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New class of modified (m, p, h)-convex functions with application to integral inequalities

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Abstract

Modified (m, p, h)-convexity is a new notion that is introduced in this study. It generalizes the classical notion of modified (p, h)-convexity and establishes several important properties for modified (m, p, h)-convex functions. Applications of modified (m, p, h)-convex functions are given in form of extensions of Hermite-Hadamard (H-H), Jensen, and Fejér inequalities. We have shown the validity of results by including several examples of modified (m, p, h)-convex functions and also inspect the viability of proved inequalities by choosing several modified (m, p, h)-convex functions. These results provide powerful tools for analyzing the behavior of modified (m, p, h)-convex functions.

Keywords: Convex function, H-H inequalities, Jensen inequality, Fejér inequality, mathematical operators, optimization

2020 MSC: 26D10, 26D15, 26B25, 26A51, 52A15, 33B15, 39B62.

1. Introduction

A fundamental concept in mathematics is convexity that plays a crucial role in various fields, including optimization [4], analysis [28], and geometry [2]. It refers to the property of a function or a set where the line segment connecting any two locations on the function or within the set lies entirely above the function or within the set respectively [20, 27]. This property leads to several important inequalities that have significant applications in different areas [13, 17, 23, 30]. One such inequality is Jensen inequality, it retains the name of Danish mathematician Johan Jensen ([6]). For a convex function $\Psi : I \subset \mathbb{R} \to \mathbb{R}$, the Jensen inequality [14] is given as

$$\Psi\left(\sum_{i=1}^{n} \lambda_i \varphi_i\right) \leqslant \sum_{i=1}^{n} \lambda_i \Psi(\varphi_i), \quad \text{where} \quad \sum_{i=1}^{n} \lambda_i = 1 \text{ and } \varphi_1, \varphi_2, \ldots, \varphi_n \in I.$$

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It states that for a convex function, the presume value of the function exercised to a contingent variable is consistently exceeding or identical to the function exercised to the presume value of the contingent variable [18]. Jensen inequality has wide-ranging applications in probability theory, statistics [1] and information theory [10].

Another important inequality related to convexity is the Hermite-Hadamard inequality, it retains the names of Charles Hermite and Jacques Hadamard [15]. For a convex function $\Psi : I \subset \mathbb{R} \to \mathbb{R}$, the H-H inequalities [26] are given as

$$\Psi\left(\frac{\varphi_1+\varphi_2}{2}\right) \leqslant \frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2} \Psi(w)\,dw \leqslant \frac{\Psi(\varphi_1)+\Psi(\varphi_2)}{2}, \quad \text{for any} \quad \varphi_1, \varphi_2 \in I.$$

Furthermore, Lipót Fejér (1880-1959) instigated an inequality in 1906 [21] by generalizing the concept of convexity called Fejér's inequality. Fejér's inequality for convex functions is a mathematical result that provides an upper bound on the discrepancy between the average of the function values and the value of the function at the average of its arguments [25]. In other words, it quantifies how close the average of a convex function to the function evaluated at the average of its inputs. This inequality is particularly useful in convex analysis and optimization, as it helps to establish convergence properties and it provides insights into the behavior of convex functions. Fejér inequalities for the convex function $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ [21] are given as

$$\Psi\left(\frac{\varphi_1+\varphi_2}{2}\right) \int_{\varphi_1}^{\varphi_2} \Phi(k)\,dk \leqslant \int_{\varphi_1}^{\varphi_2} \Psi(k)\Phi(k)\,dk \leqslant \frac{\Psi(\varphi_1)\Psi(\varphi_2)}{2} \int_{\varphi_1}^{\varphi_2} \Phi(k)\,dk,$$

where $\varphi_1, \varphi_2 \in I$ and $\Phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is a positive function. These inequalities have applications in Fourier analysis [9] and Harmonic analysis [12]. They also play a crucial role in understanding the behavior of functions and their approximations.

Convex functions exhibit distinct geometric and algebraic characteristics that establish them as a fundamental concept in the field of mathematics. Toader, established the family of m-convex functions [16] and modified h-convex function [16], Zhang et al. (2007) developed the p-convex function [29], and Feng et al. (2020) put forward the theory of modified (p, h)-convex functions [8].

The aim of this paper is to study modified (m, p, h)-convex functions and to establish their basic properties. Additionally, it aims to prove the H-H, Jensen and Fejér inequalities for the modified (m, p, h)-convex functions. The paper is organized as follows. In Section 2, we recalled the concepts of convex function, m-convex function, p-convex function, modified h-convex function, (m, p)-convex function, modified (p, h)-convex function, Gamma function, similarly ordered function, and super multiplicative function. In Section 3, we introduced the modified (m, p, h)-convex function. Some of basic properties are proved for modified (m, p, h)-convex functions. In Section 4, H-H type inequalities are proved for modified (m, p, h)-convex functions. In Section 5, Jensen type inequalities are manifested for modified (m, p, h)-convex functions. In Section 6, Fejér type inequalities are demonstrated for modified (m, p, h)-convex functions. In Section 7, the whole work is summarized.

2. Preliminaries

The certain categories of convex functions are summoned in this section.

Definition 2.1. A real valued function $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function, if

$$\Psi(\nu\varphi_1 + (1-\nu)\varphi_2) \leqslant \nu\Psi(\varphi_1) + (1-\nu)\Psi(\varphi_2) \tag{2.1}$$

holds for all $\varphi_1, \varphi_2 \in I$ and $\nu \in [0, 1]$.

Definition 2.2 ([24]). A real valued function $\Psi : [0, u] \to \mathbb{R}$ with $u > 0$ is m-convex, if

$$\Psi(\nu\varphi_1 + m(1-\nu)\varphi_2) \leqslant \nu\Psi(\varphi_1) + m(1-\nu)\Psi(\varphi_2) \tag{2.2}$$

holds for all $\varphi_1, \varphi_2 \in [0, u]$, $\nu \in [0, 1]$ and $m \in [0, 1]$.

Definition 2.3 ([22]). A function $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ is p-convex function, if

$$\Psi\left((\nu\varphi_1^p + (1-\nu)\varphi_2^p)^{1/p}\right) \leqslant \nu\Psi(\varphi_1) + (1-\nu)\Psi(\varphi_2) \tag{2.3}$$

holds $\varphi_1, \varphi_2 \in I$, $\nu \in [0, 1]$ and $p \geqslant 1$.

Definition 2.4 ([5]). Assume that $h : [0, 1] \to \mathbb{R}$ is a non-zero and non-negative function. A real valued function $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ is said to be modified h-convex function, if

$$\Psi(\nu\varphi_1 + (1-\nu)\varphi_2) \leqslant h(\nu)\Psi(\varphi_1) + (1-h(\nu))\Psi(\varphi_2) \tag{2.4}$$

holds for all $\varphi_1, \varphi_2 \in I$ and $\nu \in [0, 1]$.

Definition 2.5 ([8]). Let $h : [0, 1] \to \mathbb{R}$ be non-zero and non-negative function. A real valued function $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ is modified (p, h)-convex function, if

$$\Psi\left((\nu\varphi_1^p + (1-\nu)\varphi_2^p)^{1/p}\right) \leqslant h(\nu)\Psi(\varphi_1) + (1-h(\nu))\Psi(\varphi_2) \tag{2.5}$$

holds for all $\varphi_1, \varphi_2 \in I$, $\nu \in [0, 1]$ and $p \geqslant 1$.

Definition 2.6. Integral form of Gamma function [19] is

$$\Gamma(x) = \int_0^\infty e^{-k}k^{x-1}dk, \text{ where } x > 0.$$

Definition 2.7 ([18]). The functions Ψ and Φ are said to be similarly ordered functions on $I \subset \mathbb{R}$, if

$$\big(\Psi(\varphi_1) - \Psi(\varphi_2)\big)\big(\Phi(\varphi_1) - \Phi(\varphi_2)\big) \geqslant 0, \quad \forall \varphi_1, \varphi_2 \in I.$$

Definition 2.8 ([11]). A function $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ is super multiplicative, if

$$\Psi(\varphi_1\varphi_2) \geqslant \Psi(\varphi_1)\Psi(\varphi_2), \quad \forall \varphi_1, \varphi_2 \in I.$$

3. Main results

In this section, we introduce the modified (m, p, h)-convex function, and also present its basic properties.

Definition 3.1. Let $h : (0, 1] \to \mathbb{R}$ be non-zero and non-negative function. A real valued function $\Psi:[0, u] \to \mathbb{R}$, with $u > 0$ is modified (m, p, h)-convex function, if

$$\Psi\left((\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}\right) \leqslant h(\nu)\Psi(\varphi_1) + m(1-h(\nu))\Psi(\varphi_2) \tag{3.1}$$

holds, for $\nu \in (0, 1]$, $m \in [0, 1]$, $\varphi_1, \varphi_2 \in [0, u]$, and $p \geqslant 1$.

Remark 3.2.

(a) By choosing $m = 1$ in (3.1), we get (2.5).
(b) By substituting $m = 1, p = 1$ in (3.1), one gets (2.4).

(c) By choosing $m = 1, h(\nu) = \nu$ in (3.1), we obtain (2.3).
(d) By choosing $p = 1, h(\nu) = \nu$ in (3.1), we obtain (2.2).
(e) By choosing $m = 1, p = 1, h(\nu) = \nu$ in (3.1), one obtains (2.1).

Note: The following information is deployed throughout the figures of next examples.

(a) The values of ϕ_1 are given along the x-axis.
(b) The values of ϕ_2 are given along the y-axis.
(c) z-axis represents the values of the functions appeared on the right hand sides and the left hand sides of inequalities (3.3), (3.5), and (3.7).

Example 3.3. If we choose the function $\Psi(k) = k^{-p}$, where $p \geqslant 1$, $m \in [0,1]$, and $h(\nu) = \frac{1}{\nu}$, such that $\nu \in (0,1]$ in (3.1), we get

$$\frac{1}{\nu\varphi_1^p + m(1-\nu)\varphi_2^p} \leqslant \frac{1}{\nu\varphi_1^p} + \frac{m}{\varphi_2^p}\left(1 - \frac{1}{\nu}\right) \tag{3.2}$$

holds for all $\varphi_1, \varphi_2 \in [1,u]$ with $u > 0$ and $\varphi_1 < \varphi_2$. Particularly, if we choose $\nu = \frac{1}{4}$, $m = \frac{1}{4}$, and $p = 2$ in (3.2), we get

$$\frac{16}{4\varphi_1^2 + 3\varphi_2^2} \leqslant \frac{4}{\varphi_1^2} - \frac{3}{4\varphi_2^2}. \tag{3.3}$$

Hence, it is modified (m, p, h)-convex function, which is clear from Figure 1.

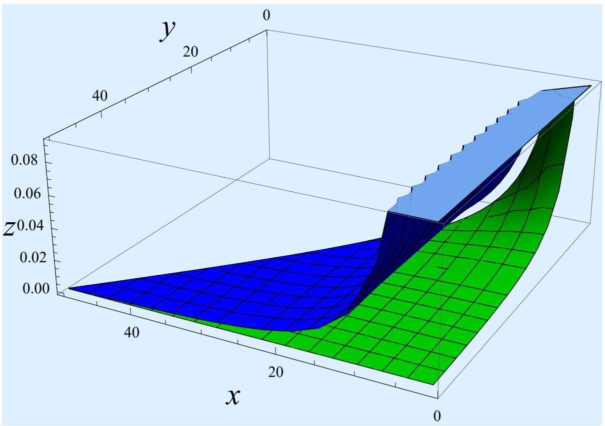

Figure 1: The graphical presentations of inequality (3.3).

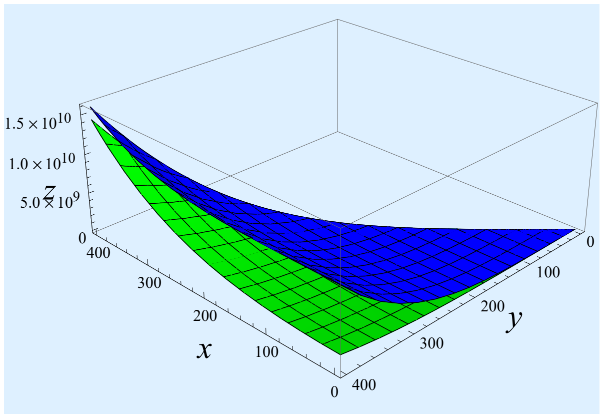

Figure 2: The graphical presentations of inequality (3.5).

• Blue and Green colours represent the right hand side and the left hand side of inequality (3.3) in Figure 1.

Example 3.4. If we choose the function $\Psi(k) = k^4$, $m \in [0,1]$, $h(\nu) = \nu^2$, such that $\nu \in (0,1]$ and $p = 2$ in (3.1), we get

$$\left(\nu\varphi_1^2 + m(1-\nu)\varphi_2^2\right)^2 \leqslant \nu^2\varphi_1^4 + m(1-\nu^2)\varphi_2^4 \tag{3.4}$$

holds for all $\varphi_1, \varphi_2 \in [0,u]$ with $u > 0$ and $\varphi_1 < \varphi_2$. Particularly, if we choose $\nu = \frac{1}{2}$ and $m = \frac{1}{2}$ in (3.4), we get

$$\frac{\varphi_1^4}{4} + \frac{\varphi_2^4}{16} + \frac{\varphi_1^2\varphi_2^2}{4} \leqslant \frac{\varphi_1^4}{4} + \frac{3\varphi_2^4}{8}. \tag{3.5}$$

Hence, it is modified (m, p, h)-convex function, which is clear from Figure 2.

• Blue and Green colors represent the right hand side and the left hand side of inequality (3.5) in Figure 2.

Example 3.5. If we choose the function $\Psi(k) = k^2$, $m \in [0, \frac{1}{4}]$, $p = 2$, and $h(\nu) = \frac{1}{\nu}$ such that $\nu \in (0, 1]$ in (3.1), we get

$$\nu\varphi_1^2 + m(1-\nu)\varphi_2^2 \leqslant \frac{\varphi_1^2}{\nu} + m\left(1 - \frac{1}{\nu}\right)\varphi_1^2 \tag{3.6}$$

holds for all $\varphi_1, \varphi_2 \in [0, u]$ with $u > 0$ and $\varphi_1 > \varphi_2$. Particularly, if we choose $\nu = \frac{1}{2}$ and $m = \frac{1}{4}$, in (3.6), we get

$$\frac{\varphi_1^2}{2} + \frac{\varphi_2^2}{8} \leqslant 2\varphi_1^2 - \frac{\varphi_2^2}{4}. \tag{3.7}$$

Hence, it is modified (m, p, h)-convex function, which is clear from Figure 3.

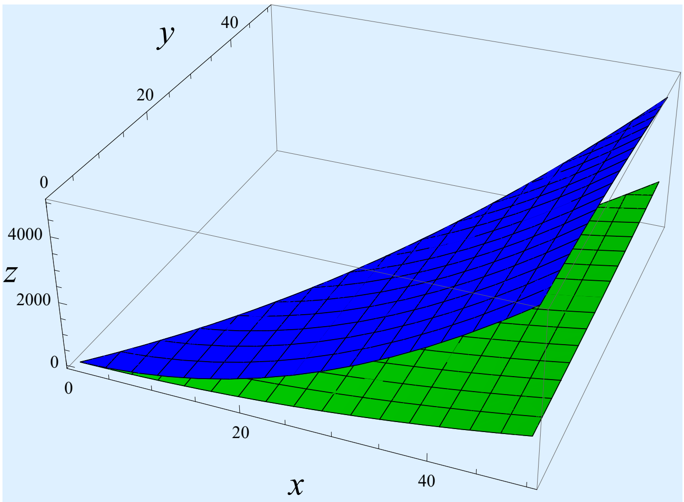

Figure 3: The graphical presentations of inequality (3.7).

• Blue and Green colors represent the right hand side and the left hand side of inequality (3.7) in Figure 3.

Now, we prove the basic properties for modified (m, p, h)-convex function.

Lemma 3.6. *Let* Ψ *and* Φ *be two modified* (m, p, h)*-convex functions. Then their sum is also modified* (m, p, h)*-convex function.*

Proof. For $\varphi_1, \varphi_2 \in [0, u]$, $u > 0$, $p \geqslant 1$, $\nu \in (0, 1]$, and $m \in [0, 1]$, we have

$$(\Psi + \Phi)\left((\nu\varphi_1^p + m(1-\nu)\varphi_1^p)^{1/p}\right) = \Psi(\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p} + \Phi(\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}.$$

Since Ψ and Φ are modified (m, p, h)-convex functions, therefore

$$\begin{aligned}(\Psi + \Phi)\left((\nu\varphi_1^p + m(1-\nu)\varphi_1^p)^{1/p}\right) &\leqslant h(\nu)\Psi(\varphi_1) + m(1-h(\nu))\Psi(\varphi_2) + h(\nu)\Phi(\varphi_1) + m(1-h(\nu))\Phi(\varphi_2)\\ &= h(\nu)(\Psi(\varphi_1) + \Phi(\varphi_1)) + m(1-h(\nu))\big(\Psi(\varphi_2) + \Phi(\varphi_2)\big)\\ &= h(\nu)((\Psi + \Phi)(\varphi_1)) + m(1-h(\nu))((\Psi + \Phi)(\varphi_2)).\end{aligned}$$

□

Lemma 3.7. *Let* Ψ *be a modified* (m, p, h)*-convex function. Then for scalar* $c > 0$, $c\Psi$ *is also modified* (m, p, h)*-convex function.*

Proof. Since Ψ is modified (m, p, h)-convex function, therefore for $\varphi_1, \varphi_2 \in [0, u]$, $u > 0$, $\nu \in (0, 1]$, $m \in [0, 1]$, and $p \geqslant 1$, we have

$$c\Psi\left((\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}\right) \leqslant c(h(\nu)\Psi(\varphi_1) + m(1-h(\nu))\Psi(\varphi_2)) = h(\nu)c\Psi(\varphi_1) + m(1-h(\nu))c\Psi(\varphi_2).$$

□

Lemma 3.8. *Let* $h : (0, 1] \to \mathbb{R}$ *and* $\Psi : [0, u] \to \mathbb{R}$ *be* m*-convex and increasing function, and* $\Phi : [0, u] \to \mathbb{R}$ *be a modified* (m, p, h)*-convex function. Then* $\Psi \circ \Phi$ *is also modified* (m, p, h)*-convex function.*

Proof. For $\varphi_1, \varphi_2 \in [0, u]$, $u > 0$, $\nu \in (0, 1]$, $m \in [0, 1]$, and $p \geqslant 1$,

$$\begin{aligned}\Psi \circ \Phi\left((\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}\right) &\leqslant \Psi\left(\Phi\left((\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}\right)\right)\\ &\leqslant \Psi(h(\nu)\Phi(\varphi_1) + m(1-h(\nu))\Phi(\varphi_2))\\ &\leqslant h(\nu)\Psi(\Phi(\varphi_1)) + m(1-h(\nu))\Psi(\Phi(\varphi_2))\\ &= h(\nu)\Psi \circ \Phi(\varphi_1) + m(1-h(\nu))\Psi \circ \Phi(\varphi_2).\end{aligned}$$

□

Lemma 3.9. *Let* $h : (0, 1] \to \mathbb{R}$*, and* $\Psi_j : [0, u] \to \mathbb{R}$; $(j \in \mathbb{N})$ *be non-empty collection of modified* (m, p, h)*-convex functions such that,* $\forall\, y \in [0, u]$*, we have* $\sup_{j\in\mathbb{N}} \Psi_j(y)$ *exists in* $\mathbb{R}$*. Then function* $\Psi : [0, u] \to \mathbb{R}$ *defined by* $\Psi(y) = \sup_{j\in\mathbb{N}} \Psi_j(y)$, $\forall\, y \in [0, u]$*, is also modified* (m, p, h)*-convex function.*

Proof. For $\varphi_1, \varphi_2 \in [0, u]$, $u > 0$, $m, \nu \in [0, 1]$, and $p \geqslant 1$, we have

$$\begin{aligned}\Psi\left((\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}\right) &= \sup_{j\in\mathbb{N}} \Psi_j\left((\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}\right)\\ &\leqslant \sup_{j\in\mathbb{N}}\left(h(\nu)\Psi_j(\varphi_1) + m(1-h(\nu))\Psi_j(\varphi_2)\right)\\ &\leqslant (h(\nu))\sup_{j\in\mathbb{N}} \Psi_j(\varphi_1) + (m(1-h(\nu))\sup_{j\in\mathbb{N}} \Psi_j(\varphi_2)\\ &\leqslant h(\nu)\Psi(\varphi_1) + m(1-h(\nu))\Psi(\varphi_2).\end{aligned}$$

□

Lemma 3.10. *Suppose that* $h(\nu) \in [0, 1]$*, and* Ψ *and* Φ *be two non-negative modified* (m, p, h)*-convex functions. If* Ψ *and* Φ *are similarly ordered functions, then* $\Psi.\Phi$ *is also modified* (m, p, h)*-convex function.*

Proof. Since Ψ and Φ are two non-negative similarly ordered functions, therefore for $b_1, b_2 \in [0, u]$ with $u > 0$ and $b_1 \leqslant b_2$, we have

$$\begin{aligned}(\Psi(b_1) - \Psi(b_2))(\Phi(b_1) - \Phi(b_2)) &\geqslant 0,\\ \Psi(b_1)\Phi(b_1) - \Psi(b_1)\Phi(b_2) - \Psi(b_2)\Phi(b_1) + \Psi(b_2)\Phi(b_2) &\geqslant 0,\\ \Psi(b_1)\Phi(b_2) + \Psi(b_2)(\Phi(b_1) &\leqslant \Psi(b_1)\Phi(b_1) + \Psi(b_2)\Psi(b_2).\end{aligned} \tag{3.8}$$

Since Ψ and Φ are also modified (m, p, h)-convex functions, therefore for $\varphi_1, \varphi_2 \in [0, u]$, $u > 0$, $\nu \in (0, 1]$, $m \in [0, 1]$, and $p \geqslant 1$, we have

$$\begin{aligned}&(\Psi \cdot \Phi)\left((\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}\right)\\ &\quad = \Psi\left((\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}\right) \cdot \Phi\left((\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}\right)\\ &\quad \leqslant (h(\nu)\Psi(\varphi_1) + m(1-h(\nu))\Psi(\varphi_2))(h(\nu)\Phi(\varphi_1) + m(1-h(\nu))\Phi(\varphi_2)).\end{aligned} \tag{3.9}$$

Choosing $h(v) = \alpha$ and $m(1-h(v)) = \beta$ in (3.9), we get

$$\begin{aligned}
&(\Psi\cdot\Phi)\Big((v\varphi_1^p + m(1-v)\varphi_2^p)^{1/p}\Big)\\
&\quad \leqslant (\alpha\Psi(\varphi_1)+\beta\Psi(\varphi_2))(\alpha\Phi(\varphi_1)+\beta\Phi(\varphi_2))\\
&\quad = \alpha^2(\Psi(\varphi_1)\Phi(\varphi_1)) + \alpha\beta(\Psi(\varphi_1)\Phi(\varphi_2)) + \alpha\beta(\Psi(\varphi_2)\Phi(\varphi_1)) + \beta^2(\Psi(\varphi_2)\Phi(\varphi_2))\\
&\quad = \alpha^2(\Psi(\varphi_1)\Phi(\varphi_1)) + \alpha\beta(\Psi(\varphi_1)\Phi(\varphi_2) + \Psi(\varphi_2)\Phi(\varphi_1)) + \beta^2(\Psi(\varphi_2)\Phi(\varphi_2)).
\end{aligned}$$

By using (3.8), we get

$$\begin{aligned}
&(\Psi\cdot\Phi)\Big((v\varphi_1^p + m(1-v)\varphi_2^p)^{1/p}\Big)\\
&\quad \leqslant \alpha^2(\Psi(\varphi_1)\Phi(\varphi_1)) + \alpha\beta(\Psi(\varphi_1)\Phi(\varphi_1) + \Psi(\varphi_2)\Phi(\varphi_2)) + \beta^2(\Psi(\varphi_2)\Phi(\varphi_2))\\
&\quad = \alpha((\alpha+\beta)(\Psi(\varphi_1)\Phi(\varphi_1))) + \beta((\alpha+\beta)(\Psi(\varphi_2)\Phi(\varphi_2)).\\
&\quad \leqslant \alpha(\Psi(\varphi_1)\Phi(\varphi_1)) + \beta(\Psi(\varphi_2)\Phi(\varphi_2)) \quad \because \alpha+\beta \leqslant 1\\
&\quad = h(v)(\Psi(\varphi_1)\Phi(\varphi_1)) + m(1-h(v))(\Psi(\varphi_2)\Phi(\varphi_2)).
\end{aligned}$$

□

4. Hermite-Hadamard inequalities

The H-H inequalities for the modified (m,p,h)-convex functions are presented in the next theorem.

Theorem 4.1. *Suppose* $h : (0,1] \to \mathbb{R}^+$. *Assume that* Ψ:$[0,u] \to \mathbb{R}$ *is a modified* (m,p,h)*-convex function. Then for* $\varphi_1, \varphi_2 \in [0,u]$ *with* $u > 0$ *and* $\varphi_1 < \varphi_2$*, we have*

$$\begin{aligned}
\Psi\left(\left(\frac{\varphi_1^p + m\varphi_2^p}{2}\right)^{1/p}\right) &\leqslant \frac{p}{(m\varphi_2^p - \varphi_1^p)}\left(h(1/2)\int_{\varphi_1}^{m^{1/p}\varphi_2}\frac{\Psi(k)}{k^{1-p}}dk + m^2(1-h(1/2))\int_{\frac{\varphi_1}{(m^{1/p})}}^{\varphi_2}\frac{\Psi(k)}{k^{1-p}}dk\right)\\
&\leqslant h(1/2)\int_0^1 \big(h(v)\Psi(\varphi_1) + m(1-h(v)\Psi(\varphi_2)\big)dv \\
&\quad + m(1-h(1/2))\int_0^1\left(m(1-h(v))\Psi\left(\frac{\varphi_1}{m^2}\right) + h(v)\Psi(\varphi_2)\right)dv.
\end{aligned} \tag{4.1}$$

Proof. Since Ψ is a modified (m,p,h)-convex function, therefore for $k,l \in [0,u]$, $m \in [0,1]$, $v \in (0,1]$, and $p \geqslant 1$, we have

$$\Psi\left((vk^p + m(1-v)l^p)^{1/p}\right) \leqslant h(v)\Psi(k) + m(1-h(v))\Psi(l). \tag{4.2}$$

By setting $v = 1/2$ in (4.2), we get

$$\Psi\left(\left(\frac{k^p + ml^p}{2}\right)^{1/p}\right) \leqslant h(1/2)\Psi(k) + m(1-h(1/2))\Psi(l). \tag{4.3}$$

By choosing $k^p = v\varphi_1^p + m(1-v)\varphi_2^p$ and $l^p = (1-v)\frac{\varphi_1^p}{m} + v\varphi_2^p$ in (4.3), we get

$$\begin{aligned}
\Psi\left(\left(\frac{\varphi_1^p + m\varphi_2^p}{2}\right)^{1/p}\right) &\leqslant h(1/2)\Psi\Big((v\varphi_1^p + m(1-v)\varphi_2^p)^{1/p}\Big)\\
&\quad + m(1-h(1/2))\Psi\left(\left((1-v)\frac{\varphi_1^p}{m} + v\varphi_2^p\right)^{1/p}\right).
\end{aligned} \tag{4.4}$$

Integrating (4.4) from 0 to 1 with respect to ν, we get

$$\begin{aligned}\Psi\left(\left(\frac{\varphi_1^p+m\varphi_2^p}{2}\right)^{1/p}\right) &\leqslant h(1/2)\int_0^1 \Psi\Big((\nu\varphi_1^p+m(1-\nu)\varphi_2^p)^{1/p}\Big)d\nu \\ &\quad + m(1-h(1/2))\int_0^1 \Psi\left(\left((1-\nu)\frac{\varphi_1^p}{m}+\nu\varphi_2^p\right)^{1/p}\right)d\nu.\end{aligned} \tag{4.5}$$

Using $k^p=\nu\varphi_1^p+m(1-\nu)\varphi_2^p$ in first integral of (4.5) and $k^p=(1-\nu)\frac{\varphi_1^p}{m}+\nu\varphi_2^p$ in second integral of (4.5), we get

$$\Psi\left(\left(\frac{\varphi_1^p+m\varphi_2^p}{2}\right)^{1/p}\right) \leqslant \frac{p}{(m\varphi_2^p-\varphi_1^p)}\left(h(1/2)\int_{\varphi_1}^{m^{1/p}\varphi_2}\frac{\Psi(k)}{k^{1-p}}dk+m^2(1-h(1/2))\int_{\frac{\varphi_1}{(m^{1/p})}}^{\varphi_2}\frac{\Psi(k)}{k^{1-p}}\,dk\right). \tag{4.6}$$

By comparing (4.5) and (4.6), we get

$$\begin{aligned}&\frac{p}{(m\varphi_2^p-\varphi_1^p)}\left(h(1/2)\int_{\varphi_1}^{m^{1/p}\varphi_2}\frac{\Psi(k)}{k^{1-p}}dk+m^2(1-h(1/2))\int_{\frac{\varphi_1}{(m^{1/p})}}^{\varphi_2}\frac{\Psi(k)}{k^{1-p}}dk\right)\\ &= h(1/2)\int_0^1\Psi\Big((\nu\varphi_1^p+m(1-\nu)\varphi_2^p)^{1/p}\Big)d\nu+m(1-h(1/2))\int_0^1\Psi\left(\left((1-\nu)\frac{\varphi_1^p}{m}+\nu\varphi_2^p\right)^{1/p}\right)d\nu.\end{aligned}$$

Since Ψ is modified (m,p,h)-convex function, therefore

$$\begin{aligned}&\frac{p}{(m\varphi_2^p-\varphi_1^p)}\left(h(1/2)\int_{\varphi_1}^{m^{1/p}\varphi_2}\frac{\Psi(k)}{k^{1-p}}dk+m^2(1-h(1/2))\int_{\frac{\varphi_1}{m^{1/p}}}^{\varphi_2}\frac{\Psi(k)}{k^{1-p}}dk\right)\\ &\leqslant h(1/2)\int_0^1(h(\nu)\Psi(\varphi_1)+m(1-h(\nu))\Psi(\varphi_2))d\nu\\ &\quad + m(1-h(1/2))\int_0^1\Big(m(1-h(\nu))\Psi\left(\frac{\varphi_1}{m^2}\right)+h(\nu)\Psi(\varphi_2)\Big)\,d\nu.\end{aligned} \tag{4.7}$$

From (4.6) and (4.7), we get

$$\begin{aligned}\Psi\left(\left(\frac{\varphi_1^p+m\varphi_2^p}{2}\right)^{1/p}\right) &\leqslant \frac{p}{(m\varphi_2^p-\varphi_1^p)}\left(h(1/2)\int_{\varphi_1}^{m^{1/p}\varphi_2}\frac{\Psi(k)}{k^{1-p}}dk+m^2(1-h(1/2))\int_{\frac{\varphi_1}{(m^{1/p})}}^{\varphi_2}\frac{\Psi(k)}{k^{1-p}}dk\right)\\ &\leqslant h(1/2)\int_0^1\big(h(\nu)\Psi(\varphi_1)+m(1-h(\nu)\Psi(\varphi_2))d\nu\end{aligned}$$

$$+m(1-h(1/2))\int_0^1\left(m(1-h(v))\Psi\left(\frac{\varphi_1}{m^2}\right)+h(v)\Psi(\varphi_2)\right)dv.$$

□

Remark 4.2.

(a) If we put $m=1$ in Theorem 4.1, we get Theorem 3.1 of [8].
(b) If we put $m=1$ and $p=1$ in Theorem 4.1, we get Theorem 3 of [16].
(c) If we put $p=1$, $h(t)=t$ and $m=1$ in Theorem 4.1, we obtain the classical Hermite-Hadamard inequality for convex function (see [7]).

Example 4.3. The 3-Dimensional figure in Figure 4 presents the existence of Theorem 4.1.

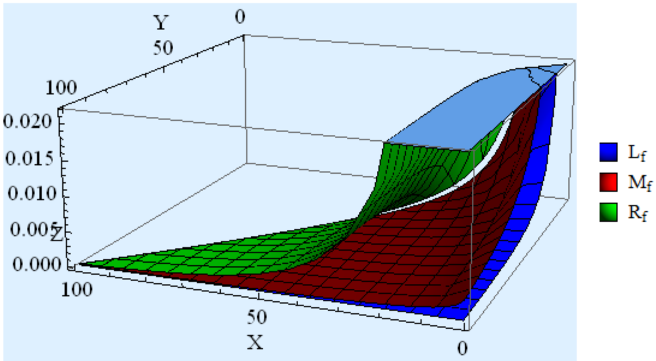

Figure 4: The graph represents the result of inequality (4.1), obtained by choosing $m=1/2$, $\nu=1/4$, $p=2$, $\Psi(k)=k^{-p}$, and $\varphi_1,\varphi_2\in[1,100]$ with $\varphi_1<\varphi_2$.

Theorem 4.4. *Suppose* $h:(0,1]\to\mathbb{R}^+$. *Let* Ψ *be a modified* (m,p,h)*-convex function and also* $\Psi\in L_1[\varphi_1,\varphi_2]$ *with* $\varphi_1<\varphi_2$ *and* $\alpha>0$, *then*

$$\frac{1}{\alpha}\Psi\left(\left(\frac{\varphi_1^p+m\varphi_2^p}{2}\right)^{1/p}\right)\leqslant(1-h(1/2))\int_{\varphi_1}^{m^{1/p}\varphi_2}\left(m\varphi_2^p-k^p\right)^{\alpha-1}\times\frac{\Psi(k)}{k^{1-p}}\times\frac{p}{(m\varphi_2^p-\varphi_1^p)^\alpha}dk$$
$$+m(h(1/2))\int_{\frac{\varphi_1}{(m^{1/p})}}^{\varphi_2}\left(mk^p-\varphi_1^p\right)^{\alpha-1}\times\frac{\Psi(k)}{k^{1-p}}\times\frac{mp}{(m\varphi_2^p-\varphi_1^p)^\alpha}dk,$$

and

$$\int_{\varphi_1}^{m^{1/p}\varphi_2}\left(m\varphi_2^p-k^p\right)^{\alpha-1}\times\frac{\Psi(k)}{k^{1-p}}\times\frac{p}{(m\varphi_2^p-\varphi_1^p)^\alpha}dk+m\int_{\frac{\varphi_1}{(m^{1/p})}}^{\varphi_2}\left(mk^p-\varphi_1^p\right)^{\alpha-1}\times\frac{\Psi(k)}{k^{1-p}}\times\frac{mp}{(m\varphi_2^p-\varphi_1^p)^\alpha}dk$$
$$\leqslant\frac{m}{\alpha}\left(m\Psi\left(\frac{\varphi_1}{m^2}\right)+\Psi(\varphi_2)\right)+\left(\Psi(\varphi_1)-m^2\Psi\left(\frac{\varphi_1}{m^2}\right)\right)\int_0^1(v^{\alpha-1}h(v))dv.$$

Proof. Since Ψ is modified (m,p,h)-convex function, therefore

$$\Psi\left((vk^p+m(1-v)l^p)^{1/p}\right)\leqslant(1-h(v))\Psi(k)+mh(v)\Psi(l). \tag{4.8}$$

By putting $\nu = 1/2$ in (4.8), we get

$$\Psi\left(\left(\frac{k^p + ml^p}{2}\right)^{1/p}\right) \leqslant (1 - h(1/2))\Psi(k) + mh(1/2)\Psi(l). \tag{4.9}$$

Assume $k^p = (\nu\varphi_1^p + m(1-\nu)\varphi_2^p)$ and $l^p = (1-\nu)\frac{\varphi_1^p}{m} + \nu\varphi_2^p$ in (4.9), to get

$$\begin{aligned}\Psi\left(\left(\frac{\varphi_1^p + m\varphi_2^p}{2}\right)^{1/p}\right) &\leqslant (1 - h(1/2))\Psi\left((\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}\right) \\ &\quad + mh(1/2)\Psi\left(\left((1-\nu)\frac{\varphi_1^p}{m} + \nu\varphi_2^p\right)^{1/p}\right).\end{aligned} \tag{4.10}$$

Multiplying (4.10) by $\nu^{\alpha-1}$ and then integrate from 0 to 1 with respect to ν, we get

$$\begin{aligned}\frac{1}{\alpha}\Psi\left(\left(\frac{\varphi_1^p + m\varphi_2^p}{2}\right)^{1/p}\right) &\leqslant (1 - h(1/2)\int_0^1 \left(\nu^{\alpha-1} \times \Psi\left((\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}\right)\right) d\nu \\ &\quad + mh(1/2)\int_0^1 \left(\nu^{\alpha-1} \times \Psi\left(\left((1-\nu)\frac{\varphi_1^p}{m} + \nu\varphi_2^p\right)^{1/p}\right)\right) d\nu.\end{aligned} \tag{4.11}$$

Using $k^p = (\nu\varphi_1^p + m(1-\nu)\varphi_2^p)$ in first integral of (4.11) and $k^p = (1-\nu)\frac{\varphi_1^p}{m} + \nu\varphi_2^p)$ in second integral of (4.11), we get

$$\begin{aligned}\frac{1}{\alpha}\Psi\left(\left(\frac{\varphi_1^p + m\varphi_2^p}{2}\right)^{1/p}\right) &\leqslant (1 - h(1/2))\int_{\varphi_1}^{m^{1/p}\varphi_2} (m\varphi_2^p - k^p)^{\alpha-1} \times \frac{\Psi(k)}{k^{1-p}} \times \frac{p}{(m\varphi_2^p - \varphi_1^p)^\alpha} dk \\ &\quad + m(h(1/2))\int_{\frac{\varphi_1}{(m^{1/p})}}^{\varphi_2} (mk^p - \varphi_1^p)^{\alpha-1} \times \frac{\Psi(k)}{k^{1-p}} \times \frac{mp}{(m\varphi_2^p - \varphi_1^p)^\alpha} dk.\end{aligned} \tag{4.12}$$

Also, Ψ is modified (m, p, h)-convex function, therefore

$$\Psi\left((\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}\right) \leqslant h(\nu)\Psi(\varphi_1) + m(1 - h(\nu)\Psi(\varphi_2) \tag{4.13}$$

and

$$\Psi\left(\left((1-\nu)\frac{\varphi_1^p}{m} + \nu\varphi_2^p\right)^{1/p}\right) \leqslant m(1 - h(\nu))\Psi\left(\frac{\varphi_1}{m^2}\right) + h(\nu)\Psi(\varphi_2). \tag{4.14}$$

From (4.13) and (4.14), we get

$$\begin{aligned}&\Psi\left((\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}\right) + m\Psi\left(\left((1-\nu)\frac{\varphi_1^p}{m} + \nu\varphi_2^p\right)^{1/p}\right) \\ &\leqslant h(\nu)\Psi(\varphi_1) + m(1 - h(\nu)\Psi(\varphi_2) + m^2(1 - h(\nu))\Psi\left(\frac{\varphi_1}{m^2}\right) + mh(\nu)\Psi(\varphi_2) \\ &\leqslant \left(\Psi(\varphi_1) - m^2\Psi\left(\frac{\varphi_1}{m^2}\right)\right) h(\nu) + m\left(m\Psi\left(\frac{\varphi_1}{m^2}\right) + \Psi(\varphi_2)\right).\end{aligned} \tag{4.15}$$

Multiplying (4.15) by $\nu^{\alpha-1}$ and then integrating from 0 to 1 with respect to ν, we get

$$\begin{aligned}&\int_0^1\left(\nu^{\alpha-1}\times\Psi\Big((\nu\varphi_1^p+m(1-\nu)\varphi_2^p)^{1/p}\Big)\right)d\nu+m\int_0^1\left(\nu^{\alpha-1}\times\Psi\left(\left((1-\nu)\frac{\varphi_1^p}{m}+\nu\varphi_2^p\right)^{1/p}\right)\right)d\nu\\&\leqslant\frac{m}{\alpha}\left(m\Psi\left(\frac{\varphi_1}{m^2}\right)+\Psi(\varphi_2)\right)+\left(\Psi(\varphi_1)-m^2\Psi\left(\frac{\varphi_1}{m^2}\right)\right)\int_0^1(\nu^{\alpha-1}\times h(\nu))d\nu.\end{aligned}\tag{4.16}$$

Using $k^p=(\nu\varphi_1^p+m(1-\nu)\varphi_2^p)$ in first integral of (4.16) and $k^p=(1-\nu)\frac{\varphi_1^p}{m}+\nu\varphi_2^p$ in second integral of (4.16), we get

$$\begin{aligned}&\int_{\varphi_1}^{m^{1/p}\varphi_2}(m\varphi_2^p-k^p)^{\alpha-1}\times\frac{\Psi(k)}{k^{1-p}}\times\frac{p}{(m\varphi_2^p-\varphi_1^p)^\alpha}dk\\&\quad+m\int_{\frac{\varphi_1}{(m^{1/p})}}^{\varphi_2}(mk^p-\varphi_1^p)^{\alpha-1}\times\frac{\Psi(k)}{k^{1-p}}\times\frac{mp}{(m\varphi_2^p-\varphi_1^p)^\alpha}dk\\&\leqslant\frac{m}{\alpha}\left(m\Psi\left(\frac{\varphi_1}{m^2}\right)+\Psi(\varphi_2)\right)+\left(\Psi(\varphi_1)-m^2\Psi\left(\frac{\varphi_1}{m^2}\right)\right)\int_0^1(\nu^{\alpha-1}h(\nu))d\nu.\end{aligned}\tag{4.17}$$

From (4.12) and (4.17), we get

$$\begin{aligned}\frac{1}{\alpha}\Psi\left(\left(\frac{\varphi_1^p+m\varphi_2^p}{2}\right)^{1/p}\right)&\leqslant(1-h(1/2))\int_{\varphi_1}^{m^{1/p}\varphi_2}(m\varphi_2^p-k^p)^{\alpha-1}\times\frac{\Psi(k)}{k^{1-p}}\times\frac{p}{(m\varphi_2^p-\varphi_1^p)^\alpha}dk\\&\quad+m(h(1/2))\int_{\frac{\varphi_1}{(m^{1/p})}}^{\varphi_2}(mk^p-\varphi_1^p)^{\alpha-1}\times\frac{\Psi(k)}{k^{1-p}}\times\frac{mp}{(m\varphi_2^p-\varphi_1^p)^\alpha}dk\end{aligned}$$

and

$$\begin{aligned}&\int_{\varphi_1}^{m^{1/p}\varphi_2}(m\varphi_2^p-k^p)^{\alpha-1}\times\frac{\Psi(k)}{k^{1-p}}\times\frac{p}{(m\varphi_2^p-\varphi_1^p)^\alpha}dk+m\int_{\frac{\varphi_1}{(m^{1/p})}}^{\varphi_2}(mk^p-\varphi_1^p)^{\alpha-1}\times\frac{\Psi(k)}{k^{1-p}}\times\frac{mp}{(m\varphi_2^p-\varphi_1^p)^\alpha}dk\\&\leqslant\frac{m}{\alpha}\left(m\Psi\left(\frac{\varphi_1}{m^2}\right)+\Psi(\varphi_2)\right)+\left(\Psi(\varphi_1)-m^2\Psi\left(\frac{\varphi_1}{m^2}\right)\right)\int_0^1(\nu^{\alpha-1}h(\nu))d\nu.\end{aligned}$$ □

Remark 4.5.

(a) If we put $m=1$ in Theorem 4.4, we get Theorem 3.2 of [8].
(b) By assuming $m=1$ and $p=1$ in Theorem 4.4, one obtains Theorem 6 of [16].

5. Jensen inequality

Let $\Psi:[0,u]\to\mathbb{R}$ be a modified (m,p,h)-convex function, then for $\varphi_1,\varphi_2\in[0,u]$, $p\geqslant1$, and $\nu_1+\nu_2=1$, we have

$$\Psi\Big((\nu_1\varphi_1^p+m\nu_2\varphi_2^p)^{1/p}\Big)=\Psi(\nu_1\varphi_1^p+m(1-\nu_1)\varphi_2^p)^{1/p}\leqslant h(\nu)\Psi(\varphi_1)+m(1-h(\nu))\Psi(\varphi_2).$$

For $n > 2$, let $\varphi_1, \varphi_2, \ldots, \varphi_n \in [0, u]$ with $u > 0$, $\sum\limits_{d=1}^{n} \nu_d = 1$, and $W_l = \sum\limits_{j=1}^{l} \nu_j$ for $l = 1, 2, \ldots, n$, then we have

$$\Psi\left(\left(\sum_{d=1}^{l-1} \nu_d \varphi_d^p + m\nu_l \varphi_l^p\right)^{1/p}\right) = \Psi\left(\left(W_{l-1} \sum_{d=1}^{l-1} \frac{\nu_d \varphi_d^p}{W_{l-1}} + m\nu_l \varphi_l^p\right)^{1/p}\right),$$

$$\Psi\left(\left(\sum_{d=1}^{l-1} \nu_d \varphi_d^p + m\nu_l \varphi_l^p\right)^{1/p}\right) \leqslant h(W_{l-1})\Psi\left(\left(\frac{\sum\limits_{d=1}^{l-1} \nu_d \varphi_d^p}{W_{l-1}}\right)^{1/p}\right) + m(1 - h(W_{l-1}))\Psi(\varphi_l).$$

Theorem 5.1. *Let $\Psi:[0, u] \to \mathbb{R}$ be a modified (m, p, h)-convex function and $h:(0, 1] \to \mathbb{R}^+$ be a non-negative super multiplicative function. If $\sum\limits_{j=1}^{l} \nu_j = W_l$ for $l = 1, 2, \ldots, n$ and $\sum\limits_{d=1}^{n} \nu_d = 1$ for $n \in \mathbb{N}$, then*

$$\begin{aligned}\Psi\left(\left(\sum_{d=1}^{l-1} \nu_d \varphi_d^p + m\nu_l \varphi_l^p\right)^{1/p}\right) &\leqslant m\Psi(\varphi_l) + h(W_{l-1})(\Psi(\varphi_{l-1}) - m\Psi(\varphi_l)) \\ &\quad + \sum_{d=1}^{l-2} h(W_d)(\Psi(\varphi_d) - \Psi(\varphi_{d+1})).\end{aligned} \tag{5.1}$$

Proof.

$$\begin{aligned}&\Psi\left(\left(\sum_{d=1}^{l-1} \nu_d \varphi_d^p + m\nu_l \varphi_l^p\right)^{1/p}\right) \\ &= \Psi\left(\left(\frac{\sum\limits_{d=1}^{l-1} \nu_d \varphi_d^p}{W_{l-1}} W_{l-1} + m\nu_l \varphi_l^p\right)^{1/p}\right) \\ &\leqslant h(W_{l-1})\Psi\left(\left(\frac{\sum\limits_{d=1}^{l-1} \nu_d \varphi_d^p}{W_{l-1}}\right)^{1/p}\right) + m(1 - h(W_{l-1}))\Psi(\varphi_l) \\ &= m(1 - h(W_{l-1}))\Psi(\varphi_l) + h(W_{l-1})\Psi\left(\left(\frac{W_{l-2}}{W_{l-1}} \frac{\sum\limits_{d=1}^{l-2} \nu_d \varphi_d^p}{W_{l-2}} + \frac{\nu_{l-1} \varphi_{l-1}^p}{W_{l-1}}\right)^{1/p}\right).\end{aligned}$$

Since Ψ is also modified (p, h)-convex function, therefore

$$m(1 - h(W_{l-1}))\Psi(\varphi_l) + h(W_{l-1})\Psi\left(\left(\frac{W_{l-2}}{W_{l-1}} \frac{\sum\limits_{d=1}^{l-2} \nu_d \varphi_d^p}{W_{l-2}} + \frac{\nu_{l-1} \varphi_{l-1}^p}{W_{l-1}}\right)^{1/p}\right)$$

$$\leqslant m(1-h(W_{l-1}))\Psi(\varphi_l)+h(W_{l-1})\left(h\left(\frac{W_{l-2}}{W_{l-1}}\right)\right)\Psi\left(\left(\frac{\sum\limits_{d=1}^{l-2}\nu_d\varphi_d^p}{W_{l-2}}\right)^{1/p}\right)$$
$$+h(W_{l-1})\left(1-h\left(\frac{W_{l-2}}{W_{l-1}}\right)\right)\Psi(\varphi_{l-1}).$$

Since h is super multiplicative, therefore

$$m(1-h(W_{l-1}))\Psi(\varphi_l)+h(W_{l-1})\left(h\left(\frac{W_{l-2}}{W_{l-1}}\right)\right)\Psi\left(\left(\frac{\sum\limits_{d=1}^{l-2}\nu_d\varphi_d^p}{W_{l-2}}\right)^{1/p}\right)$$
$$+h(W_{l-1})\left(1-h\left(\frac{W_{l-2}}{W_{l-1}}\right)\right)\Psi(\varphi_{l-1})$$
$$\leqslant m\Psi(\varphi_l)+h(W_{l-1})(\Psi(\varphi_{l-1})-m\Psi(\varphi_l))$$
$$-h(W_{l-2})\Psi(\varphi_{l-1})+h(W_{l-2})\Psi\left(\left(\frac{\sum\limits_{d=1}^{l-2}\nu_d\varphi_d^p}{W_{l-2}}\right)^{1/p}\right)$$
$$\leqslant m\Psi(\varphi_l)+h(W_{l-1})(\Psi(\varphi_{l-1})-m\Psi(\varphi_l))-h(W_{l-2})\Psi(\varphi_{l-1})$$
$$+h(W_{l-2})\Psi\left(\left(\frac{W_{l-3}}{W_{l-2}}\frac{\sum\limits_{d=1}^{l-3}\nu_d\varphi_d^p}{W_{l-3}}+\frac{\nu_{l-2}\varphi_{l-2}^p}{W_{l-2}}\right)^{1/p}\right).$$

Since Ψ is also modified (p,h)-convex function and h is super multiplicative, therefore we get

$$m\Psi(\varphi_l)+h(W_{l-1})(\Psi(\varphi_{l-1})-m\Psi(\varphi_l))-h(W_{l-2})\Psi(\varphi_{l-1})$$
$$+h(W_{l-2})\Psi\left(\left(\frac{W_{l-3}}{W_{l-2}}\frac{\sum\limits_{d=1}^{l-3}\nu_d\varphi_d^p}{W_{l-3}}+\frac{\nu_{l-2}\varphi_{l-2}^p}{W_{l-2}}\right)^{1/p}\right)$$
$$\leqslant m\Psi(\varphi_l)+h(W_{l-1})(\Psi(\varphi_{l-1})-m\Psi(\varphi_l))-h(W_{l-2})\Psi(\varphi_{l-1})$$
$$+h(W_{l-2})h\left(\frac{W_{l-3}}{W_{l-2}}\right)\Psi\left(\left(\frac{\sum\limits_{d=1}^{l-3}\nu_d\varphi_d^p}{W_{l-3}}\right)^{1/p}\right)+h(W_{l-2})\left(1-h\left(\frac{W_{l-3}}{W_{l-2}}\right)\right)\Psi(\varphi_{l-2})$$
$$=m\Psi(\varphi_l)+h(W_{l-1})(\Psi(\varphi_{l-1})-m\Psi(\varphi_l))+h(W_{l-2})(\Psi(\varphi_{l-2})-\Psi(\varphi_{l-1}))$$
$$-h(W_{l-3})\Psi(\varphi_{l-2})+h(W_{l-3})\Psi\left(\left(\frac{\sum\limits_{d=1}^{l-3}\nu_d\varphi_d^p}{W_{l-3}}\right)^{1/p}\right).$$

By repeating the same procedure, we get

$$\Psi\left(\left(\sum_{d=1}^{l-1} \nu_d \varphi_d^p + m\nu_l \varphi_l^p\right)^{1/p}\right) \leqslant m\Psi(\varphi_l) + h(W_{l-1})(\Psi(\varphi_{l-1}) - m\Psi(\varphi_l)) + \sum_{d=1}^{l-2} h(W_d)(\Psi(\varphi_d) - \Psi(\varphi_{d+1}).$$

□

Remark 5.2.

(a) By putting $m = 1$ in inequality (5.1) of Theorem 5.1, we get Theorem 4.1 of [8].
(b) By putting $m = 1$, $p = 1$, and $h(t) = t$ in inequality (5.1) of Theorem 5.1, we get classical Jensen type inequality.

Example 5.3. If we choose $\varphi_1 = 1$, $\varphi_2 = 2$, $\varphi_3 = 3$, $\varphi_4 = 4$, $\nu_1 = \frac{1}{4}$, $\nu_2 = \frac{1}{4}$, $\nu_3 = \frac{1}{4}$, $\nu_4 = \frac{1}{4}$, $p = 2$, $m = \frac{1}{4}$, and $h(\nu) = \frac{1}{\nu}$ such that $\nu \in (0, 1]$ and $\Psi(k) = k^{-p}$ in inequality (5.1), we get $\Longrightarrow 0.2222 \leqslant 3.423$. Also, the following graph solidifies the existence of Theorem 5.1.

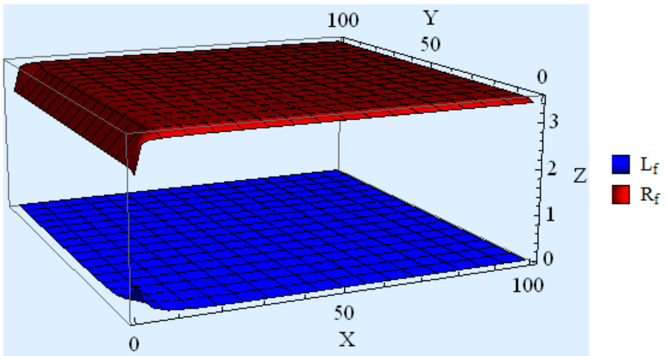

Figure 5: The graphical representation of the conclusion of inequality (5.1), obtained by putting $\nu_1 = \frac{1}{4}$, $\nu_2 = \frac{1}{4}$, $\nu_3 = \frac{1}{4}$, $\nu_4 = \frac{1}{4}$, $p = 2$, $m = \frac{1}{4}$, $h(\nu) = \frac{1}{\nu}$, $\Psi(k) = k^{-p}$, and $\varphi_3, \varphi_4 \in [1, 100]$.

6. Fejér inequalities

Theorem 6.1. *Let* $\Psi : [0, u] \to \mathbb{R}$ *be a non-negative modified* (m, p, h_1)*-convex function and* $\Phi : [0, u] \to \mathbb{R}$ *be a non-negative modified* (m, p, h_2)*-convex function. Also* $\Psi.\Phi \in L_1[\varphi_1, \varphi_2]$ *and* $h_1.h_2 \in L_1[0, 1]$ *and* $\varphi_1, \varphi_2 \in [0, u]$ *with* $\varphi_1 < \varphi_2$*. Then we have*

$$\begin{aligned} \frac{p}{(m\varphi_2^p - \varphi_1^p)} \int_{\varphi_1}^{m^{1/p}\varphi_2} \Psi(x)\Phi(x)dx &\leqslant m^2\Psi(\varphi_2)\Phi(\varphi_2) + i(\varphi_1, \varphi_2)\int_0^1 (h_1(\nu)h_2(\nu))d\nu \\ &\quad + j(\varphi_1, \varphi_2)\int_0^1 h_1(\nu)d\nu + k(\varphi_1, \varphi_2)\int_0^1 h_2(\nu)d\nu, \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} i(\varphi_1, \varphi_2) &= \Psi(\varphi_1)\Phi(\varphi_1) - m\Psi(\varphi_1)\Phi(\varphi_2) - m\Psi(\varphi_2)\Phi(\varphi_1) + m^2\Psi(\varphi_2)\Phi(\varphi_2), \\ j(\varphi_1, \varphi_2) &= m\Psi(\varphi_1)\Phi(\varphi_2) - m^2\Psi(\varphi_2)\Phi(\varphi_2), \\ k(\varphi_1, \varphi_2) &= m\Psi(\varphi_2)\Phi(\varphi_1) - m^2\Psi(\varphi_2)\Phi(\varphi_2). \end{aligned}$$

Proof. Since Ψ and Φ are modified (m,p,h_1)-convex function and modified (m,p,h_2)-convex function, respectively, therefore

$$\Psi\Big((\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}\Big) \leqslant h_1(\nu)\Psi(\varphi_1) + (m(1-h_1(\nu))\Psi(\varphi_2)),$$

and

$$\Phi\Big((\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}\Big) \leqslant h_2(\nu)\Phi(\varphi_1) + (m(1-h_2(\nu))\Phi(\varphi_2)).$$

Since Ψ and Φ are non-negative functions, therefore

$$\begin{aligned}\Psi\Big((\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}\Big)\Phi\Big((\nu\varphi_1^p + m(1-\nu)\varphi_2^p)^{1/p}\Big) &\leqslant m^2\Psi(\varphi_2)\Phi(\varphi_2)\\ &+ \big(\Psi(\varphi_1)\Phi(\varphi_1) - m\Psi(\varphi_1)\Phi(\varphi_2) - m\Psi(\varphi_2)\Phi(\varphi_1) + m^2\Psi(\varphi_2)\Phi(\varphi_2)\big)\big(h_1(\nu)h_2(\nu)\big)\\ &+ \big(m\Psi(\varphi_1)\Phi(\varphi_2) - m^2\Psi(\varphi_2)\Phi(\varphi_2)\big)\big(h_1(\nu)\big) + \big(m\Psi(\varphi_2)\Phi(\varphi_1) - m^2\Psi(\varphi_2)\Phi(\varphi_2)\big)\big(h_2(\nu)\big).\end{aligned} \tag{6.2}$$

Integrating (6.2) from 0 to 1 with respect to ν, we get

$$\begin{aligned}&\int_0^1 \Big(\Psi\Big((\nu_1\varphi_1^p + m(1-\nu_1)\varphi_2^p)^{1/p}\Big)\cdot\Phi\Big((\nu_1\varphi_1^p + m(1-\nu_1)\varphi_2^p)^{1/p}\Big)\Big)\,d\nu\\ &\leqslant m^2\Psi(\varphi_2)\Phi(\varphi_2) + \big(\Psi(\varphi_1)\Phi(\varphi_1) - m\Psi(\varphi_1)\Phi(\varphi_2) - m\Psi(\varphi_2)\Phi(\varphi_1)\\ &\quad + m^2\Psi(\varphi_2)\Phi(\varphi_2)\big)\int_0^1 (h_1(\nu)h_2(\nu))d\nu + \big(m\Psi(\varphi_1)\Phi(\varphi_2) - m^2\Psi(\varphi_2)\Phi(\varphi_2)\big)\int_0^1 h_1(\nu)d\nu\\ &\quad + \big(m\Psi(\varphi_2)\Phi(\varphi_1) - m^2\Psi(\varphi_2)\Phi(\varphi_2)\big)\int_0^1 h_2(\nu)d\nu.\end{aligned} \tag{6.3}$$

By setting $x^p = \nu\varphi_1^p + m(1-\nu)\varphi_2^p$, in (6.3), we get

$$\begin{aligned}\frac{p}{(m\varphi_2^p - \varphi_1^p)}\int_{\varphi_1}^{m^{1/p}\varphi_2} \Psi(x)\Phi(x)dx &\leqslant m^2\Psi(\varphi_2)\Phi(\varphi_2) + i(\varphi_1,\varphi_2)\int_0^1 (h_1(\nu)h_2(\nu))d\nu\\ &\quad + j(\varphi_1,\varphi_2)\int_0^1 h_1(\nu)d\nu + k(\varphi_1,\varphi_2)\int_0^1 h_2(\nu)d\nu.\end{aligned}$$

□

Remark 6.2.

(a) If we put $m = 1$ in inequality (6.1) of Theorem 6.1, we get Theorem 5.1 of [8].
(b) If we put $m = 1$, $p = 1$ and $h(t) = t$ in inequality (6.1) of Theorem 6.1, we get result for convex function.

Example 6.3. Choosing $\varphi_1 = 2$, $\varphi_2 = 6$, $\nu = \frac{1}{4}$, $p = 2$, $m = \frac{1}{2}$, and $h_1(\nu) = \frac{1}{\nu^2}$ such that $\nu \in (0,1]$, and $h_2(\nu) = \frac{1}{\nu}$ such that $\nu \in (0,1]$, and $\Psi(k) = k^{-2}$ and $\Phi(k) = k^4$ in inequality (6.1), we get $3.25 \leqslant 1801.72$.

7. Conclusion

In this study, a novel concept of convexity known as modified (m, p, h)-convexity is introduced. We have begun by establishing the fundamentals of this novel concept of convexity. Then H-H, Jensen, and Fejér inequalities are proved for it. Several examples and graphs are included to demonstrate the validity of modified (m, p, h)-convex functions. The present results extend the H-H inequalities presented in [7, 8, 16], Jensen inequality introduced in [8] and the Fejér inequalities presented in [8]. Numerical examples are provided to present the sustainability of proved inequalities. In future, it is possible to extend the H-H weighted integral inequalities for (h, m)-convex modified functions presented in [3], for the obtained class of convexity.

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