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Wijsman asymptotically ideal statistical sequences of order (α, β) under decision making

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Abstract

The statistical convergence of big data analysis is an essential tool for studying the stability of economic models and many computer vision and machine learning problems. In this paper, we propose the concepts of asymptotically ideal ϕ-statistical equivalent sequences of order (α, β) in Wijsman sense by using Musielak-Orlicz function $\mathcal{M} = (\mathfrak{F}_n)$. We also make an effort to define the concept of asymptotic equivalence, statistical equivalent, ϕ-statistical equivalent sequences of order (α, β) in Wijsman sense. We also define the concept of Cesáro Musielak-Orlicz asymptoticallly ϕ-equivalent sequences of order (α, β) in Wijsman sense and examine some algebraic and topological relationship between these concepts.

Keywords: Statistical convergence, asymptotically equivalence, Wijsman convergence, Orlicz function, Musielak-Orlicz function.

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1. Introduction

The statistically Cauchy sequence concept was introduced by Fridy [6], who demonstrated it using statistical convergence. Kostyrko et al. [9] invented the concept of $\mathcal{I}$-convergence as an extension of statistical convergence and developed it using an admissible ideal $\mathcal{I}$, also known as the ideal of $\mathbb{N}$-subsets. Das et al. [4] proposed the concepts of $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence. They also looked at some of its consequences. For more details see [7, 15, 26] and references therein.

A family $\mathcal{I} \subset P(\mathbb{N})$ (where $P(\mathbb{N})$ is power set) is known to be an ideal if and only if the following properties holds:

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(i) $A \cup B \in \mathcal{I}$, for each $A, B \in \mathcal{I}$;
(ii) for each $A \in \mathcal{I}$ and each $B \subset A$, we have $B \in \mathcal{I}$.

For following properties, a non-empty family $\mathcal{F} \subset P(\mathbb{N})$ is a filter on $\mathbb{N}$ if

(i) $\phi \in \mathcal{F}$;
(ii) $A \cap B \in \mathcal{F}$, for each $A, B \in \mathcal{F}$;
(iii) for each $A \in \mathcal{F}$ and each $B \supset A$, we have $B \in \mathcal{F}$.

The ideal $\mathcal{I}$ is known to be non-trivial ideal if it is non-empty and $\mathbb{N} \notin \mathcal{I}$. An ideal $\mathcal{I}$ is known to be admissible if and only if $\{\{\xi\} : \xi \in \mathbb{N}\} \subset \mathcal{I}$, where $(\mathcal{I} \subset P(\mathbb{N}))$. Unless otherwise specified, we write $\mathcal{I}$ is an admissible ideal of subsets of $\mathbb{N}$.

In recent studies, the concept of lacunary sequences has been merged with ideal convergence to explore new forms of summability theory. By applying ideal-based methods to lacunary sequences, researchers can explore complex behaviors in sequence spaces, contributing to advancements in fuzzy number theory and Orlicz functions. $\mathcal{I}$-convergent sequence spaces, a concept introduced to generalize classical convergence, play a key role in modern summability theory. The idea of $\mathcal{I}$-convergence originates from ideal theory, where a sequence is said to be $\mathcal{I}$-convergent if the set of indices for which the sequence deviates from its limit is contained in an ideal. This approach was further extended to different contexts, including fuzzy numbers and sequences defined by Orlicz functions, as studied by Tripathy and Hazarika [27], and lacunary sequences by Tripathy and Dutta [25]. Additionally, Tripathy and Mahanta [28] investigated $\mathcal{I}$-acceleration convergence, enhancing the understanding of sequence behaviors in these spaces. These works provide a robust theoretical foundation for further research in the field. These developments offer a rich framework for examining different types of convergence across a variety of sequence spaces.

Pobyvanets [20] was the first to suggest asymptotically regular matrices, which maintain the asymptotic equivalence of two nonnegative sequences. Marouf [13] introduced the concept of asymptotically equivalent and asymptotic regular matrices. Li [12] used summability to propose the idea of asymptotic equivalent sequences. By providing an analogue of these definitions that is asymptotically statistically identical and natural consistency for nonnegative summability, Patterson [18] expanded on these ideas. Patterson and Savaş [19] firstly proposed asymptotically lacunary statistical equivalent sequence. The idea of Δ^m-lacunary statistical sequences was proposed by Braha [2]. Savaş [22] developed the idea of asymptotically lacunary statistical equivalent sequences using ideals. Several authors have expanded the idea of the convergence of sequences of points to include the convergence of sequences of sets. For more information about the idea of statistical convergence see [21, 23] and references therein. The idea of Wijsman convergence is one of these expansions that was taken into consideration in this paper. Wijsman statistical convergence, introduced by Nuray and Rhoades [17], extends the idea of statistical convergence to sequences of sets. The idea of statistical almost λ-convergence for sequences of sets firstly proposed by Hazarika and Esi [8]. Further information about this we refer to [1, 29] and references therein. Nuray and Rhoades [17] introduced boundedness for the sequences (A_n) and Wijsman statistical convergence. Recently, some authors studied sequence spaces of order (α, β), we may refer to [3, 10, 11, 14, 24] and references therein. We continue in this direction and studied Wijsman asymptotically ideal statistical sequences of order (α, β). Throughout the article we will use α, β such that $0 < \alpha \leqslant \beta \leqslant 1$.

Suppose $P \subseteq \mathbb{N}$ and $P_m = \{n \leqslant m : n \in P\}$. Thus, the natural density P is given by $\rho(P) = \lim_m m^{-1}|P_m|$, if limit exists, by vertical bar we mean the elements in set. For a sequence of real numbers, Fast [5] provided definitions of statistical convergence. The sequence $\xi = (\xi_n)$ is statistically convergent to L of order (α, β) if for every $\varepsilon > 0$, the set $P_\varepsilon = \{n \in \mathbb{N} : |\xi_n - L| \geqslant \varepsilon\}$ has natural density zero, that is, for each $\varepsilon > 0$,

$$\left\{ m \in \mathbb{N} : \frac{1}{m^\alpha} \left[\sum_{n=1}^{m} \left| \xi_n - L \right| \geqslant \varepsilon \right]^\beta \right\} = 0.$$

It is expressed as $\mathcal{S} - \lim \xi = L$ or $\xi_n \to L(\mathcal{S})$, where $\mathcal{S}$ is the set of all statistically convergent sequences.

Definition 1.1. The nonnegative sequences $\xi = (\xi_n)$ and $\chi = (\chi_n)$ are asymptotically statistical equivalent to L of order (α, β), for every $\varepsilon > 0$,

$$\left\{m \in \mathbb{N} : \frac{1}{m^\alpha}\left[\sum_{n=1}^{m}\left|\frac{\xi_n}{\chi_n} - L\right| \geqslant \varepsilon\right]^\beta\right\} = 0,$$

we can write it as $\xi_{(\alpha,\beta)} \overset{\mathcal{S}^L}{\sim} \chi_{(\alpha,\beta)}$ and known to be simply asymptotically statistical equivalent of order (α, β) if $L = 1$.

Definition 1.2. The nonnegative sequences $\xi = (\xi_n)$ and $\chi = (\chi_n)$ are strongly asymptotically $\mathcal{I}$-equivalent to L of order (α, β), for every $\varepsilon > 0$,

$$\left\{m \in \mathbb{N} : \frac{1}{m^\alpha}\left[\sum_{n=1}^{m}\left|\frac{\xi_n}{\chi_n} - L\right| \geqslant \varepsilon\right]^\beta\right\} \in \mathcal{I}.$$

It is denoted by $\xi_{(\alpha,\beta)} \overset{\mathcal{I}-[C_1]^L}{\sim} \chi_{(\alpha,\beta)}$ and known to be simply strongly asymptotically $\mathcal{I}$-equivalent of order (α, β) if $L = 1$.

Definition 1.3. The nonnegative sequences $\xi = (\xi_n)$ and $\chi = (\chi_n)$ are $\mathcal{I}$-asymptotically statistical equivalent to L of order (α, β), for every $\varepsilon > 0$ and $\rho > 0$,

$$\left\{m \in \mathbb{N} : \frac{1}{m^\alpha}\left[\sum_{n=1}^{m}\left|\frac{\xi_n}{\chi_n} - L\right| \geqslant \varepsilon\right]^\beta \geqslant \rho\right\} \in \mathcal{I},$$

written as $(\xi_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{S}^L}{\sim} (\chi_n)_{(\alpha,\beta)}$ and known as simply $\mathcal{I}$-asymptotically statistical equivalent of order (α, β) if $L = 1$.

Definition 1.4. A real number sequence (ξ_n) is known to be $\mathcal{I}$-statistical convergent to real number ξ_0 of order (α, β), for each $\varepsilon > 0$ and $\rho > 0$,

$$\left\{m \in \mathbb{N} : \frac{1}{m^\alpha}\left[\sum_{n=1}^{m}\left|\xi_n - \xi_0\right| \geqslant \varepsilon\right]^\beta \geqslant \rho\right\} \in \mathcal{I}.$$

It is represented by $\mathcal{I} - \mathcal{S} - \lim \xi_n = \xi_0$.

Consider a metric space $(Z, \mathcal{J})$. For any point $\xi \in Z$ and $\mathcal{A}$ be any non-empty subset of Z, the distance between ξ and $\mathcal{A}$ is given by

$$\delta(\xi, \mathcal{A}) = \inf_{a \in \mathcal{A}} \mathcal{J}(\xi, a).$$

Definition 1.5. Consider a metric space $(Z, \mathcal{J})$. For A, A_n be any closed subsets of Z, which are non-empty, then the sequence (A_n) is Wijsman statistical convergent to A of order (α, β) if the sequence $(\delta(\xi, A_n))$ is statistically convergent to $\delta(\xi, A)$ of order (α, β), i.e., for $\varepsilon > 0$ and for each $\xi \in Z$,

$$\left\{m \in \mathbb{N} : \frac{1}{m^\alpha}\left[\sum_{n=1}^{m}\left|\delta(\xi, A_n) - \delta(\xi, A)\right| \geqslant \varepsilon\right]^\beta\right\} = 0.$$

It is denoted by $\mathcal{WS} - \lim_n A_n = A$ or $A_n \to A(\mathcal{WS})$.

The sequence (A_n) is said to be bounded if $\sup_n \delta(\xi, A_n) < \infty$ for each $\xi \in Z$. By L_∞ we mean the set of all bounded sequences of sets.

Definition 1.6. Consider a metric space $(Z, \mathcal{J})$. For A_n, B_n be any closed subsets of Z, which are non-empty, such that $\delta(\xi, A_n) > 0$ and $\delta(\xi, B_n) > 0$ for each $\xi \in Z$, then the sequences (A_n) and (B_n) are known to be asymptotically statistical equivalent of order (α, β) to L, if, for every $\varepsilon > 0$ and for each $\xi \in Z$,

$$\left\{m \in \mathbb{N} : \frac{1}{m^\alpha}\left[\sum_{n=1}^{m}\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right]^\beta\right\} = 0.$$

It is represented by $(A_n)_{(\alpha,\beta)} \overset{\mathcal{WS}^L}{\sim} (B_k)_{(\alpha,\beta)}$ and known to be simply asymptotically statistical equivalent of order (α, β) if $L = 1$.

2. Wijsman Musielak-Orlicz asymptotically ideal ϕ-statistical equivalent sequences of order (α, β)

In this section, we define in Wijsman sense the idea of Cesàro Musielak-Orlicz asymptotically ϕ-statistical equivalent sequences of order (α, β) and Musielak-Orlicz asymptotically ideal ϕ-statistical equivalent sequences of order (α, β). Also, we make an effort to study some inclusion relation between these concepts. Throughout the article we will use α, β such that $0 < \alpha \leqslant \beta \leqslant 1$.

Consider P as finite collection of distinct positive integers. For any element ω of P, by $p(\omega)$ we mean the sequence $\{p_m(\omega)\}$ such that $p_m(\omega) = 1$ for $m \in \omega$ and $p_m(\omega) = 0$ in otherways. Additionally,

$$P_r = \left\{\omega \in P : \sum_{m=1}^{\infty} P_m(\omega) \leqslant r\right\},$$

i.e., P_r is the set of those ω whose support has cardinality atmost r, and we have

$$\Psi = \{\phi = (\phi_m) : 0 < \phi_1 \leqslant \phi_m \leqslant \phi_{m+1}, m\phi_{m+1} \leqslant (m+1)\phi_m\}.$$

Consider a metric space $(Z, \mathcal{J})$. For A_n be any closed subset of Z, which is non-empty, we define

$$\tau_r = \frac{1}{\phi_r}\sum_{n\in\omega,\omega\in P_r} \delta(\xi, A_n).$$

Definition 2.1. Consider a metric space $(Z, \mathcal{J})$. For A, A_n be closed subsets of Z, which are non-empty, then the sequence (A_n) is known to be strongly ϕ-summable to A of order (α, β) if

$$\lim_r \frac{1}{\phi_r^\alpha}\left[\sum_{n\in\omega,\omega\in P_r}\left|\delta(\xi, A_n) - \delta(\xi, A)\right|\right]^\beta = 0,$$

we write it as $\mathcal{W}[\phi] - \lim_n A_{n(\alpha,\beta)} = A$ or $A_{n(\alpha,\beta)} \overset{\mathcal{W}[\phi]}{\to} A$, where $\mathcal{W}[\phi]$ stands for the set of all strongly ϕ-summable Wijsman sequences.

Definition 2.2. For $G \subseteq \mathbb{N}$. The number

$$\varphi_\phi(G) = \lim_{r\to\infty} \frac{1}{\phi_r} |\{n \in \omega, \omega \in P_r : n \in G\}|$$

is known as ϕ-density of G. Clearly $\varphi_\phi(G) \leqslant \varphi(G)$.

Definition 2.3. Consider a metric space $(Z, \mathcal{J})$. For A_n, B_n be any closed subsets of Z, which are non-empty, such that $\delta(\xi, A_n) > 0$ and $\delta(\xi, B_n) > 0$ for each $\xi \in Z$, then the sequences (A_n) and (B_n) are known to be $\mathcal{I}$-asymptotically ϕ-statistical equivalent to L of order(α, β) such that, for every $\varepsilon > 0$ and $\rho > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{\phi_r^\alpha}\left|\left\{n \in \omega, \omega \in P_r : \left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^\beta \geqslant \rho\right\} \in \mathcal{I},$$

denoted as $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{WS}_\phi^L}{\sim} (B_n)_{(\alpha,\beta)}$, and known to be simply Wijsman $\mathcal{I}$-asymptotically ϕ-statistical equivalent of order (α, β) if $L = 1$.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, convex, nondecreasing with $M(0) = 0$, $M(z) > 0$ and $M(z) \to \infty$ as $z \to \infty$. If convexity of Orlicz function is replaced by $M(z + y) \leqslant M(z) + M(y)$, thus it is known as modulus function. An Orlicz function M is known to satisfy Δ_2-condition for all values of t, if there exists $Q \geqslant 1$ such that $M(2t) \leqslant QM(t), t \geqslant 0$. Note that, if $0 < \lambda < 1$, then $M(\lambda z) \leqslant \lambda M(z)$, for all $z \geqslant 0$. A sequence $\mathcal{M} = (\mathfrak{F}_n)$ of Orlicz functions is known to be Musielak-Orlicz function [17]. A sequence $\mathcal{N} = (\mathfrak{N}_n)$ is given by

$$\mathfrak{N}_n(v) = \sup\{|v|t - \mathfrak{F}_n(t) : t \geqslant 0\}, \ n = 1, 2, \ldots,$$

is known to be complementary function of a Musielak-Orlicz function $\mathcal{M}$. The Musielak-Orlicz sequence space $p_{\mathcal{M}}$ and its subspace $q_{\mathcal{M}}$ are given by

$$p_{\mathcal{M}} = \{z \in w : \mathcal{I}_{\mathcal{M}}(cz) < \infty \text{ for some } c > 0\}, \quad q_{\mathcal{M}} = \{z \in w : \mathcal{I}_{\mathcal{M}}(cz) < \infty \text{ for all } c > 0\},$$

where $\mathcal{I}_{\mathcal{M}}$ is a convex modular given as

$$\mathcal{I}_{\mathcal{M}}(z) = \sum_{n=1}^{\infty} \mathcal{M}_n(z_n), z = (z_n) \in p_{\mathcal{M}}.$$

Definition 2.4. Consider a metric space $(Z, \mathcal{J})$ and $\mathcal{M}$ be a Musielak-Orlicz function. Let A_n, B_n be any closed subsets of Z, which are non-empty, such that $\delta(\xi, A_n) > 0$ and $\delta(\xi, B_n) > 0$ for each $\xi \in Z$. Then the sequences (A_n) and (B_n) are known to be

(i) Cesàro Musielak-Orlicz asymptotically equivalent to L of order (α, β) provided that

$$\lim_m \frac{1}{m^\alpha}\left[\sum_{n=1}^{m} \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right|\right]^\beta = 0,$$

denoted as $(A_n)_{(\alpha,\beta)} \overset{\mathcal{W}[C_1]^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$ and known as simply Wijsman Cesàro Musielak-Orlicz asymptotically equivalent of order (α, β) if $L = 1$;

(ii) Cesàro Musielak-Orlicz $\mathcal{I}$-asymptotically equivalent to L of order (α, β) such that, for every $\rho > 0$,

$$\left\{m \in \mathbb{N} : \frac{1}{m^\alpha}\left[\sum_{n=1}^{m} \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right|\right]^\beta \geqslant \rho\right\} \in \mathcal{I},$$

denoted as $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{W}[C_1]^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$ and known as simply Wijsman Cesàro Musielak-Orlicz $\mathcal{I}$-asymptotically equivalent of order (α, β) if $L = 1$;

(iii) Musielak-Orlicz asymptotically ϕ-equivalent to L of order (α, β) provided that

$$\lim_r \frac{1}{\phi_r^\alpha}\left[\sum_{n\in\omega,\omega\in P_r} \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right|\right]^\beta = 0,$$

denoted as $(A_n)_{(\alpha,\beta)} \overset{\mathcal{W}[\phi]^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$ and known as simply Wijsman Musielak-Orlicz asymptotically ϕ-equivalent of order (α, β) if $L = 1$;

(iv) Musielak-Orlicz $\mathcal{I}$-asymptotically ϕ-equivalent to L of order (α, β) such that, for every $\rho > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{\phi_r^\alpha}\left[\sum_{n\in\omega,\omega\in P_r} \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right|\right]^\beta \geqslant \rho\right\} \in \mathcal{I},$$

denoted as $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{W}[\phi]^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$ and known as simply Wijsman Musielak-Orlicz $\mathcal{I}$- asymptotically ϕ-equivalent of order (α, β) if $L = 1$;

(v) Musielak-Orlicz asymptotically ϕ-statistical equivalent to L of order (α, β) such that, for every $\varepsilon > 0$,

$$\lim_r \frac{1}{\phi_r^\alpha}\left|\left\{n \in \omega, \omega \in P_r : \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^\beta = 0,$$

denoted as $(A_n)_{(\alpha,\beta)} \overset{\mathcal{WS}_\phi^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$ and known as simply Wijsman Musielak-Orlicz asymptotically ϕ-statistical equivalent of order (α, β) if $L = 1$;

(vi) Musielak-Orlicz $\mathcal{I}$-asymptotically ϕ-statistical equivalent to L of order (α, β) such that, for every $\varepsilon > 0$ and $\rho > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{\phi_r^\alpha}\left|\left\{n \in \omega, \omega \in P_r : \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^\beta \geqslant \rho\right\} \in \mathcal{I},$$

denoted as $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{WS}_\phi^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$ and known as simply Wijsman Musielak-Orlicz $\mathcal{I}$- asymptotically ϕ-statistical equivalent of order (α, β) if $L = 1$.

Theorem 2.5. *Consider a metric space $(Z, \mathcal{J})$ and $\mathcal{M}$ be a Musielak-Orlicz function. Let A_n, B_n be any closed subsets of Z which are non-empty such that $\delta(\xi, A_n) > 0$ and $\delta(\xi, B_n) > 0$ for each $\xi \in Z$. Thus we have*

(i) *if* $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{W}[C_1]^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$, *then* $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{WS}^L}{\sim} (B_n)_{(\alpha,\beta)}$;

(ii) *if $\mathcal{M} = (\mathfrak{F}_n)$ satisfies Δ_2-condition and $(A_n)_{(\alpha,\beta)} \in L_\infty(\mathcal{M})$ such that* $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{WS}^L}{\sim} (B_n)_{(\alpha,\beta)}$, *then* $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{W}[C_1]^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$;

(iii) *if $\mathcal{M} = (\mathfrak{F}_n)$ satisfies Δ_2-condition, then $\mathcal{I} - \mathcal{W}[C_1]^L(\mathcal{M}) \cap L_\infty(\mathcal{M}) = \mathcal{I} - \mathcal{WS} \cap L_\infty(\mathcal{M})$, where $L_\infty(\mathcal{M}) = \{(A_n) : \mathfrak{F}_n(\delta(\xi, A_n)) \in L_\infty, \xi \in Z\}$.*

Proof.

(i) Assume that $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{W}[C_1]^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$. For given $\varepsilon > 0$, we can write

$$\begin{aligned}\frac{1}{m^\alpha}\left[\sum_{n=1}^m \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right|\right]^\beta &\geqslant \frac{1}{m^\alpha}\left[\sum_{\substack{n=1 \\ \left|\frac{\delta(\xi,A_k)}{\delta(\xi,B_k)}-L\right| \geqslant \varepsilon}}^m \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right|\right]^\beta \\ &\geqslant \frac{\mathfrak{F}_n(\varepsilon)}{m^\alpha}\left|\left\{n \leqslant m : \left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^\beta.\end{aligned}$$

Therefore, for any $\rho > 0$, we have

$$\begin{aligned}&\left\{m \in \mathbb{N} : \frac{1}{m^\alpha}\left|\left\{n \leqslant m : \left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^\beta \geqslant \frac{\rho}{\mathfrak{F}_n(\varepsilon)}\right\} \\ &\subseteq \left\{m \in \mathbb{N} : \frac{1}{m^\alpha}\left[\sum_{n=1}^m \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right|\right]^\beta \geqslant \rho\right\} \in \mathcal{I}.\end{aligned}$$

Therefore, $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{WS}^L}{\sim} (B_n)_{(\alpha,\beta)}$.

(ii) Assume that $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{WS}^L}{\sim} (B_n)_{(\alpha,\beta)}$. Since $\mathcal{M}$ is bounded, so there exists a positive real number Q such that $\sup_t \mathcal{M}(t) \leqslant Q$. Thus for any $\varepsilon > 0$, we have

$$\frac{1}{m^\alpha}\left[\sum_{n=1}^m \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right|\right]^\beta = \frac{1}{m^\alpha}\left[\sum_{\substack{n=1 \\ \left|\frac{\delta(\xi,A_k)}{\delta(\xi,B_k)}-L\right| < \varepsilon}}^m \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right|^\beta + \sum_{\substack{n=1 \\ \left|\frac{\delta(\xi,A_k)}{\delta(\xi,B_k)}-L\right| < \varepsilon}}^m \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right|^\beta\right]$$

$$\leqslant \frac{Q}{m^{\alpha}}\left|\left\{n \leqslant m : \left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^{\beta} + \mathcal{M}(\varepsilon).$$

Now, for any $\rho > 0$, we have

$$\left\{m \in \mathbb{N} : \frac{1}{m^{\alpha}}\left[\sum_{n=1}^{m} \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right|\right]^{\beta} \geqslant \rho\right\}$$
$$\subseteq \left\{m \in \mathbb{N} : \frac{1}{m^{\alpha}}\left|\left\{n \leqslant m : \left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^{\beta} \geqslant \frac{\rho}{Q}\right\} \in \mathcal{I}.$$

Hence $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{W}[C_1]^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$.

(iii) The proof of (iii) directly follows from (i) and (ii). □

Theorem 2.6. *Consider a metric space $(Z, \mathcal{J})$. For A_n, B_n be any closed subsets of Z which are non-empty, such that $\delta(\xi, A_n) > 0$ and $\delta(\xi, B_n) > 0$ for each $\xi \in Z$ and positive non decreasing sequence (ϕ_r) of real numbers such that $\phi_r \to \infty$ as $r \to \infty$ and $\phi_r \leqslant r$, for every $r \in \mathbb{N}$, then $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{WS}^L}{\sim} (B_n)_{(\alpha,\beta)}$ implies $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{WS}^L_{\phi}}{\sim} (B_n)_{(\alpha,\beta)}$.*

Proof. Since by given sequence (ϕ_r), it follows that $\inf_r(\frac{r}{r-\phi_r}) \geqslant 1$, then there exists $q > 0$ such that

$$\frac{r^{\alpha}}{\phi_r^{\alpha}} \leqslant \frac{1+q}{q}.$$

Since $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{WS}^L}{\sim} (B_n)_{(\alpha,\beta)}$, thus for every $\varepsilon > 0$ and for adequately large r, we get

$$\frac{1}{\phi_r^{\alpha}}\left|\left\{n \in \omega, \omega \in P_r : \left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^{\beta}$$
$$= \frac{1}{r^{\alpha}}.\frac{r^{\alpha}}{\phi_r^{\alpha}}\left|\left\{n \leqslant r : \left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^{\beta} - \frac{1}{\phi_r^{\alpha}}\left|\left\{n \in \{1, 2, \ldots, r\} - \omega, \omega \in P_r : \left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^{\beta}$$
$$\leqslant \frac{1+q}{q}\frac{1}{r^{\alpha}}\left|\left\{n \leqslant r : \left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^{\beta} - \frac{1}{\phi_r^{\alpha}}\left|\left\{n_0 \in \{1, 2, \ldots, r\} - \omega, \omega \in P_r : \left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^{\beta}$$
$$\leqslant \frac{1+q}{q}\frac{1}{r^{\alpha}}\left|\left\{n \leqslant r : \left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^{\beta}.$$

Thus, for any $\eta > 0$, we get

$$\left\{r \in \mathbb{N} : \frac{1}{\phi_r^{\alpha}}\left|\left\{n \in \omega, \omega \in P_r : \left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^{\beta} \geqslant \eta\right.$$
$$\left.\subseteq \left\{r \in \mathbb{N} : \frac{1}{r^{\alpha}}\left|\left\{n \leqslant r : \left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^{\beta} \geqslant \frac{q\eta}{1+q}\right\} \in \mathcal{I}.\right.$$

This completes the proof. □

Theorem 2.7. *Consider a metric space $(Z, \mathcal{J})$. For A_n, B_n be any closed subsets of Z, which are non-empty, such that $\delta(\xi, A_n) > 0$ and $\delta(\xi, B_n) > 0$ for each $\xi \in Z$ and $\mathcal{M}$ satisfies Δ_2 condition, then $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{WS}^L}{\sim} (B_n)_{(\alpha,\beta)}$ implies $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{WS}^L_{\phi}(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$.*

Proof. Since by given sequence (ϕ_r), it follows that $\inf_r(\frac{r}{r-\phi_r}) \geqslant 1$, then there exists $q > 0$ such that

$$\frac{r^\alpha}{\phi_r^\alpha} \leqslant \frac{1+q}{q}.$$

Since $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{WS}^L}{\sim} (B_n)_{(\alpha,\beta)}$, then for every $\varepsilon > 0$ and for adequately large r, we get

$$\begin{aligned}
&\frac{1}{\phi_r^\alpha}\left|\left\{n \in \omega, \omega \in P_r : \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^\beta \\
&\quad = \frac{1}{r^\alpha}.\frac{r^\alpha}{\phi_r^\alpha}\left|\left\{n \leqslant r : \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^\beta \\
&\qquad - \frac{1}{\phi_r^\alpha}\left|\left\{n \in \{1,2,\dots,r\} - \omega, \omega \in P_r : \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^\beta \\
&\quad \leqslant \frac{1+q}{q}\frac{1}{r^\alpha}\left|\left\{n \leqslant r : \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^\beta \\
&\qquad - \frac{1}{\phi_r^\alpha}\left|\left\{n_0 \in \{1,2,\dots,r\} - \omega, \omega \in P_r : \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^\beta \\
&\quad \leqslant \frac{1+q}{q}\frac{1}{r^\alpha}\left|\left\{n \leqslant r : \mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^\beta.
\end{aligned}$$

Since $\mathcal{M}$ satisfies Δ_2-condition, it follows that

$$\mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \leqslant Q.\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right|,$$

where $Q > 0$ is constant in both cases $\left|\frac{\delta(\xi,A_n)}{\delta(\xi,B_n)} - L\right| \leqslant 1$ and $\left|\frac{\delta(\xi,A_n)}{\delta(\xi,B_n)} - L\right| \geqslant 1$.

First condition holds by Musielak-Orlicz function, and for second, we have $\left|\frac{\delta(\xi,A_n)}{\delta(\xi,B_n)} - L\right| = 2.L^{(1)} = 2^2.L^{(2)} = \cdots = 2^r.L^{(r)}$ such that $L^{(r)} \leqslant 1$. Since $\mathcal{M}$ satisfies Δ_2-condition, we get

$$\mathfrak{F}_n\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \leqslant M.L^{(r)}.\mathcal{M}(1) = Q.\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right|,$$

where Q and M are constants. The proof of theorem follows from above inequality. □

Theorem 2.8. *Consider a metric space $(Z, \mathcal{J})$. For A_n, B_n be any closed subsets of Z, which are non-empty, such that $\delta(\xi, A_n) > 0$ and $\delta(\xi, B_n) > 0$ for each $\xi, \in Z$, suppose that $\mathcal{M}$ be a Musielak-Orlicz function and $n \in \mathbb{Z}$ such that $\phi_r^\alpha \leqslant [\phi_r^\alpha] + n$, $\sup_r(([\phi_r^\alpha] + k)/\phi_{r-1}^\alpha) < \infty$. Then $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{WS}^L_\phi(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$ implies $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{WS}^L}{\sim} (B_n)_{(\alpha,\beta)}$.*

Proof. If $\sup_r(([\phi_r^\alpha] + n)/\phi_{r-1}) < \infty$, then there exists $Q > 0$ such that $(([\phi_r^\alpha] + n)/\phi_{r-1}^\alpha) < Q$ for all $r \geqslant 1$. Assume that m be an integer such that $\phi_{r-1}^\alpha < m \leqslant \phi_r^\alpha$. Thus for every $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{m^\alpha}\left|\left\{n \leqslant m : \left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \varepsilon\right\}\right|^\beta \\
&\quad \leqslant \frac{1}{m^\alpha}\left|\left\{n \leqslant m : \mathcal{M}\left|\frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L\right| \geqslant \mathcal{M}(\varepsilon)\right\}\right|^\beta
\end{aligned}$$

$$\leqslant \frac{1}{[\phi_r^\alpha]+n}.\frac{[\phi_r^\alpha]+n}{\phi_{r-1}^\alpha}\left|\left\{n \leqslant \phi_r^\alpha : \mathcal{M}\left|\frac{\delta(\xi,A_n)}{\delta(\xi,B_n)}-L\right| \geqslant \mathcal{M}(\varepsilon)\right\}\right|^\beta$$
$$\leqslant \frac{1}{[\phi_r^\alpha]+n}.\frac{[\phi_r^\alpha]+n}{\phi_{r-1}^\alpha}\left|\left\{n \in \omega, \omega \in P_{[\phi_r^\alpha]+n} : \mathcal{M}\left|\frac{\delta(\xi,A_n)}{\delta(\xi,B_n)}-L\right| \geqslant \mathcal{M}(\varepsilon)\right\}\right|^\beta$$
$$\leqslant \frac{Q}{[\phi_r^\alpha]+n}\left|\left\{n \in \omega, \omega \in P_{[\phi_r^\alpha]+n} : \mathcal{M}\left|\frac{\delta(\xi,A_n)}{\delta(\xi,B_n)}-L\right| \geqslant \mathcal{M}(\varepsilon)\right\}\right|^\beta$$
$$\leqslant \frac{Q}{\phi_r^\alpha}\left|\left\{n \in \omega, \omega \in P_r : \mathcal{M}\left|\frac{\delta(\xi,A_n)}{\delta(\xi,B_n)}-L\right| \geqslant \mathcal{M}(\varepsilon)\right\}\right|^\beta.$$

Therefore, for any $\rho > 0$, we get

$$\left\{m \in \mathbb{N} : \frac{1}{m^\alpha}\left[\left|\left\{n \leqslant m : |\frac{\delta(\xi,A_n)}{\delta(\xi,B_n)}-L\right| \geqslant \varepsilon\right\}\right| \geqslant \rho\right\}\Bigg]^\beta$$
$$\subseteq \left\{r \in \mathbb{N} : \frac{1}{\phi_r^\alpha}\left[\left|\left\{n \in \omega, \omega \in P_{[\phi_r^\alpha]+n} : \mathcal{M}\left|\frac{\delta(\xi,A_n)}{\delta(\xi,B_n)}-L\right| \geqslant \mathcal{M}(\varepsilon)\right\}\right| \geqslant \frac{\rho}{Q}\right\}\right]^\beta \in \mathcal{I}.$$

This completes the proof. □

Theorem 2.9. *Consider a metric space $(Z, \mathcal{J})$ and $\mathcal{M}$ be a Musielak-Orlicz function. For A_n, B_n be any closed subsets of Z, which are non-empty, such that $\delta(\xi, A_n) > 0$ and $\delta(\xi, B_n) > 0$ for each $\xi \in Z$, then we have*

(i) *if* $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{W}[C_1]^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$ *implies* $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{W}[\phi]^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$;

(ii) *if* $\sup_r(\phi_r/\phi_{r-1}) < \infty$ *and for every* $r \in \mathbb{N}$, *then* $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{W}[\phi]^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$ *implies* $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{W}[C_1]^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$.

Proof.

(i) By given sequence (ϕ_r), it follows that $\inf_r(r/(r-\phi_r)) \geqslant 1$. Then there exists $l > 0$ such that

$$\frac{r^\alpha}{\phi_r^\alpha} \leqslant \frac{1+l}{l}.$$

Now, we have

$$\frac{1}{\phi_r^\alpha}\left[\sum_{n\in\omega,\omega\in P_r}\mathcal{M}\left|\frac{\delta(\xi,A_n)}{\delta(\xi,B_n)}-L\right|\right]^\beta$$
$$= \frac{r^\alpha}{\phi_r^\alpha}.\frac{1}{r^\alpha}\left[\sum_{n=1}^{m}\mathcal{M}\left|\frac{\delta(\xi,A_n)}{\delta(\xi,B_n)}-L\right|\right]^\beta - \frac{1}{\phi_r^\alpha}\left[\sum_{n\in\{1,2,\dots,r\}-\omega,\omega\in P_r}\mathcal{M}\left|\frac{\delta(\xi,A_n)}{\delta(\xi,B_n)}-L\right|\right]^\beta$$
$$\leqslant \frac{1+l}{l}\frac{1}{r^\alpha}\left[\sum_{n=1}^{r}\mathcal{M}\left|\frac{\delta(\xi,A_n)}{\delta(\xi,B_n)}-L\right|\right]^\beta - \frac{1}{\phi_r^\alpha}\left[\sum_{n_0\in\{1,2,\dots,r\}-\omega,\omega\in P_r}\mathcal{M}\left|\frac{\delta(\xi,A_n)}{\delta(\xi,B_n)}-L\right|\right]^\beta$$
$$\leqslant \frac{1+l}{l}\frac{1}{r^\alpha}\left[\sum_{n=1}^{r}\mathcal{M}\left|\frac{\delta(\xi,A_n)}{\delta(\xi,B_n)}-L\right|\right]^\beta.$$

Since $\mathcal{M}$ are continuous and $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{W}[C_1]^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$, then for any $\rho > 0$ and from last relation we have

$$\left\{r \in \mathbb{N} : \frac{1}{r^\alpha}\left[\sum_{n=1}^{r}\mathcal{M}\left|\frac{\delta(\xi,A_n)}{\delta(\xi,B_n)}-L\right| \geqslant \frac{l\rho}{1+l}\right\}\right]^\beta$$

$$\subseteq \left\{ r \in \mathbb{N} : \frac{1}{\phi_r^{\alpha}} \left[\sum_{n \in \omega, \omega \in P_r} \mathcal{M} \left| \frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L \right| \geqslant \rho \right\} \right]^{\beta} \in \mathcal{I}.$$

Therefore, $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{W}[\phi]^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$.

(ii) Suppose that the sequence (ϕ_r) satisfies the relation that for any set $V \in \mathcal{F}$, $\cup\{m : \phi_{r-1} < m \leqslant [\phi_r], r \in V\} \in \mathcal{F}$. Assume that $\sup_r(\phi_r/\phi_{r-1}) < \infty$, thus there exists $Q > 0$ such that $\phi_r/\phi_{r-1}^{\alpha} < Q$ for all $r \geqslant 1$. Since $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{W}[\phi]^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$. Thus for every $\varepsilon > 0, \rho > 0$, we put

$$V = \left\{ r \in \mathbb{N} : \frac{1}{\phi_r^{\alpha}} \left[\sum_{n \in \omega, \omega \in P_r} \mathcal{M} \left| \frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L \right| < \varepsilon \right\} \right]^{\beta},$$

$$Y = \left\{ m \in \mathbb{N} : \frac{1}{m^{\alpha}} \left[\sum_{n=1}^{m} \mathcal{M} \left| \frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L \right| < \rho \right\} \right]^{\beta}.$$

From our supposition it follows that $V \in \mathcal{F}$. Also, note that

$$\mathcal{V}_r = \frac{1}{\phi_r^{\alpha}} \left[\sum_{n \in \omega, \omega \in P_r} \mathcal{M} \left| \frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L \right| < \varepsilon \right]^{\beta}, \ \forall r \in V.$$

Suppose m be an integer with $\phi_{r-1} < m \leqslant [\phi_r]$ for some $r \in V$. Thus we have

$$\begin{aligned}
&\frac{1}{m^{\alpha}} \left[\sum_{n=1}^{m} \mathcal{M} \left| \frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L \right| \right]^{\beta} \\
&\leqslant \frac{1}{\phi_{r-1}^{\alpha}} \left[\sum_{n=1}^{[\phi_r]} \mathcal{M} \left| \frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L \right| \right]^{\beta} \\
&= \frac{1}{\phi_{r-1}^{\alpha}} \left[\sum_{n=1}^{[\phi_1]} \mathcal{M} \left| \frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L \right|^{\beta} + \sum_{[\phi_1]}^{[\phi_2]} \mathcal{M} \left| \frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L \right|^{\beta} + \cdots + \sum_{\phi_{r-1}}^{[\phi_r]} \mathcal{M} \left| \frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L \right| \right]^{\beta} \\
&\leqslant \frac{\phi_1}{\phi_{r-1}^{\alpha}} \left[\frac{1}{\phi_1} \sum_{n \in \omega, \omega \in P^{(1)}} \mathcal{M} \left| \frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L \right| \right]^{\beta} + \frac{\phi_2}{\phi_{r-1}^{\alpha}} \left[\frac{1}{\phi_2} \sum_{n \in \omega, \omega \in P^{(2)}} \mathcal{M} \left| \frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L \right| \right]^{\beta} \\
&\quad + \cdots + \frac{\phi_R}{\phi_{r-1}^{\alpha}} \left[\frac{1}{\phi_R} \sum_{n \in \omega, \omega \in P^{(R)}} \mathcal{M} \left| \frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L \right| \right]^{\beta} + \cdots + \frac{\phi_r}{\phi_{r-1}^{\alpha}} \left[\frac{1}{\phi_r} \sum_{n \in \omega, \omega \in P^{(r)}} \mathcal{M} \left| \frac{\delta(\xi, A_n)}{\delta(\xi, B_n)} - L \right| \right]^{\beta} \\
&\leqslant \sup_{r \in V} \mathcal{V}_r \frac{\phi_r}{\phi_{r-1}^{\alpha}} < Q\varepsilon,
\end{aligned}$$

where $P^{(q)}$ are the set of integers having more elements than $[\phi_q]$ for $q \in \{1, 2, \ldots, r\}$. Selecting $\rho = \varepsilon/Q$ is because of the fact that $\cup\{m : \phi_{r-1} < m \leqslant [\phi_r], r \in V\} \subset Y$, where $V \in \mathcal{F}$. Thus, based on supposition the set $Y \in \mathcal{F}$. □

Theorem 2.10. *Consider a metric space $(Z, \mathcal{J})$ and $\mathcal{M}$ be a Musielak-Orlicz function. For A_n, B_n be any closed subsets of Z, which are non-empty, such that $\delta(\xi, A_n) > 0$ and $\delta(\xi, B_n) > 0$ for each $\xi \in Z$, thus we have*

(i) *if $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{W}[\phi]^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$, then $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{WS}_{\phi}^L}{\sim} (B_n)_{(\alpha,\beta)}$;*

(ii) *if $\mathcal{M}$ satisfies Δ_2-condition and $(A_n), (B_n) \in L_{\infty}(\mathcal{M})$ such that $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{WS}_{\phi}^L}{\sim} (B_n)_{(\alpha,\beta)}$, then $(A_n)_{(\alpha,\beta)} \overset{\mathcal{I}-\mathcal{W}[\phi]^L(\mathcal{M})}{\sim} (B_n)_{(\alpha,\beta)}$;*

(iii) *if $\mathcal{M}$ satisfies Δ_2-condition, then $\mathcal{I} - \mathcal{W}[\phi]^L(\mathcal{M}) \cap L_{\infty}(\mathcal{M}) = \mathcal{I} - \mathcal{WS}_{\phi}^L \cap L_{\infty}(\mathcal{M})$.*

Proof. The proof is on the similar lines of the Theorems 2.5 and 2.9, so we can omit the details. □

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Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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