of this class of functions, including fundamental inequalities over fractal set.

In [15] authors investigated Ostrowski-type inequalities from the perspective of generalized convexity. In 2018, Noor et al. [11] explored the harmonic convex functions to derive weighted Hadamard-like inequalities. Authors in [13] presented a stronger version of convex functions defined over $\mathbb{R}^{\sigma}$ and its characterization. By utilizing this class of convexity, new bounds of Jensen's like were acquired. Luo et al. [8] analyzed the weighted form of Hadamard-like inequalities related to h-convex functions defined over $\mathbb{R}^{\sigma}$. Sun et al. [21] established Hadamard-like inequalities considering harmonic convex functions and their connection with generalized fractional calculus. In 2021, Weibing Sun [20] used generalized convex functions to derive new Ostrowski's bounds with applications to information theory. Vivas-Cortez et al. [23] examined some novel extended forms of Bullen's like inequalities along with applications to non-linear analysis. Sarikaya and Budak [14] analyzed the weighted Hadamard's like inequalities by utilizing the concepts of Yang calculus. Du et al. [4] constructed several integral inequalities incorporated with m-convexity within fractal space. Du and Yuan [5] studied the unified integral inequalities by means of parametric identity and generalized convexity along with applications. In [22] authors discussed some new improved forms of fractal Ostrowsli's inequality pertaining to generalized convexity.

In 2013 [1], Alomari and Dragomir proposed the unified kernel and derived error inequalities for several Newton-Cotes formulas, including the trapezoidal, midpoint, Simpson's, Milne, and Maclaurin's inequalities. Meftah et al. [9] constructed error bounds for Maclaurin's method by using the concept of generalized convexity in the fractal domain, supported with applications. In 2023, Hezenci et al. [7] analyzed the fractional analogues of Maclaurin's-like inequalities via convex functions, with results verified by numerical and visual analysis. In [18], authors explored Maclaurin's type inequalities in the framework of quantum calculus. Penga and Duo [12] used multiplicative calculus to propose new error estimations concerning Milne's formula. For more detail, see [6, 16, 17, 19].

The motivation of this research is to acquire new versions of Maclaurin's inequality using generalized fractional calculus and a generalized concept of convexity. To provide new error estimates for Maclaurin's schemes, we establish a new identity that allows us to determine bounds on the remainder in the open methods we are investigating. We derive several new estimates and their specific cases, which have proven to be very beneficial. By selecting certain values of $\sigma \in (0, 1]$, we can obtain several Maclaurin's like inequalities. In addition, we validate our main results through numerical and visual analysis as well as through interesting applications involving probability distributions, quadrature schemes, and special means.

1.1. *Local fractional calculus*

We now present the facts of local calculus introduced by Yang [24]. First, we provide the σ type set of element sets:

1. $\mathbb{Z}^\sigma := \{\pm 0^\sigma, \pm 1^\sigma, \pm 2^\sigma, \ldots\}$;
2. $\mathbb{Q}^\sigma := \{\nu^\sigma = \left(\frac{p}{q}\right)^\sigma : p, q \in \mathbb{Z}, q \neq 0\}$;
3. $\mathbb{Q}'^\sigma := \{\nu^\sigma \neq \left(\frac{p}{q}\right)^\sigma : p, q \in \mathbb{Z}, q \neq 0\}$;
4. $\mathbb{R}^\sigma := \mathbb{Q}^\sigma \bigcup \mathbb{Q}'^\sigma$.

We discuss operations $+$ and $*$ (multiplication) on the σ-set $\mathbb{R}^\sigma$ of real numbers

$$c^\sigma + d^\sigma := (c + d)^\sigma \quad \text{and} \quad c^\sigma * d^\sigma = c^\sigma d^\sigma := (cd)^\sigma,$$

and both $c^\sigma + d^\sigma$, $c^\sigma d^\sigma \in \mathbb{R}^\sigma$.

• Also $(\mathbb{R}^\sigma, +)$ is a commutative group. For any $c^\sigma, d^\sigma, e^\sigma \in \mathbb{R}^\sigma$, $c^\sigma + d^\sigma = d^\sigma + c^\sigma$ and $(c^\sigma + d^\sigma) + e^\sigma = c^\sigma + (d^\sigma + e^\sigma)$. Also 0^σ is the additive identity of $\mathbb{R}^\sigma$, $0^\sigma + c^\sigma = c^\sigma + 0^\sigma$, $\forall c^\sigma \in \mathbb{R}^\sigma$. For any c^σ, there exists $(-c)^\sigma \in \mathbb{R}^\sigma$ such that $c^\sigma + (-c)^\sigma = 0^\sigma$.

• Also $(\mathbb{R}^\sigma, *) - \{0\}$ is a commutative group. For any $c^\sigma, d^\sigma, e^\sigma \in \mathbb{R}^\sigma$, $c^\sigma d^\sigma = d^\sigma c^\sigma$ and $(c^\sigma d^\sigma)e^\sigma = c^\sigma(d^\sigma e^\sigma)$. $1^\sigma \in \mathbb{R}^\sigma$, then for each $c^\sigma \in \mathbb{R}^\sigma$ such that $1^\sigma c^\sigma = c^\sigma 1^\sigma = c^\sigma$. For each $c^\sigma \in \mathbb{R}^\sigma$ there exists $\left(\frac{1}{c}\right)^\sigma$ such that $c^\sigma \left(\frac{1}{c}\right)^\sigma = \left(c\frac{1}{c}\right)^\sigma = 1^\sigma$.

Local fractional continuity is demonstrated as follows.

Definition 1.2 ([24])**.** Any function $\Psi : \mathbb{R} \to \mathbb{R}^\sigma,\ \mu \to \Psi(\mu)$ is considered to be L.F continuous at μ_0, if for any $\epsilon > 0$, $\exists \delta > 0$ such that $|\Psi(\mu) - \Psi(\mu_0)| < \epsilon^\sigma$, $|\mu - \mu_0| < \delta$. If $\Psi(\mu)$ is local continuous at (a, ρ), then $\Psi(\mu) \in C_\sigma(\vartheta, \rho)$.

Now we revisit the differentiability of functions defined by Yang [24].

Definition 1.3 ([24])**.** The L.F derivative of $\Psi(\mu)$ of order σ at $\mu = \mu_0$ is described as:

$$\Psi^{\sigma}(\mu) = {}_{\mu_0}D_{\mu}^{\sigma}\Psi(\mu) = \left|\frac{d^{\sigma}\Psi(\mu)}{(d\mu)^{\sigma}}\right|_{\mu=\mu_0} = \lim_{\mu\to\mu_0}\frac{\triangle^{\sigma}(\Psi(\mu)-\Psi(\mu_0))}{(\mu-\mu_0)^{\sigma}},$$

where $\triangle^{\sigma}(\Psi(\mu)-\Psi(\mu_0)) = \Gamma(1+\sigma)(\Psi(\mu)-\Psi(\mu_0))$.

Let $\Psi^{\sigma}(\mu) = D_{\mu}^{\sigma}\Psi(\mu)$. If there exists $\Psi^{(k+1)\sigma}(\mu) = \overbrace{D_{\mu}^{\sigma}\Psi(\mu).D_{\mu}^{\sigma}\Psi(\mu)\ldots D_{\mu}^{\sigma}\Psi(\mu)}^{(k+1)\text{times}}$ for any $\mu \in [\vartheta,\rho]$, then it is specified by $\Psi \in D_{(k+1)\sigma}$, where $k = 1,2,3,\ldots$.

Now, we report the L.F integration of $\Psi(\mu) \in C_{\sigma}(\vartheta,\rho)$.

Definition 1.4. Let $\triangle = \{\mu_0,\mu_1,\mu_2,\ldots,\mu_{\gamma}\}$ where $\gamma \in \mathbb{N}$ be a division of $[\vartheta,\rho]$ such that $\mu_0 < \mu_1 < \mu_2 < \cdots < \mu_{\gamma}$. Then L.F integral of Ψ on $[\vartheta,\rho]$ is explored as

$${}_{\vartheta}I_{\rho}^{\sigma}\Psi(\mu) = \frac{1}{\Gamma(1+\sigma)}\int_{\vartheta}^{\rho}\Psi(\mu)(d\mu)^{\sigma} = \frac{1}{\Gamma(1+\sigma)}\lim_{\triangle\mu_i\to 0}\sum_{i=1}^{\gamma}\Psi(\mu_i(\triangle\mu_i)^{\sigma},$$

where $\triangle\mu_i = \mu_i - \mu_{i-1}$ for $i = 1,2,3,\ldots,\gamma$.

The above statement is evident that ${}_{\vartheta}I_{\rho}^{\sigma}\Psi(\mu) = 0$ if $\vartheta = \rho$ and ${}_{\vartheta}I_{\rho}^{\sigma}\Psi(\mu) = -{}_{\rho}I_{\vartheta}^{\sigma}\Psi(\mu)$, when $\vartheta < \rho$. For any $\mu \in [\vartheta,\rho]$, if there exists ${}_{\vartheta}I_{\rho}^{\sigma}\Psi(\mu)$, then it is denoted by $\Psi(\mu) \in I_{\mu}^{\sigma}[\vartheta,\rho]$.

Lemma 1.5. *The subsequent equations are satisfied.*

1. *If* $\Psi(\mu) = r^{\sigma}(\mu) \in C_{\sigma}[\vartheta,\rho]$, *then* ${}_{\vartheta}I_{\rho}^{\sigma}r^{\sigma}(\mu) = r(\rho) - r(\vartheta)$ *(L.F integration).*
2. *L.F derivative of* $\mu^{k\sigma}$ *is such that*
$$\frac{d^{\sigma}\mu^{k\sigma}}{(d\mu)^{\sigma}} = \frac{\Gamma(1+k\sigma)}{\Gamma(1+(k-1)\sigma)}\mu^{(k-1)\sigma}.$$
3. *L.F integration of* $\mu^{k\sigma}$ *is such that:*
$$\frac{1}{\Gamma(1+\sigma)}\int_{\vartheta}^{\rho}\mu^{k\sigma}(d\mu)^{\sigma} = \frac{\Gamma(1+k\sigma)}{\Gamma(1+(k+1)\sigma)}(\rho^{(k+1)\sigma} - \vartheta^{(k+1)\sigma}).$$

In 2014 Mo et al. [10] explored the idea of convex functions in fractal domain, and is demonstrated as follows.

Definition 1.6. Any function $\Psi : [\vartheta,\rho] \to \mathbb{R}^{\sigma}$ is said to be generalized convex functions, if

$$\Psi(\ell\vartheta + (1-\ell)\rho) \leqslant \ell^{\sigma}\Psi(\vartheta) + (1-\ell)^{\sigma}\Psi(\rho),$$

$\ell \in [0,1]$ and $0 < \sigma \leqslant 1$.

The trapezium inequality for the fractal set is given as follows.

Theorem 1.7. *If* $\Psi : [\vartheta,\rho] \to \mathbb{R}^{\sigma}\quad (0 < \sigma \leqslant 1)$ *is a generalized convex function, then*

$$\Psi\left(\frac{\vartheta+\rho}{2}\right) \leqslant \frac{\Gamma(1+\sigma)}{(\rho-\vartheta)^{\sigma}}\,{}_{\vartheta}I_{\rho}^{\sigma}\Psi(\mu) \leqslant \frac{\Psi(\vartheta)+\Psi(\rho)}{2^{\sigma}},$$

where $0 < \sigma \leqslant 1$.

For more detail see [10]. We have organized our investigations in the following segments. In the very first, we discuss the background of the problem and some essential facts about convexity and inequalities both in the classical and fractal domains. The second part contains the principle findings regarding Maclaurin's inequality. The third section contains the applicable and visual analysis of primary outcomes. Finally, the concluding remarks are added.

2. Main results

In this part, we demonstrate our key findings.

Lemma 2.1. *Assume that* $\Psi : I \to \mathbb{R}^\sigma$ *be a differentiable function on* $I^o, \vartheta, \rho \in I^o$ *with* $\vartheta < \rho$ *and* $\Psi^\sigma \in C_\sigma[\vartheta, \rho]$*, then*

$$\begin{aligned}&\frac{1}{8^\sigma}\left(3^\sigma\Psi\left(\frac{5\vartheta+\rho}{6}\right)+2^\sigma\Psi\left(\frac{\vartheta+\rho}{2}\right)+3^\sigma\Psi\left(\frac{\vartheta+5\rho}{6}\right)\right)-\frac{\Gamma(1+\sigma)}{(\rho-\vartheta)^\sigma}{}_\vartheta I\rho^\sigma\Psi(u)\\&=(\rho-\vartheta)^\sigma\left(\frac{1}{\Gamma(1+\sigma)}\int_0^{\frac16}\ell^\sigma\Psi^\sigma((1-\ell)\vartheta+\ell\rho)(d\ell)^\sigma+\frac{1}{\Gamma(1+\sigma)}\int_{\frac16}^{\frac12}\left(\ell-\frac38\right)^\sigma\Psi^\sigma((1-\ell)\vartheta+\ell\rho)(d\ell)^\sigma\right.\\&\left.+\frac{1}{\Gamma(1+\sigma)}\int_{\frac12}^{\frac56}\left(\ell-\frac58\right)^\sigma\Psi^\sigma((1-\ell)\vartheta+\ell\rho)(d\ell)^\sigma+\frac{1}{\Gamma(1+\sigma)}\int_{\frac56}^{1}(\ell-1)^\sigma\Psi^\sigma((1-\ell)\vartheta+\ell\rho)(d\ell)^\sigma\right).\end{aligned}\tag{2.1}$$

Proof. Consider

$$I_1=\frac{1}{\Gamma(1+\sigma)}\int_0^{\frac16}\ell^\sigma\Psi^\sigma((1-\ell)\vartheta+\ell\rho)(d\ell)^\sigma,\qquad I_2=\frac{1}{\Gamma(1+\sigma)}\int_{\frac16}^{\frac12}\left(\ell-\frac38\right)^\sigma\Psi^\sigma((1-\ell)\vartheta+\ell\rho)(d\ell)^\sigma,$$

$$I_3=\frac{1}{\Gamma(1+\sigma)}\int_{\frac12}^{\frac56}\left(\ell-\frac58\right)^\sigma\Psi^\sigma((1-\ell)\vartheta+\ell\rho)(d\ell)^\sigma,\quad I_4=\frac{1}{\Gamma(1+\sigma)}\int_{\frac56}^{1}(\ell-1)^\sigma\Psi^\sigma((1-\ell)\vartheta+\ell\rho)(d\ell)^\sigma.$$

Implementing the integration by parts, we have

$$\begin{aligned}I_1&=\ell^\sigma\left(\frac{\Psi((1-\ell)\vartheta+\ell\rho)}{(\rho-\vartheta)^\sigma}\right)\Bigg|_0^{\frac16}-\frac{1}{\Gamma(1+\sigma)}\int_0^{\frac16}\Gamma(1+\sigma)\frac{\Psi((1-\ell)\vartheta+\ell\rho)}{(\rho-\vartheta)^\sigma}(d\ell)^\sigma\\&=\frac{1}{(\rho-\vartheta)^\sigma}\left[\frac{1}{6^\sigma}\Psi\left(\frac{5\vartheta+\rho}{6}\right)-\int_0^{\frac16}\frac{\Gamma(1+\sigma)}{\Gamma(1+\sigma)}\Psi((1-\ell)\vartheta+\ell\rho)(d\ell)^\sigma\right]\\&=\frac{1}{(\rho-\vartheta)^\sigma}\left[\frac{1}{6^\sigma}\Psi\left(\frac{5\vartheta+\rho}{6}\right)-\frac{\Gamma(1+\sigma)}{(\rho-\vartheta)^\sigma}{}_a I^\sigma_{\frac{5\vartheta+\rho}{6}}\Psi(u)\right].\end{aligned}\tag{2.2}$$

Similarly, we obtain

$$\begin{aligned}I_2&=\left(\ell-\frac38\right)^\sigma\frac{\Psi((1-\ell)\vartheta+\ell\rho)}{(\rho-\vartheta)^\sigma}\Bigg|_{\frac16}^{\frac12}-\frac{1}{\Gamma(1+\sigma)}\int_{\frac16}^{\frac12}\Gamma(1+\sigma)\frac{\Psi((1-\ell)\vartheta+\ell\rho)}{(\rho-\vartheta)^\sigma}(d\ell)^\sigma\\&=\frac{1}{(\rho-\vartheta)^\sigma}\left[\frac{1}{8^\sigma}\Psi\left(\frac{\vartheta+\rho}{2}\right)+\left(\frac{5}{24}\right)^\sigma\Psi\left(\frac{5\vartheta+\rho}{6}\right)-\frac{\Gamma(1+\sigma)}{(\rho-\vartheta)^\sigma}{}_{\frac{5\vartheta+\rho}{6}}I^\sigma_{\frac{\vartheta+\rho}{2}}\Psi(u)\right],\end{aligned}\tag{2.3}$$

$$\begin{aligned}I_3&=\left(\ell-\frac58\right)^\sigma\frac{\Psi((1-\ell)\vartheta+\ell\rho)}{(\rho-\vartheta)^\sigma}\Bigg|_{\frac12}^{\frac56}-\frac{1}{\Gamma(1+\sigma)}\int_{\frac12}^{\frac56}\Gamma(1+\sigma)\frac{\Psi((1-\ell)\vartheta+\ell\rho)}{(\rho-\vartheta)^\sigma}(d\ell)^\sigma\\&=\frac{1}{(\rho-\vartheta)^\sigma}\left[\frac{1}{8^\sigma}\Psi\left(\frac{\vartheta+\rho}{2}\right)+\left(\frac{5}{24}\right)^\sigma\Psi\left(\frac{\vartheta+5\rho}{6}\right)-\frac{\Gamma(1+\sigma)}{(\rho-\vartheta)^\sigma}{}_{\frac{\vartheta+\rho}{2}}I^\sigma_{\frac{5\vartheta+\rho}{6}}\Psi(u)\right],\end{aligned}\tag{2.4}$$

$$\begin{aligned}I_4&=(\ell-1)^\sigma\frac{\Psi((1-\ell)\vartheta+\ell\rho)}{(\rho-\vartheta)^\sigma}\Bigg|_{\frac56}^{1}-\frac{1}{\Gamma(1+\sigma)}\int_{\frac56}^{1}\Gamma(1+\sigma)\frac{\Psi((1-\ell)\vartheta+\ell\rho)}{(\rho-\vartheta)^\sigma}(d\ell)^\sigma\\&=\frac{1}{(\rho-\vartheta)^\sigma}\left[\frac{1}{6}\Psi\left(\frac{\vartheta+5\rho}{6}\right)-\frac{\Gamma(1+\sigma)}{(\rho-\vartheta)^\sigma}{}_{\frac{\vartheta+5\rho}{6}}I^\sigma_\rho\Psi(u)\right].\end{aligned}\tag{2.5}$$

Adding (2.2)-(2.5) and then taking the product of resulting equality by $(\rho-\vartheta)^\sigma$, we get (2.1). □

Theorem 2.2. *Suppose that all the conditions of Lemma 2.1 are admitted. If $|\Psi^\sigma|$ is a generalized convex function on $[\vartheta, \rho]$, then for all $\ell \in [0,1]$, we have*

$$\left|\frac{1}{8^\sigma}\left(3^\sigma\Psi\left(\frac{5\vartheta+\rho}{6}\right)+2^\sigma\Psi\left(\frac{\vartheta+\rho}{2}\right)+3^\sigma\Psi\left(\frac{\vartheta+5\rho}{6}\right)\right)-\frac{\Gamma(1+\sigma)}{(\rho-\vartheta)^\sigma}{}_\vartheta I\rho^\sigma\Psi(u)\right| \leqslant \frac{(\rho-\vartheta)^\sigma\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\frac{25}{288}\right)^\sigma(|\Psi^\sigma(\vartheta)|+|\Psi^\sigma(\rho)|).$$

Proof. Through Lemma 2.1 and taking into account the generalized convexity of $|\Psi^\sigma|$, we have

$$\begin{aligned}
&\left|\frac{1}{8^\sigma}\left(3^\sigma\Psi\left(\frac{5\vartheta+\rho}{6}\right)+2^\sigma\Psi\left(\frac{\vartheta+\rho}{2}\right)+3^\sigma\Psi\left(\frac{\vartheta+5\rho}{6}\right)\right)-\frac{\Gamma(1+\sigma)}{(\rho-\vartheta)^\sigma}{}_\vartheta I\rho^\sigma\Psi(u)\right|\\
&\leqslant(\rho-\vartheta)^\sigma\left(\frac{1}{\Gamma(1+\sigma)}\int_0^{\frac{1}{6}}\ell^\sigma|\Psi^\sigma((1-\ell)\vartheta+\ell\rho)|(d\ell)^\sigma+\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\ell-\frac{3}{8}\right|^\sigma|\Psi^\sigma((1-\ell)\vartheta+\ell\rho)|(d\ell)^\sigma\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\ell-\frac{5}{8}\right|^\sigma|\Psi^\sigma((1-\ell)\vartheta+\ell\rho)|(d\ell)^\sigma+\frac{1}{\Gamma(1+\sigma)}\int_{\frac{5}{6}}^{1}|\ell-1|^\sigma|\Psi^\sigma((1-\ell)\vartheta+\ell\rho)|(d\ell)^\sigma\right)\\
&\leqslant(\rho-\vartheta)^\sigma\left(\frac{1}{\Gamma(1+\sigma)}\int_0^{\frac{1}{6}}\ell^\sigma((1-\ell)^\sigma|\Psi^\sigma(\vartheta)|+\ell^\sigma|\Psi^\sigma(\rho)|)(d\ell)^\sigma\right.\\
&\quad+\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\ell-\frac{3}{8}\right|^\sigma((1-\ell)^\sigma|\Psi^\sigma(\vartheta)|+\ell^\sigma|\Psi'(\rho)|)(d\ell)^\sigma\\
&\quad+\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\ell-\frac{5}{8}\right|^\sigma((1-\ell)^\sigma|\Psi^\sigma(\vartheta)|+\ell^\sigma|\Psi^\sigma(\rho)|)(d\ell)^\sigma\\
&\quad\left.+\frac{1}{\Gamma(1+\sigma)}\int_{\frac{5}{6}}^{1}|\ell-1|^\sigma((1-\ell)^\sigma|\Psi^\sigma(\vartheta)|+\ell^\sigma|\Psi^\sigma(\rho)|)(d\ell)^\sigma\right)\\
&=(\rho-\vartheta)^\sigma\left[\frac{1}{\Gamma(1+\sigma)}\int_0^{\frac{1}{6}}\ell^\sigma((1-\ell)^\sigma|\Psi^\sigma(\vartheta)|+\ell^\sigma|\Psi^\sigma(\rho)|)(d\ell)^\sigma\right.\\
&\quad+\frac{1}{\Gamma(1+\sigma)}\left(\int_{\frac{1}{6}}^{\frac{3}{8}}\left(\frac{3}{8}-\ell\right)^\sigma+\int_{\frac{3}{8}}^{\frac{1}{2}}\left(\ell-\frac{3}{8}\right)^\sigma\right)((1-\ell)^\sigma|\Psi^\sigma(\vartheta)|+\ell^\sigma|\Psi^\sigma(\rho)|)(d\ell)^\sigma\\
&\quad+\frac{1}{\Gamma(1+\sigma)}\left(\int_{\frac{1}{2}}^{\frac{5}{8}}\left(\frac{5}{8}-\ell\right)^\sigma+\int_{\frac{5}{8}}^{\frac{5}{6}}\left(\ell-\frac{5}{8}\right)^\sigma\right)((1-\ell)^\sigma|\Psi^\sigma(\vartheta)|+\ell^\sigma|\Psi^\sigma(\rho)|)(d\ell)^\sigma\\
&\quad\left.+\frac{1}{\Gamma(1+\sigma)}\int_{\frac{5}{6}}^{1}|\ell-1|^\sigma((1-\ell)^\sigma|\Psi^\sigma(\vartheta)|+\ell^\sigma|\Psi^\sigma(\rho)|)(d\ell)^\sigma\right]\\
&=\frac{(\rho-\vartheta)^\sigma\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\frac{25}{288}\right)^\sigma(|\Psi^\sigma(\vartheta)|+|\Psi^\sigma(\rho)|).
\end{aligned}$$

The proof is completed. □

Now we give special case of Theorem 2.2.

- Substituting $\sigma = 1$ in Theorem 2.2, we obtain the following inequality

$$\left|\frac{1}{8}\left(3\Psi\left(\frac{5\vartheta+\rho}{6}\right)+2\Psi\left(\frac{\vartheta+\rho}{2}\right)+3\Psi\left(\frac{\vartheta+5\rho}{6}\right)\right)-\frac{1}{\rho-\vartheta}\int_\vartheta^b f(x)dx\right|\leqslant\frac{25(\rho-\vartheta)}{576}(|\Psi'(\vartheta)|+|\Psi'(\rho)|).$$

Remark 2.3. By different values of $\alpha \in (0, 1]$, we can generate blend of new Maclaurin's like inequalities.

Theorem 2.4. *Suppose that all the conditions of Lemma 2.1 are admitted. If $|\Psi^{\sigma}|^{q}$ is a generalized convex function on $[\vartheta,\rho]$, then for all $\ell\in[0,1]$ and $q>1$ with $\frac{1}{r}+\frac{1}{q}=1$, we have*

$$\begin{aligned}
&\left|\frac{1}{8^{\sigma}}\left(3^{\sigma}\Psi\left(\frac{5\vartheta+\rho}{6}\right)+2^{\sigma}\Psi\left(\frac{\vartheta+\rho}{2}\right)+3^{\sigma}\Psi\left(\frac{\vartheta+5\rho}{6}\right)\right)-\frac{\Gamma(1+\sigma)}{(\rho-\vartheta)^{\sigma}}{}_{\vartheta}I_{\rho}^{\sigma}\Psi(u)\right|\\
&\leqslant(\rho-\vartheta)^{\sigma}\left[\left(\frac{\Gamma(1+r\sigma)}{6^{(r+1)\sigma}\Gamma(1+(r+1)\sigma)}\right)^{\frac{1}{r}}\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\left(\frac{11}{36}\right)^{\sigma}|\Psi^{\sigma}(\vartheta)|^{q}+\left(\frac{1}{36}\right)^{\sigma}|\Psi^{\sigma}(\rho)|^{q}\right)\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\frac{\Gamma(1+r\sigma)}{\Gamma(1+(r+1)\sigma)}\left(\left(\frac{5}{24}\right)^{(r+1)\sigma}+\left(\frac{1}{8}\right)^{(r+1)\sigma}\right)\right)^{\frac{1}{r}}\\
&\quad\times\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\left(\frac{4}{9}\right)^{\sigma}|\Psi^{\sigma}(\vartheta)|^{q}+\left(\frac{2}{9}\right)^{\sigma}|\Psi^{\sigma}(\rho)|^{q}\right)\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{\Gamma(1+r\sigma)}{\Gamma(1+(r+1)\sigma)}\left(\left(\frac{1}{8}\right)^{(r+1)\sigma}+\left(\frac{5}{24}\right)^{(r+1)\sigma}\right)\right)^{\frac{1}{r}}\\
&\quad\times\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\left(\frac{2}{9}\right)^{\sigma}|\Psi^{\sigma}(\vartheta)|^{q}+\left(\frac{4}{9}\right)^{\sigma}|\Psi^{\sigma}(\rho)|^{q}\right)\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\frac{\Gamma(1+r\sigma)}{6^{(r+1)\sigma}\Gamma(1+(r+1)\sigma)}\right)^{\frac{1}{r}}\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\left(\frac{1}{36}\right)^{\sigma}|\Psi^{\sigma}(\vartheta)|^{q}+\left(\frac{11}{36}\right)^{\sigma}|\Psi^{\sigma}(\rho)|^{q}\right)\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. Through Lemma 2.1 and taking into account Hölder integral inequality and generalized convexity of $|\Psi^{\sigma}|$, we have

$$\begin{aligned}
&\left|\frac{1}{8^{\sigma}}\left(3^{\sigma}\Psi\left(\frac{5\vartheta+\rho}{6}\right)+2^{\sigma}\Psi\left(\frac{\vartheta+\rho}{2}\right)+3^{\sigma}\Psi\left(\frac{\vartheta+5\rho}{6}\right)\right)-\frac{\Gamma(1+\sigma)}{(\rho-\vartheta)^{\sigma}}{}_{\vartheta}I_{\rho}^{\sigma}\Psi(u)\right|\\
&\leqslant(\rho-\vartheta)^{\sigma}\left(\frac{1}{\Gamma(1+\sigma)}\int_{0}^{\frac{1}{6}}\ell^{\sigma}|\Psi^{\sigma}((1-\ell)\vartheta+\ell\rho)|(d\ell)^{\sigma}+\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\ell-\frac{3}{8}\right|^{\sigma}|\Psi^{\sigma}((1-\ell)\vartheta+\ell\rho)|(d\ell)^{\sigma}\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\ell-\frac{5}{8}\right|^{\sigma}|\Psi^{\sigma}((1-\ell)\vartheta+\ell\rho)|(d\ell)^{\sigma}+\frac{1}{\Gamma(1+\sigma)}\int_{\frac{5}{6}}^{1}|\ell-1|^{\sigma}|\Psi^{\sigma}((1-\ell)\vartheta+\ell\rho)|(d\ell)^{\sigma}\right)\\
&\leqslant(\rho-\vartheta)^{\sigma}\left[\left(\frac{1}{\Gamma(1+\sigma)}\int_{0}^{\frac{1}{6}}\ell^{r\sigma}(d\ell)^{\sigma}\right)^{\frac{1}{r}}\left(\frac{1}{\Gamma(1+\sigma)}\int_{0}^{\frac{1}{6}}|\Psi^{\sigma}((1-\ell)\vartheta+\ell\rho)|^{q}(d\ell)^{\sigma}\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\ell-\frac{3}{8}\right|^{r\sigma}(d\ell)^{\sigma}\right)^{\frac{1}{r}}\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{6}}^{\frac{1}{2}}|\Psi^{\sigma}((1-\ell)\vartheta+\ell\rho)|^{q}(d\ell)^{\sigma}\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\ell-\frac{5}{8}\right|^{r\sigma}(d\ell)^{\sigma}\right)^{\frac{1}{r}}\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{2}}^{\frac{5}{6}}|\Psi^{\sigma}((1-\ell)\vartheta+\ell\rho)|^{q}(d\ell)^{\sigma}\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{5}{6}}^{1}|\ell-1|^{r\sigma}(d\ell)^{\sigma}\right)^{\frac{1}{r}}\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{5}{6}}^{1}|\Psi^{\sigma}((1-\ell)\vartheta+\ell\rho)|^{q}(d\ell)^{\sigma}\right)^{\frac{1}{q}}\right]\\
&\leqslant(\rho-\vartheta)^{\sigma}\left[\left(\frac{1}{\Gamma(1+\sigma)}\int_{0}^{\frac{1}{6}}\ell^{r\sigma}(d\ell)^{\sigma}\right)^{\frac{1}{r}}\left(\frac{1}{\Gamma(1+\sigma)}\int_{0}^{\frac{1}{6}}((1-\ell)^{\sigma}|\Psi^{\sigma}(\vartheta)|^{q}+\ell^{\sigma}|\Psi^{\sigma}(\rho)|^{q})(d\ell)^{\sigma}\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\ell-\frac{3}{8}\right|^{r\sigma}(d\ell)^{\sigma}\right)^{\frac{1}{r}}\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{6}}^{\frac{1}{2}}((1-\ell)^{\sigma}|\Psi^{\sigma}(\vartheta)|^{q}+\ell^{\sigma}|\Psi^{\sigma}(\rho)|^{q})(d\ell)^{\sigma}\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\ell-\frac{5}{8}\right|^{r\sigma}(d\ell)^{\sigma}\right)^{\frac{1}{r}}\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{2}}^{\frac{5}{6}}((1-\ell)^{\sigma}|\Psi^{\sigma}(\vartheta)|^{q}+\ell^{\sigma}|\Psi^{\sigma}(\rho)|^{q})(d\ell)^{\sigma}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{5}{6}}^{1}|\ell-1|^{r\sigma}(d\ell)^{\sigma}\right)^{\frac{1}{r}}\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{5}{6}}^{1}((1-\ell)^{\sigma}|\Psi^{\sigma}(\vartheta)|^{q}+\ell^{\sigma}|\Psi^{\sigma}(\rho)|^{q})(d\ell)^{\sigma}\right)^{\frac{1}{q}}\Bigg]$$
$$=(\rho-\vartheta)^{\sigma}\Bigg[\left(\frac{\Gamma(1+r\sigma)}{6^{(r+1)\sigma}\Gamma(1+(r+1)\sigma)}\right)^{\frac{1}{r}}\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\left(\frac{11}{36}\right)^{\sigma}|\Psi^{\sigma}(\vartheta)|^{q}+\left(\frac{1}{36}\right)^{\sigma}|\Psi^{\sigma}(\rho)|^{q}\right)\right)^{\frac{1}{q}}$$
$$+\left(\frac{\Gamma(1+r\sigma)}{\Gamma(1+(r+1)\sigma)}\left(\left(\frac{5}{24}\right)^{(r+1)\sigma}+\left(\frac{1}{8}\right)^{(r+1)\sigma}\right)\right)^{\frac{1}{r}}$$
$$\times\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\left(\frac{4}{9}\right)^{\sigma}|\Psi^{\sigma}(\vartheta)|^{q}+\left(\frac{2}{9}\right)^{\sigma}|\Psi^{\sigma}(\rho)|^{q}\right)\right)^{\frac{1}{q}}$$
$$+\left(\frac{\Gamma(1+r\sigma)}{\Gamma(1+(r+1)\sigma)}\left(\left(\frac{1}{8}\right)^{(r+1)\sigma}+\left(\frac{5}{24}\right)^{(r+1)\sigma}\right)\right)^{\frac{1}{r}}$$
$$\times\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\left(\frac{2}{9}\right)^{\sigma}|\Psi^{\sigma}(\vartheta)|^{q}+\left(\frac{4}{9}\right)^{\sigma}|\Psi^{\sigma}(\rho)|^{q}\right)\right)^{\frac{1}{q}}$$
$$+\left(\frac{\Gamma(1+r\sigma)}{6^{(r+1)\sigma}\Gamma(1+(r+1)\sigma)}\right)^{\frac{1}{r}}\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\left(\frac{1}{36}\right)^{\sigma}|\Psi^{\sigma}(\vartheta)|^{q}+\left(\frac{11}{36}\right)^{\sigma}|\Psi^{\sigma}(\rho)|^{q}\right)\right)^{\frac{1}{q}}\Bigg],$$

which ends the proof. □

- By substituting $\sigma = 1$ in Theorem 2.4, we obtain following bound:

$$\left|\frac{1}{8}\left(3\Psi\left(\frac{5\vartheta+\rho}{6}\right)+2\Psi\left(\frac{\vartheta+\rho}{2}\right)+3\Psi\left(\frac{\vartheta+5\rho}{6}\right)\right)-\frac{1}{\rho-\vartheta}\int_{\vartheta}^{b}f(x)dx\right|$$
$$\leqslant(\rho-\vartheta)\Bigg[\left(\frac{1}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}\left(\frac{11}{72}|\Psi'(\vartheta)|^{q}+\frac{1}{72}|\Psi'(\rho)|^{q}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{r+1}\left(\left(\frac{5}{24}\right)^{r+1}+\left(\frac{1}{8}\right)^{r+1}\right)\right)^{\frac{1}{r}}\left(\frac{2}{9}|\Psi'(\vartheta)|^{q}+\frac{1}{9}|\Psi'(\rho)^{q}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{r+1}\left(\left(\frac{1}{8}\right)^{r+1}+\left(\frac{5}{24}\right)^{r+1}\right)\right)^{\frac{1}{r}}\left(\frac{1}{9}|\Psi'(\vartheta)|^{q}+\frac{2}{9}|\Psi'(\rho)^{q}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}\left(\frac{1}{72}|\Psi'(\vartheta)|^{q}+\frac{11}{72}|\Psi'(\rho)^{q}\right)^{\frac{1}{q}}\Bigg].$$

Theorem 2.5. *Assume that all the hypothesises of the Lemma 2.1 are fulfilled. If $|\Psi'|^{q}$ is generalized convex function on $[\vartheta, \rho]$, $q \geqslant 1$, then*

$$\left|\frac{1}{8^{\sigma}}\left(3^{\sigma}\Psi\left(\frac{5\vartheta+\rho}{6}\right)+2^{\sigma}\Psi\left(\frac{\vartheta+\rho}{2}\right)+3^{\sigma}\Psi\left(\frac{\vartheta+5\rho}{6}\right)\right)-\frac{\Gamma(1+\sigma)}{(\rho-\vartheta)^{\sigma}}{}_{\vartheta}I\rho^{\sigma}\Psi(u)\right|$$
$$\leqslant(\rho-\vartheta)^{\sigma}\Bigg\{\left(\frac{\Gamma(1+\sigma)}{36^{\sigma}\Gamma(1+2\sigma)}\right)^{1-\frac{1}{q}}\left(\left(\frac{\Gamma(1+\sigma)}{36^{\sigma}\Gamma(1+2\sigma)}-\frac{\Gamma(1+2\sigma)}{216^{\sigma}\Gamma(1+3\sigma)}\right)|\Psi^{\sigma}(\vartheta)|^{q}+\frac{\Gamma(1+2\sigma)}{216^{\sigma}\Gamma(1+3\sigma)}|\Psi^{\sigma}(\rho)|^{q}\right)^{\frac{1}{q}}$$

$$+\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\frac{17}{288}\right)^{\sigma}\right)^{1-\frac{1}{q}}\left[\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\frac{133}{2304}\right)^{\sigma}-\frac{\Gamma(1+2\sigma)}{\Gamma(1+3\sigma)}\left(\frac{167}{6912}\right)^{\sigma}\right)|\Psi^{\sigma}(\vartheta)|^{q}\right.$$
$$\left.+\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\frac{1}{768}\right)^{\sigma}+\frac{\Gamma(1+2\sigma)}{\Gamma(1+3\sigma)}\left(\frac{167}{6912}\right)^{\sigma}\right)|\Psi^{\sigma}(\rho)|^{q}\right]^{\frac{1}{q}}$$
$$+\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\frac{17}{288}\right)^{\sigma}\right)^{1-\frac{1}{q}}\left[\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\frac{1}{768}\right)^{\sigma}+\frac{\Gamma(1+2\sigma)}{\Gamma(1+3\sigma)}\left(\frac{167}{6912}\right)^{\sigma}\right)|\Psi^{\sigma}(\vartheta)|^{q}\right.$$
$$\left.+\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\frac{133}{2304}\right)^{\sigma}-\frac{\Gamma(1+2\sigma)}{\Gamma(1+3\sigma)}\left(\frac{167}{6912}\right)^{\sigma}\right)|\Psi^{\sigma}(\rho)|^{q}\right]^{\frac{1}{q}}$$
$$\left.+\left(\frac{\Gamma(1+\sigma)}{36^{\sigma}\Gamma(1+2\sigma)}\right)^{1-\frac{1}{q}}\left(\frac{\Gamma(1+2\sigma)}{216^{\sigma}\Gamma(1+3\sigma)}|\Psi^{\sigma}(\vartheta)|^{q}+\left(\frac{\Gamma(1+\sigma)}{36^{\sigma}\Gamma(1+2\sigma)}-\frac{\Gamma(1+2\sigma)}{216^{\sigma}\Gamma(1+3\sigma)}\right)|\Psi^{\sigma}(\rho)|^{q}\right)^{\frac{1}{q}}\right\},$$

satisfies for $\ell \in [0,1]$, where $\frac{1}{r}+\frac{1}{q}=1$.

Proof. Suppose that $q \geqslant 1$. From Lemma 2.1, by using the power-mean integral inequality and the convexity of $|f'|^{q}$, we obtain

$$\left|\frac{1}{8^{\sigma}}\left(3^{\sigma}\Psi\left(\frac{5\vartheta+\rho}{6}\right)+2^{\sigma}\Psi\left(\frac{\vartheta+\rho}{2}\right)+3^{\sigma}\Psi\left(\frac{\vartheta+5\rho}{6}\right)\right)-\frac{\Gamma(1+\sigma)}{(\rho-\vartheta)^{\sigma}}{}_{\vartheta}I\rho^{\sigma}\Psi(u)\right|$$
$$\leqslant(\rho-\vartheta)^{\sigma}\left(\frac{1}{\Gamma(1+\sigma)}\int_{0}^{\frac{1}{6}}\ell^{\sigma}|\Psi^{\sigma}((1-\ell)\vartheta+\ell\rho)|(d\ell)^{\sigma}+\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\ell-\frac{3}{8}\right|^{\sigma}|\Psi^{\sigma}((1-\ell)\vartheta+\ell\rho)|(d\ell)^{\sigma}\right.$$
$$\left.+\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\ell-\frac{5}{8}\right|^{\sigma}|\Psi^{\sigma}((1-\ell)\vartheta+\ell\rho)|(d\ell)^{\sigma}+\frac{1}{\Gamma(1+\sigma)}\int_{\frac{5}{6}}^{1}|\ell-1|^{\sigma}|\Psi^{\sigma}((1-\ell)\vartheta+\ell\rho)|(d\ell)^{\sigma}\right)$$
$$\leqslant(\rho-\vartheta)^{\sigma}\left[\left(\frac{1}{\Gamma(1+\sigma)}\int_{0}^{\frac{1}{6}}\ell^{\sigma}(d\ell)^{\sigma}\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(1+\sigma)}\int_{0}^{\frac{1}{6}}\ell^{\sigma}((1-\ell)^{\sigma}|\Psi^{\sigma}(\vartheta)|^{q}+\ell^{\sigma}|\Psi^{\sigma}(\rho)|^{q})(d\ell)^{\sigma}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\ell-\frac{3}{8}\right|^{\sigma}(d\ell)^{\sigma}\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\ell-\frac{3}{8}\right|^{\sigma}((1-\ell)^{\sigma}|\Psi^{\sigma}(\vartheta)|^{q}+\ell^{\sigma}|\Psi^{\sigma}(\rho)|^{q})(d\ell)^{\sigma}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\ell-\frac{5}{8}\right|^{\sigma}(d\ell)^{\sigma}\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\ell-\frac{5}{8}\right|^{\sigma}((1-\ell)^{\sigma}|\Psi^{\sigma}(\vartheta)|^{q}+\ell^{\sigma}|\Psi^{\sigma}(\rho)|^{q})(d\ell)^{\sigma}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{5}{6}}^{1}|\ell-1|^{\sigma}(d\ell)^{\sigma}\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(1+\sigma)}\int_{\frac{5}{6}}^{1}|\ell-1|^{\sigma}((1-\ell)^{\sigma}|\Psi^{\sigma}(\vartheta)|^{q}+\ell^{\sigma}|\Psi^{\sigma}(\rho)|^{q})(d\ell)^{\sigma}\right)^{\frac{1}{q}}\right]$$
$$=(\rho-\vartheta)^{\sigma}\left\{\left(\frac{\Gamma(1+\sigma)}{36^{\sigma}\Gamma(1+2\sigma)}\right)^{1-\frac{1}{q}}\left(\left(\frac{\Gamma(1+\sigma)}{36^{\sigma}\Gamma(1+2\sigma)}-\frac{\Gamma(1+2\sigma)}{216^{\sigma}\Gamma(1+3\sigma)}\right)|\Psi^{\sigma}(\vartheta)|^{q}+\frac{\Gamma(1+2\sigma)}{216^{\sigma}\Gamma(1+3\sigma)}|\Psi^{\sigma}(\rho)|^{q}\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\frac{17}{288}\right)^{\sigma}\right)^{1-\frac{1}{q}}\left[\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\frac{133}{2304}\right)^{\sigma}-\frac{\Gamma(1+2\sigma)}{\Gamma(1+3\sigma)}\left(\frac{167}{6912}\right)^{\sigma}\right)|\Psi^{\sigma}(\vartheta)|^{q}\right.$$
$$\left.+\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\frac{1}{768}\right)^{\sigma}+\frac{\Gamma(1+2\sigma)}{\Gamma(1+3\sigma)}\left(\frac{167}{6912}\right)^{\sigma}\right)|\Psi^{\sigma}(\rho)|^{q}\right]^{\frac{1}{q}}$$
$$+\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\frac{17}{288}\right)^{\sigma}\right)^{1-\frac{1}{q}}\left[\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\frac{1}{768}\right)^{\sigma}+\frac{\Gamma(1+2\sigma)}{\Gamma(1+3\sigma)}\left(\frac{167}{6912}\right)^{\sigma}\right)|\Psi^{\sigma}(\vartheta)|^{q}\right.$$
$$\left.+\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\frac{133}{2304}\right)^{\sigma}-\frac{\Gamma(1+2\sigma)}{\Gamma(1+3\sigma)}\left(\frac{167}{6912}\right)^{\sigma}\right)|\Psi^{\sigma}(\rho)|^{q}\right]^{\frac{1}{q}}$$

$$+\left(\frac{\Gamma(1+\sigma)}{36^{\sigma}\Gamma(1+2\sigma)}\right)^{1-\frac{1}{q}}\left(\frac{\Gamma(1+2\sigma)}{216^{\sigma}\Gamma(1+3\sigma)}|\Psi^{\sigma}(\vartheta)|^{q}+\left(\frac{\Gamma(1+\sigma)}{36^{\sigma}\Gamma(1+2\sigma)}-\frac{\Gamma(1+2\sigma)}{216^{\sigma}\Gamma(1+3\sigma)}\right)|\Psi^{\sigma}(\rho)|^{q}\right)^{\frac{1}{q}}\Bigg\}.$$

Hence, we achieve our desired finding. □

Theorem 2.6. *Suppose that all the hypothesises of the Lemma 2.1 are admitted. If $|\Psi'| \leqslant M$, $M > 0$ is generalized convex function on $[\vartheta, \rho]$, then*

$$\left|\frac{1}{8^{\sigma}}\left(3^{\sigma}\Psi\left(\frac{5\vartheta+\rho}{6}\right)+2^{\sigma}\Psi\left(\frac{\vartheta+\rho}{2}\right)+3^{\sigma}\Psi\left(\frac{\vartheta+5\rho}{6}\right)\right)-\frac{\Gamma(1+\sigma)}{(\rho-\vartheta)^{\sigma}}{}_{\vartheta}I\rho^{\sigma}\Psi(u)\right|$$
$$\leqslant \frac{2^{\sigma}M\Gamma(1+\sigma)(\rho-\vartheta)^{\sigma}}{\Gamma(1+2\sigma)}\left(\frac{25}{288}\right)^{\sigma}.$$

Proof. Using Lemma 2.1, and utilizing that $|\Psi'|\leqslant M$, we have

$$\begin{aligned}
&\left|\frac{1}{8^{\sigma}}\left(3^{\sigma}\Psi\left(\frac{5\vartheta+\rho}{6}\right)+2^{\sigma}\Psi\left(\frac{\vartheta+\rho}{2}\right)+3^{\sigma}\Psi\left(\frac{\vartheta+5\rho}{6}\right)\right)-\frac{\Gamma(1+\sigma)}{(\rho-\vartheta)^{\sigma}}{}_{\vartheta}I\rho^{\sigma}\Psi(u)\right|\\
&\leqslant (\rho-\vartheta)^{\sigma}\left(\int_0^{\frac{1}{6}}\ell^{\sigma}|\Psi^{\sigma}((1-\ell)\vartheta+\ell\rho)|(d\ell)^{\sigma}+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\ell-\frac{3}{8}\right|^{\sigma}|\Psi^{\sigma}((1-\ell)\vartheta+\ell\rho)|(d\ell)^{\sigma}\right.\\
&\quad\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\ell-\frac{5}{8}\right|^{\sigma}|\Psi^{\sigma}((1-\ell)\vartheta+\ell\rho)|(d\ell)^{\sigma}+\int_{\frac{5}{6}}^{1}|\ell-1|^{\sigma}|\Psi^{\sigma}((1-\ell)\vartheta+\ell\rho)|(d\ell)^{\sigma}\right)\\
&\leqslant (\rho-\vartheta)^{\sigma}\left(\int_0^{\frac{1}{6}}\ell^{\sigma}((1-\ell)^{\sigma}|\Psi^{\sigma}(\vartheta)|+\ell^{\sigma}|\Psi^{\sigma}(\rho)|)(d\ell)^{\sigma}+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\ell-\frac{3}{8}\right|^{\sigma}((1-\ell)^{\sigma}|\Psi^{\sigma}(\vartheta)|+\ell^{\sigma}|\Psi^{\sigma}(\rho)|)(d\ell)^{\sigma}\right.\\
&\quad\left.+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\ell-\frac{5}{8}\right|^{\sigma}((1-\ell)^{\sigma}|\Psi^{\sigma}(\vartheta)|+\ell^{\sigma}|\Psi^{\sigma}(\rho)|)(d\ell)^{\sigma}+\int_{\frac{5}{6}}^{1}|\ell-1|^{\sigma}((1-\ell)^{\sigma}|\Psi^{\sigma}(\vartheta)|+\ell^{\sigma}|\Psi^{\sigma}(\rho)|)(d\ell)^{\sigma}\right)\\
&\leqslant M(\rho-\vartheta)^{\sigma}\left(\int_0^{\frac{1}{6}}\ell^{\sigma}(d\ell)^{\sigma}+\left(\int_{\frac{1}{6}}^{\frac{3}{8}}\left(\frac{3}{8}-\ell\right)^{\sigma}+\int_{\frac{3}{8}}^{\frac{1}{2}}\left(\ell-\frac{3}{8}\right)^{\sigma}\right)(d\ell)^{\sigma}\right.\\
&\quad\left.+\left(\int_{\frac{1}{2}}^{\frac{5}{8}}\left(\frac{5}{8}-\ell\right)^{\sigma}+\int_{\frac{5}{8}}^{\frac{5}{6}}\left(\ell-\frac{5}{8}\right)^{\sigma}\right)(d\ell)^{\sigma}+\int_{\frac{5}{6}}^{1}|\ell-1|^{\sigma}(d\ell)^{\sigma}\right)=\frac{2^{g}M\Gamma(1+\sigma)(\rho-\vartheta)^{\sigma}}{\Gamma(1+2\sigma)}\left(\frac{25}{288}\right)^{\sigma}.
\end{aligned}$$

The proof is completed. □

3. Applications

We now explore various applications of our main results. Initially, we establish connections between different methods, considering the findings from the previous section. This section also features numerical and graphical demonstrations of our main results.

3.1. *The quadrature formula*

If a partition $\mathcal{P}: \vartheta = \mu_0 < \mu_1 < \cdots < \mu_{\gamma-1} < \mu_{\gamma} = \rho$ is found by subdividing the $[\vartheta, \rho]$ into γ subintervals μ_i, μ_{i+1} with $i = 0, 1, \ldots, \gamma-1$, then

$${}_{a}I_{\rho}^{\sigma}\Psi(u)=\frac{1}{\Gamma(1+\sigma)}\int_{\vartheta}^{\rho}\Psi(u)(d(u))^{\sigma}=T(\mu)+R(\mu).$$

Here

$$T(\mu)=\frac{(\rho-\vartheta)^{\sigma}}{8^{\sigma}\Gamma(1+\sigma)}\left(3^{\sigma}\Psi\left(\frac{5\vartheta+\rho}{6}\right)+2^{\sigma}\Psi\left(\frac{\vartheta+\rho}{2}\right)+3^{\sigma}\Psi\left(\frac{\vartheta+5\rho}{6}\right)\right)$$

and $R(\mu)$ denotes the remainder of quadrature rule.

Proposition 3.1. *Suppose that all the hypothesises of Theorem 2.2 are held, then*

$$|R(\mu)| \leqslant \sum_{i=0}^{\theta_1-1} \frac{\Gamma(1+\sigma)(\mu_{i+1}-\mu_i)}{\Gamma(1+2\sigma)} \left(\frac{25}{288}\right)^{\sigma} (|\Psi'(\mu_i)|+|\Psi'(\mu_{i+1})|).$$

Proof. The claim arises straight away by employing the sum from $i=0$ to $i=\theta_1-1$ across subinterval $[\mu_i,\mu_{i+1}]$ in Theorem 2.2. □

Proposition 3.2. *Suppose that the hypothesises of Theorem 2.4 are held, then*

$$\begin{aligned}|R(\mu)| \leqslant \sum_{i=0}^{\theta_1-1}(\mu_{i+1}-\mu_i)^{\sigma}\Bigg[&\left(\frac{\Gamma(1+r\sigma)}{6^{(r+1)\sigma}\Gamma(1+(r+1)\sigma)}\right)^{\frac{1}{r}}\\
&\times\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\left(\frac{11}{36}\right)^{\sigma}|\Psi^{\sigma}(\mu_i)|^q+\left(\frac{1}{36}\right)^{\sigma}|\Psi^{\sigma}(\mu_{i+1})|^q\right)\right)^{\frac{1}{q}}\\
&+\left(\frac{\Gamma(1+r\sigma)}{\Gamma(1+(r+1)\sigma)}\left(\left(\frac{5}{24}\right)^{(r+1)\sigma}+\left(\frac{1}{8}\right)^{(r+1)\sigma}\right)\right)^{\frac{1}{r}}\\
&\times\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\left(\frac{4}{9}\right)^{\sigma}|\Psi^{\sigma}(\mu_i)|^q+\left(\frac{2}{9}\right)^{\sigma}|\Psi^{\sigma}(\mu_{i+1})|^q\right)\right)^{\frac{1}{q}}\\
&+\left(\frac{\Gamma(1+r\sigma)}{\Gamma(1+(r+1)\sigma)}\left(\left(\frac{1}{8}\right)^{(r+1)\sigma}+\left(\frac{5}{24}\right)^{(r+1)\sigma}\right)\right)^{\frac{1}{r}}\\
&\times\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\left(\frac{2}{9}\right)^{\sigma}|\Psi^{\sigma}(\mu_i)|^q+\left(\frac{4}{9}\right)^{\sigma}|\Psi^{\sigma}(\mu_{i+1})|^q\right)\right)^{\frac{1}{q}}\\
&+\left(\frac{\Gamma(1+r\sigma)}{6^{(r+1)\sigma}\Gamma(1+(r+1)\sigma)}\right)^{\frac{1}{r}}\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\left(\frac{1}{36}\right)^{\sigma}|\Psi^{\sigma}(\mu_i)|^q+\left(\frac{11}{36}\right)^{\sigma}|\Psi^{\sigma}(\mu_{i+1})|^q\right)\right)^{\frac{1}{q}}\Bigg].\end{aligned}$$

Proof. The claim arises straight away by employing the sum from $i=0$ to $i=\theta_1-1$ across subinterval $[\mu_i,\mu_{i+1}]$ in Theorem 2.4. □

3.2. Generalized special means

Here, we recreate the well-known generalized binary mean ${\omega_3}^{\sigma},{\omega_4}^{\sigma}\in\mathbb{R}^{\sigma}$.

1. The generalized arithmetic mean:

$$A_{\sigma}(\omega_3,\omega_4)=\frac{{\omega_3}^{\sigma}+{\omega_4}^{\sigma}}{2^{\sigma}}=\left(\frac{\omega_3+\omega_4}{2}\right)^{\sigma}.$$

2. The generalized weighted arithmetic mean:

$${}_{\omega}A_{\sigma}(\omega_3,\omega_4;m_1,m_2)=\frac{m_1^{\sigma}{\omega_3}^{\sigma}+m_2^{\sigma}\omega_4^{\sigma}}{(m_1+m_2)^{\sigma}}.$$

3. The generalized log-r-mean:

$$L_{\sigma,r}(\omega_3,\omega_4)=\left[\frac{\Gamma(1+r\sigma)}{\Gamma(1+(1+r)\sigma)}\frac{{\omega_4}^{\sigma}-{\omega_3}^{(r+1)\sigma}}{(r+1)(\omega_4-\omega_3)^{\sigma}}\right]^{\frac{1}{r}},\quad r\in\mathbb{R}\setminus\{-1,0\}.$$

Proposition 3.3. *Suppose that all the hypothesises of Theorem 2.2 are fulfilled, then:*

$$\begin{aligned}&\left|\frac{1}{8}\left(3^{\sigma}\,{}_{\omega}A_{\sigma}^{3}\left(\vartheta,\rho,\frac{5}{6},\frac{1}{6}\right)+2^{\sigma}A_{\sigma}^{3}(\vartheta,\rho)+3^{\sigma}\,{}_{\omega}A_{\sigma}^{3}\left(\vartheta,\rho,\frac{1}{6},\frac{5}{6}\right)\right)-\Gamma(1+\sigma)L_{\sigma,3}^{3}(\vartheta,\rho)\right|\\&\quad\leqslant\frac{(\rho-\vartheta)^{\sigma}\Gamma(1+\sigma)\Gamma(1+3\sigma)}{(\Gamma(1+2\sigma))^{2}}\left(\frac{25}{144}\right)^{9}a_{\sigma}^{2}(|\vartheta^{2}|,|\rho^{2}|).\end{aligned}$$

Proof. The statement of claim is directly followed by substitution $\Psi(\mu)=\mu^{3\sigma}$ in Theorem 2.2. □

Proposition 3.4. *Suppose that all the suppositions of Theorem 2.4 are fulfilled, then*

$$\begin{aligned}&\left|\frac{1}{8}\left(3^{\sigma}\,{}_{\omega}A_{\sigma}^{3}\left(\vartheta,\rho,\frac{5}{6},\frac{1}{6}\right)+2^{\sigma}A_{\sigma}^{3}(\vartheta,\rho)+3^{\sigma}\,{}_{\omega}A_{\sigma}^{3}\left(\vartheta,\rho,\frac{1}{6},\frac{5}{6}\right)\right)-\Gamma(1+\sigma)L_{\sigma,3}^{3}(\vartheta,\rho)\right|\\
&\leqslant(\rho-\vartheta)^{\sigma}\left[\left(\frac{\Gamma(1+r\sigma)}{6^{(r+1)\sigma}\Gamma(1+(r+1)\sigma)}\right)^{\frac{1}{r}}\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\frac{1}{18^{\sigma}}\left|\frac{\Gamma(1+3\sigma)}{\Gamma(1+2\sigma)}\right|^{q}A_{\sigma}(11|\vartheta^{2}|^{q},|\rho^{2}|^{q})\right)\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\frac{\Gamma(1+r\sigma)}{\Gamma(1+(r+1)\sigma)}\left(\left(\frac{5}{24}\right)^{(r+1)\sigma}+\left(\frac{1}{8}\right)^{(r+1)\sigma}\right)\right)^{\frac{1}{r}}\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\left(\frac{2}{9}\right)^{\sigma}\left|\frac{\Gamma(1+3\sigma)}{\Gamma(1+2\sigma)}\right|^{q}A_{\sigma}(4|\vartheta^{2}|^{q},2|\rho^{2}|^{q})\right)\right)^{\frac{1}{q}}\\
&\quad+\left(\frac{\Gamma(1+r\sigma)}{\Gamma(1+(r+1)\sigma)}\left(\left(\frac{1}{8}\right)^{(r+1)\sigma}+\left(\frac{5}{24}\right)^{(r+1)\sigma}\right)\right)^{\frac{1}{r}}\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\left(\frac{2}{9}\right)^{\sigma}\left|\frac{\Gamma(1+3\sigma)}{\Gamma(1+2\sigma)}\right|^{q}A_{\sigma}(2|\vartheta^{2}|^{q},4|\rho^{2}|^{q})\right)\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\frac{\Gamma(1+r\sigma)}{6^{(r+1)\sigma}\Gamma(1+(r+1)\sigma)}\right)^{\frac{1}{r}}\left(\frac{\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\frac{1}{18^{\sigma}}\left|\frac{\Gamma(1+3\sigma)}{\Gamma(1+2\sigma)}\right|^{q}A_{\sigma}(|\vartheta^{2}|^{q},11|\rho^{2}|^{q})\right)\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof. By substituting $\Psi(\mu)=\mu^{3\sigma}$ in Theorem 2.4, we attain our desired result. □

3.3. *Probability density mappings*

Suppose that X is generalized convex with respect to generalized probability density function $r:[\vartheta,\rho]\to[0^{\sigma},1^{\sigma}]$, and the notion of cumulative distribution functions is explored as follows:

$$Pr_{\sigma}(X\leqslant\rho)=F_{\sigma}(\rho):=\frac{1}{\Gamma(1+\sigma)}\int_{\vartheta}^{b}p(\sigma)(d\sigma)^{\sigma}.$$

We take advantage of the fact that

$$E_{\sigma}(X)=\frac{1}{\Gamma(1+\sigma)}\int_{\vartheta}^{\rho}\sigma^{\sigma}r(\sigma)(d\sigma)^{\sigma}=\frac{1}{\Gamma(1+\sigma)}\int_{\vartheta}^{\rho}\sigma^{\sigma}dF_{\sigma}(\sigma)=\rho^{\sigma}-\frac{1}{\Gamma(1+\sigma)}\int_{\vartheta}^{\rho}\Gamma(1+\sigma)F_{\sigma}(\sigma)(d\sigma)^{\sigma}.$$

Now we give a relation between probability distribution and expectation in generalized framework.

Proposition 3.5. *From Theorem 2.2, we acquire*

$$\begin{aligned}&\left|\frac{1}{8^{\sigma}}\left(3^{9}Pr_{\sigma}\left(X\leqslant\frac{5\vartheta+\rho}{6}\right)+2^{9}Pr_{\sigma}\left(X\leqslant\frac{\vartheta+\rho}{2}\right)+3^{9}Pr_{\sigma}\left(X\leqslant\frac{\vartheta+5\rho}{6}\right)\right)-\frac{\Gamma(1+\sigma)}{(\rho-\vartheta)^{\sigma}}(\rho^{\sigma}-E_{\sigma}(X))\right|\\&\quad\leqslant\frac{(\rho-\vartheta)^{\sigma}\Gamma(1+\sigma)}{\Gamma(1+2\sigma)}\left(\frac{25}{288}\right)^{\sigma}(|r(\vartheta)|+|r(\rho)|).\end{aligned}$$

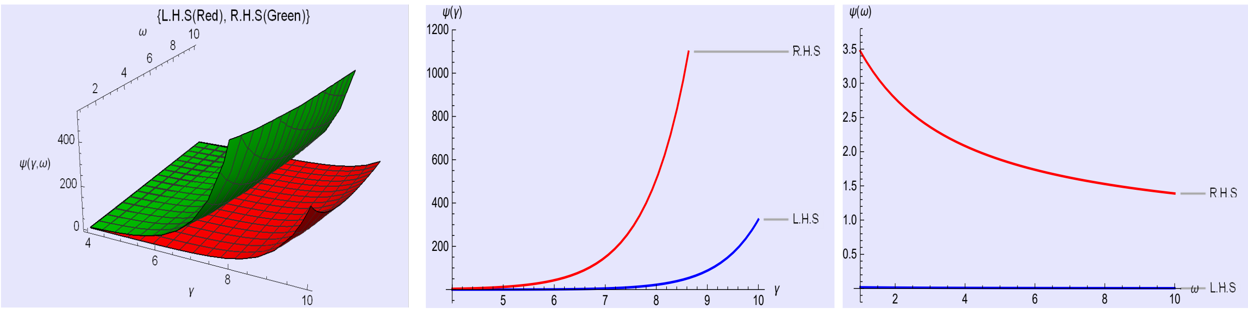
3.4. *Visual analysis*

In this section, we present a graphical analysis of our main results.

Example 3.1. Suppose all the hypothesises of Theorem 2.2 are admitted. Consider $\Psi(\mu) = \frac{\mu^{\gamma\sigma}+\omega^{g}x^{\frac{\gamma\sigma}{2\sigma}}+1^{\sigma}}{\omega^{\sigma}+2^{\sigma}}$ defined on $\mathbb{R}^{+}$ with $\sigma = 1$, $\gamma \geqslant 4$, and $\omega \geqslant 1$ be a generalized convex function, then

$$\left|\frac{1}{8}\left[\frac{3}{\omega+2}\left(\left(\frac{4}{3}\right)^{\gamma}+\omega\left(\frac{4}{3}\right)^{\frac{\gamma}{2}}+2+\left(\frac{8}{3}\right)^{\gamma}+\omega\left(\frac{8}{3}\right)^{\frac{\gamma}{2}}\right)+\frac{2(2^{\gamma}+\omega 2^{\frac{\gamma}{2}}+1)}{\omega+2}\right]\right.$$
$$\left.-\frac{1}{2(\omega+2)}\left(\frac{3^{\gamma+1}-1}{\gamma+1}+\frac{2\omega(3^{\frac{\gamma+2}{2}}-1)}{\gamma+2}+2\right)\right| \leqslant \frac{50\gamma}{576}\left[\frac{1}{2}+\frac{1}{\omega+2}\left(3^{\gamma-1}+\frac{\omega 3^{\frac{\gamma-2}{2}}}{2}\right)\right].$$

Selecting $\gamma = 4$ and $\omega = 2$ in aforementioned inequality, we achieve $0.0130 < 2.7779$. Under similar conditions of Theorem 3.2 proved in [9], we get $0.0130 < 2.5187$. One can easily visualize that our results provides upper bounds of previously established results.



(a) (b) (c)

Figure 1: Visuals analysis of Theorem 2.2.

- For Figure 1 (a) we use $\gamma \in [4, 10]$ and $\omega \in [1, 10]$ as variables to demonstrate the validity of Theorem 2.2.
- For Figure 1 (b) we use $\gamma \in [4, 10]$ as variable to depict a visual that describe the precision of Theorem 2.2.
- For Figure 1 (c) we use $\omega \in [1, 10]$ as variable to construct a visual that ensure the reliability of Theorem 2.2.

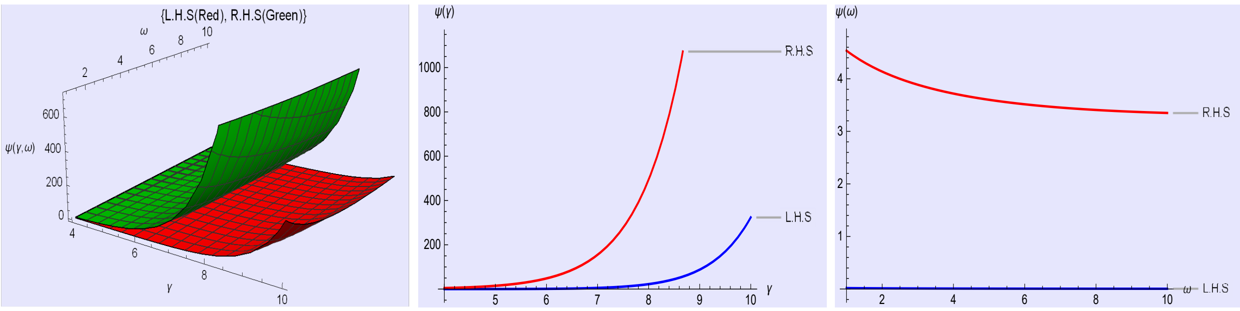
All figures clearly describe that the left hand of side is less than right hand side of Theorem 2.2.

Example 3.2. Suppose all the hypothesises of Theorem 2.4 are admitted. Consider $\Psi(\mu) = \frac{\mu^{\gamma\sigma}+\omega^{g}x^{\frac{\gamma\sigma}{2\sigma}}+1^{\sigma}}{\omega^{\sigma}+2^{\sigma}}$ defined on $\mathbb{R}^{+}$ with $\sigma = 1$, $\gamma \geqslant 4$, and $\omega \geqslant 1$ be a generalized convex function, then

$$\left|\frac{1}{8}\left[\frac{3}{\omega+2}\left(\left(\frac{4}{3}\right)^{\gamma}+\omega\left(\frac{4}{3}\right)^{\frac{\gamma}{2}}+2+\left(\frac{8}{3}\right)^{\gamma}+\omega\left(\frac{8}{3}\right)^{\frac{\gamma}{2}}\right)+\frac{2(2^{\gamma}+\omega 2^{\frac{\gamma}{2}}+1)}{\omega+2}\right]\right.$$
$$\left.-\frac{1}{2(\omega+2)}\left(\frac{3^{\gamma+1}-1}{\gamma+1}+\frac{2\omega(3^{\frac{\gamma+2}{2}}-1)}{\gamma+2}+2\right)\right|$$
$$\leqslant 2\left[\sqrt{\frac{1}{648}}\left(\sqrt{\frac{11}{72}\left(\frac{\gamma}{\omega+2}\left(1+\frac{\omega}{2}\right)\right)^{2}+\frac{1}{72}\left(\frac{\gamma}{\omega+2}\left(3^{\gamma-1}+\frac{\omega}{2}3^{\frac{\gamma-2}{2}}\right)^{2}\right)}\right)\right.$$
$$+\sqrt{\frac{19}{5184}}\left(\sqrt{\frac{2}{9}\left(\frac{\gamma}{\omega+2}\left(1+\frac{\omega}{2}\right)\right)^{2}+\frac{1}{9}\left(\frac{\gamma}{\omega+2}\left(3^{\gamma-1}+\frac{\omega}{2}3^{\frac{\gamma-2}{2}}\right)^{2}\right)}\right)$$

$$+\sqrt{\frac{19}{5184}}\left(\sqrt{\frac{1}{9}\left(\frac{\gamma}{\omega+2}\left(1+\frac{\omega}{2}\right)\right)^2+\frac{2}{9}\left(\frac{\gamma}{\omega+2}\left(3^{\gamma-1}+\frac{\omega}{2}3^{\frac{\gamma-2}{2}}\right)^2\right)}\right)$$
$$+\sqrt{\frac{1}{648}}\left(\sqrt{\frac{1}{72}\left(\frac{\gamma}{\omega+2}\left(1+\frac{\omega}{2}\right)\right)^2+\frac{11}{72}\left(\frac{\gamma}{\omega+2}\left(3^{\gamma-1}+\frac{\omega}{2}3^{\frac{\gamma-2}{2}}\right)^2\right)}\right)\Bigg].$$

Selecting $\gamma = 4$ and $\omega = 2$ in aforementioned inequality, we achieve $0.0130 < 4.1364$.



(a) (b) (c)

Figure 2: Visuals analysis of Theorem 2.4.

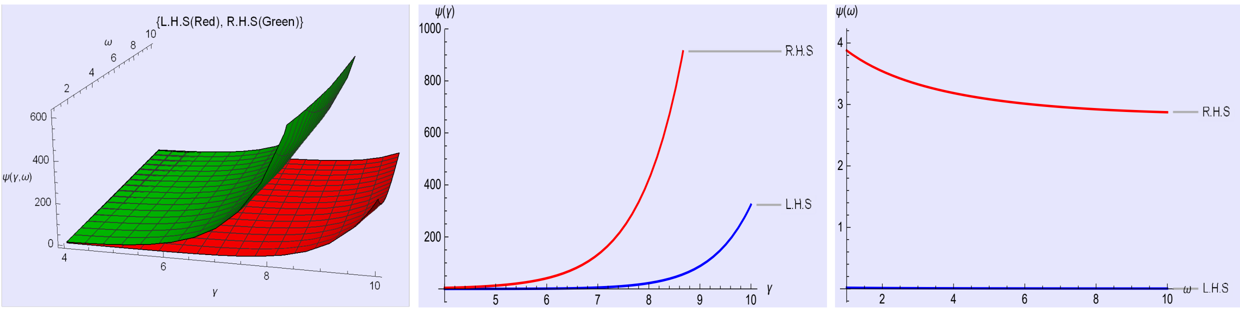
- For Figure 2 (a) we use $\gamma \in [4, 10]$ and $\omega \in [1, 10]$ as variables to demonstrate the validity of Theorem 2.4.
- For Figure 2 (b) we use $\gamma \in [4, 10]$ as variable to depict a visual that describe the precision of Theorem 2.4.
- For Figure 2(c) we use $\omega \in [1, 10]$ as variable to construct a visual that ensure the reliability of Theorem 2.4.

All figures clearly describe that the left hand of side is less than right hand side of Theorem 2.4.

Example 3.3. Suppose all the hypothesises of Theorem 2.5 are admitted. Consider $\Psi(\mu) = \frac{\mu^{\gamma\sigma}+\omega^{g}x^{\frac{\gamma\sigma}{2\sigma}}+1^{\sigma}}{\omega^{\sigma}+2^{\sigma}}$ defined on $\mathbb{R}^+$ with $\sigma = 1$, $\gamma \geqslant 4$, and $\omega \geqslant 1$ be a generalized convex function, then

$$\left|\frac{1}{8}\left[\frac{3}{\omega+2}\left(\left(\frac{4}{3}\right)^{\gamma}+\omega\left(\frac{4}{3}\right)^{\frac{\gamma}{2}}+2+\left(\frac{8}{3}\right)^{\gamma}+\omega\left(\frac{8}{3}\right)^{\frac{\gamma}{2}}\right)+\frac{2(2^{\gamma}+\omega 2^{\frac{\gamma}{2}}+1)}{\omega+2}\right]\right.$$
$$\left.-\frac{1}{2(\omega+2)}\left(\frac{3^{\gamma+1}-1}{\gamma+1}+\frac{2\omega(3^{\frac{\gamma+2}{2}}-1)}{\gamma+2}+2\right)\right|$$
$$\leqslant 2\left[\sqrt{\frac{1}{648}}\left(\sqrt{\frac{1}{81}\left(\frac{\gamma}{\omega+2}\left(1+\frac{\omega}{2}\right)\right)^2+\frac{1}{648}\left(\frac{\gamma}{\omega+2}\left(3^{\gamma-1}+\frac{\omega}{2}3^{\frac{\gamma-2}{2}}\right)^2\right)}\right)\right.$$
$$+\sqrt{\frac{19}{5184}}\left(\sqrt{\frac{863}{41472}\left(\frac{\gamma}{\omega+2}\left(1+\frac{\omega}{2}\right)\right)^2+\frac{361}{41472}\left(\frac{\gamma}{\omega+2}\left(3^{\gamma-1}+\frac{\omega}{2}3^{\frac{\gamma-2}{2}}\right)^2\right)}\right)$$
$$+\sqrt{\frac{19}{5184}}\left(\sqrt{\frac{361}{41472}\left(\frac{\gamma}{\omega+2}\left(1+\frac{\omega}{2}\right)\right)^2+\frac{863}{41472}\left(\frac{\gamma}{\omega+2}\left(3^{\gamma-1}+\frac{\omega}{2}3^{\frac{\gamma-2}{2}}\right)^2\right)}\right)$$
$$\left.+\sqrt{\frac{1}{648}}\left(\sqrt{\frac{1}{648}\left(\frac{\gamma}{\omega+2}\left(1+\frac{\omega}{2}\right)\right)^2+\frac{1}{81}\left(\frac{\gamma}{\omega+2}\left(3^{\gamma-1}+\frac{\omega}{2}3^{\frac{\gamma-2}{2}}\right)^2\right)}\right)\right].$$

Selecting $\gamma = 4$ and $\omega = 2$ in aforementioned inequality, we achieve $0.0130 < 3.5402$.



(a) (b) (c)

Figure 3: Visuals analysis of Theorem 2.5.

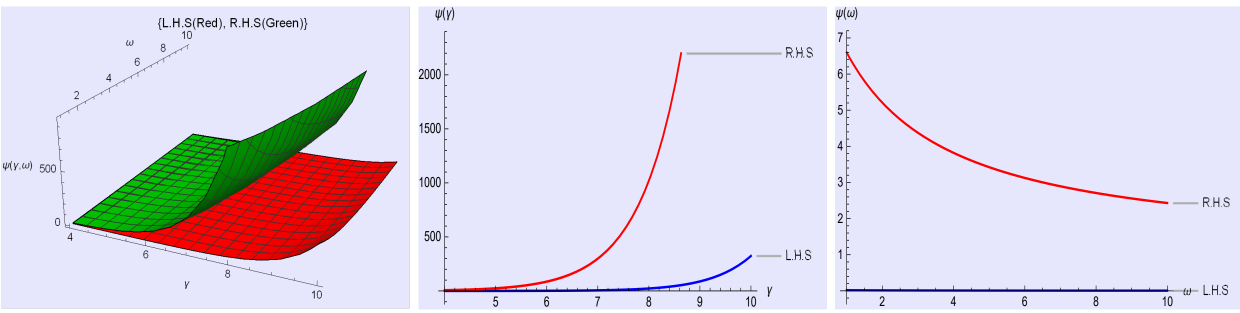
- For Figure 3 (a) we use $\gamma \in [4, 10]$ and $\omega \in [1, 10]$ as variables to demonstrate the validity of Theorem 2.5.
- For Figure 3 (b) we use $\gamma \in [4, 10]$ as variable to depict a visual that describe the precision of Theorem 2.5.
- For Figure 3 (c) we use $\omega \in [1, 10]$ as variable to construct a visual that ensure the reliability of Theorem 2.5.

All figures clearly describe that the left hand of side is less than right hand side of Theorem 2.5.

Example 3.4. Suppose all the hypothesises of Theorem 2.6 are admitted. Consider $\Psi(\mu) = \frac{\mu^{\gamma\sigma}+\omega^{g}x^{\frac{\gamma\sigma}{2\sigma}}+1^{\sigma}}{\omega^{\sigma}+2^{\sigma}}$ defined on $\mathbb{R}^+$ with $\sigma = 1$, $\gamma \geqslant 4$, and $\omega \geqslant 1$ be a generalized convex function, then

$$\left|\frac{1}{8}\left[\frac{3}{\omega+2}\left(\left(\frac{4}{3}\right)^{\gamma}+\omega\left(\frac{4}{3}\right)^{\frac{\gamma}{2}}+2+\left(\frac{8}{3}\right)^{\gamma}+\omega\left(\frac{8}{3}\right)^{\frac{\gamma}{2}}\right)+\frac{2(2^{\gamma}+\omega 2^{\frac{\gamma}{2}}+1)}{\omega+2}\right]\right.$$
$$\left.-\frac{1}{2(\omega+2)}\left(\frac{3^{\gamma+1}-1}{\gamma+1}+\frac{2\omega(3^{\frac{\gamma+2}{2}}-1)}{\gamma+2}+2\right)\right| \leqslant \frac{50\gamma}{288(\omega+2)}\left[1+3^{\gamma-1}+\frac{\omega}{2}\left(1+3^{\frac{\gamma-2}{2}}\right)\right].$$

Selecting $\gamma = 4$ and $\omega = 2$ in aforementioned inequality, we achieve $0.0130 < 5.2083$.



(a) (b) (c)

Figure 4: Visuals analysis of Theorem 2.6.

- For Figure 4 (a) we use $\gamma \in [4, 10]$ and $\omega \in [1, 10]$ as variables to demonstrate the validity of Theorem 2.6.

- For Figure 4 (b) we use $\gamma \in [4, 10]$ as variable to depict a visual that describe the precision of Theorem 2.6.
- For Figure 4 (c) we use $\omega \in [1, 10]$ as variable to construct a visual that ensure the reliability of Theorem 2.6.

All figures clearly describe that the left hand of side is less than right hand side of Theorem 2.6.

4. Conclusion

Inequalities are studied from various aspects by implementing different techniques in different frameworks. L.F calculus plays a vital role in the advancement of convex analysis. Various classes of convexity have been introduced and investigated within fractal concepts. It is known fact that inequalities heavily rely on convexity and its generalizations. In this article, we studied the Maclaurin's inequality incorporated with first order local differentiable generalized mappings. We have proved four different estimates of Maclaurin's inequality and its special cases. Moreover, the results are also new in classical sense. Later on, we provided the validity of primary outcomes through applicable and visual analysis. It is worth to mention that this inequality will serve as base point for further investigation on Maclaurin's like inequalities for different functions classes. In future, we will try to explore these results to find applications in information theory and will try to develop the quantum and fractional variants of these kinds of inequalities. Another interesting problem is that how can we develop the Riemann-Liouville L.F versions of Maclaurin's like inequalities. By using similar strategies, we can derive the bounds for Euler-corrected Maclaurin's inequality, Bools inequality, etc. We hope this study will increase the interest of researchers.

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