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Iterative computational results for SEIAR worm propagation model using fractal-fractional approach

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Abstract

The current work introduces a new fractal-fractional modeling approach for simulating worm-attacking dynamics in computer networks, all while keeping these motivations in mind.

1. **Problem statement.** A computational and theoretical analysis of SEIR worm propagation model with the use of fractal-fractional operators.
2. **Main results.** The work includes existence, uniqueness, stability, and numerical simulation results. Existence results are analyzed using applications and fixed-point theory. Lagrange's interpolation polynomial forms are used to solve the non-dimensional fractal-fractional model of worm propagation in computer networks numerically.
3. **Implications.** The solution is tested for a particular case using numerical values from readily and freely accessible sources. In comparison to classical and fractional solutions, the FF dynamical systems are more general, as can be seen from the graphical findings. These problems can further be analysed in the stochastic version for further deeper and scientific studies.

Keywords: Computer networks, fractal-fractional derivative, results of existence, solutions of stability, computational simulations.

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1. Introduction

Network security has grown to be a significant issue for the internet throughout the years, along with the quick development of computer network applications and technology for communication ([14]). In particular, since the first known worm, known as Morris, appeared for the first time in 1988, a significant

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amount of computer worms have regularly entered the internet. Computer worms are self-replicating programmes designed to carry out tasks. Without the owners' authorization they can swiftly infect millions of electronic devices (computers, smartphones, etc.), causing significant financial losses and having a significant negative societal impact [7, 12]. Consequently, to reduce the potential threat caused by computer worms, it is essential to examine the laws governing their spread. The widespread use of computer networks has led to an increase in the complexity and remarkable nature of network security issues, such as unauthorized access, malicious worm attacks, and virus dissemination. A worm is a type of malicious code that can propagate automatically over networks, self-replicate, and generally do not require human interaction [40]. The enormous threats to the Internet have brought considerable attention to the problem of controlling malicious worms on the network. Malicious worms are spreading quickly, so it's important to investigate integrated measures that can stop their spread automatically over the Internet. Due to their extensive influence and quick spread, worms cause more harm than other network security threats. As a result, preventing the spread of internet worms in a large-scale network is imperative. With the creation of a method for monitoring network worm propagation, the technologies of detection, recovery, and interdiction have advanced somewhat. Further defenses to get eliminated of the worm and immunize hosts before the worm attack should be developed in order to effectively control the propagation of malicious worms. The security of computer networks is now seriously threatened by computer worms, which demand a lot of resources and take time to recover from aggressive attacks. Mathematical modelling plays a major role in the epidemiological component of computer virus/worm propagation in networks. A mathematical model of propagation may assist both in controlling transmission and understanding the pattern of worm transmission from one node to another. Epidemiology is one of the most important areas of research. Numerous academics have considered researching modelling and analysis of infectious illnesses. The biggest risk to computer network security, functionality, and assets is from worm threats. Computer worms that are launched from the outside and spread maliciously throughout networks are one of the biggest and most dangerous security threats that the business, political, military, and research communities are currently facing [37]. Malware, also referred to as harmful software, such as viruses, worms, Trojan horses, or root kits, presents a significant risk to the computer user community by obtaining unauthorized access to computer resources. Computer security researchers are primarily interested in studying worms because they are among the types of malware that have the potential to infect millions of machines quickly and cause harm worth hundreds of millions of dollars. A number of mathematical models have been proposed to analyze the transmission law and transmission trend of various worms in various real-life networks based on the propagation mechanism. Worm modelling can help us better understand how worms spread and propagate as well as how to prevent and mitigate the effects of worm attacks [17, 29].

In practice, the identification of an outbreak is a necessary first step towards developing a treatment for the unidentified disease. From a network security standpoint, one must first be aware that a worm epidemic is taking place in order to assess the malicious traffic and then develop a suitable remedy. Undiscovered worms are the worst kind because they can do a great deal of damage without the user ever discovering it. To propagate, a worm of this kind will still increase network traffic in the target port even though it won't leave any evidence of its existence on the target hosts. Both biological and computer viruses exhibit remarkably similar behavioural traits. As a result, by introducing factors important to worm/virus propagation, mathematical models of computer worm/virus propagation in computer networks will be built based on the epidemic models. The creation of mathematical models in epidemiology was greatly influenced by historical studies on epidemic models. Many epidemic models, including SIRS model [37], SEIR model [15], SIR-A model [5], SIQR model [9], SEIQR model [39], SEIRS-V model [31], VEIQS model [11], and SIRD model [30] have been used to analyse and describe the role of the internet in the spread of computer worms. This is because of a novel observation that the spread of worms among computers is closely similar to the spread of an infectious disease among a population. This researches the pattern of worm/malicious code transmission and spread in computer networks based on a review of the literature.

This article is organized as follows. Section 2 examined the fractal-fractional order SEIAR worm propagation model, and Section 3 presented some basic preliminary findings. Iterative formulations and limit points were used to analyze the existence criteria and uniqueness in sections 4 and 5. In Section 6, the HU-stability of the solutions is evaluated for the SEIAR model (2.1). Additionally, a scheme is created and put through simulation in Section 7 to determine the various effects of the fractal and fractional orders. This is done with the aid of Lagrange polynomials. The study is concluded with conclusions in the last section.

2. Fractal-fractional SEIAR model

Many real-world issues can be better understood by using fractional calculus and the better memory effect. The generalisation of integer order calculus, known as fractional calculus (FC), was insufficient to explain certain memory effects in certain engineering and real-world problems. The FC has drawn the attention of researchers over the past three decades because of its vast applications in numerous scientific domains. Stated differently, FC describes the past occurrences of various phenomena that we refer to as memory. Different definitions of fractional derivatives were developed by numerous researchers for various physical scenarios. A few FC applications are listed in Tenreiro et al. [33]. Several FC applications in the applied sciences were examined by Dalir and Bashour [8]. Tavazoei et al. [32] investigated the application of FC in contemporary science and the suppression of chaotic oscillations. Corresponding to this, Sabatier et al. [26] covered a few sophisticated uses of FC in contemporary science.

Atangana [3] recently created the novel concept of the fractal-fractional derivative in FC. This novel concept works incredibly well for solving some challenging issues in a variety of contexts. There are two orders in the operator: the fractal dimension is the second operator, and the fractional order is the first. Both the classical and the new concept of the fractal fractional derivative are superior to fractional derivatives. The reason for this is that we can simultaneously study the fractional operator and fractal dimension when working with fractal-fractional derivatives. Spurred by the novel and distinctive characteristics, fractal-fractional operators are attracting the attention of numerous researchers. Fractional calculus is the extension of classical calculus to encompass derivatives and integrals of any order. Additionally, fractional calculus is challenging in several ways both discrete and continuous part. In recent decades, it has become increasingly clear that this mathematical theory provides benefits both in pure mathematics and applications [2, 6, 13, 24, 27–29]. Many researchers discussed the stability of the system for different fractional orders including fractal-fractional order model [1, 4].

A computer worm propagates over a network by sending self-replication and taking advantage of holes in other computer programs, just like a biological virus does. With the use of epidemiological models, numerous researchers are attempting to comprehend the behavior of worm propagation in order to create effective strategies for slowing the worms' rapid spread.

As shown in Figure 1, computer networks are classified into five categories based on known characteristics: susceptible (S), exposed (E), symptomatic infected (I), asymptomatic infected (A), or recovered class (R). Consider about the SEIAR model with FF order as follows:

$$\begin{aligned}
{}_0^{FFM}D_\ell^{\psi,\phi}S(\ell) &= \Pi - \delta S(\ell)I(\ell) - \sigma S(\ell)E(\ell) - \chi S(\ell)A(\ell) - (\epsilon+\nu)S(\ell),\\
{}_0^{FFM}D_\ell^{\psi,\phi}E(\ell) &= \sigma S(\ell)E(\ell) - (\mu+\rho+\nu)E(\ell),\\
{}_0^{FFM}D_\ell^{\psi,\phi}I(\ell) &= \delta S(\ell)I(\ell) - \xi^{\circledast}I(\ell)A(\ell) + \mu E(\ell) - (\beta^{\circledast}+\nu)I(\ell),\\
{}_0^{FFM}D_\ell^{\psi,\phi}A(\ell) &= \xi^{\circledast}I(\ell)A(\ell) + \chi S(\ell)A(\ell) - (\gamma+\nu)A(\ell),\\
{}_0^{FFM}D_\ell^{\psi,\phi}R(\ell) &= \epsilon S(\ell) + \rho E(\ell) + \beta^{\circledast}I(\ell) + \gamma A(\ell) - \nu R(\ell),
\end{aligned} \tag{2.1}$$

with $S(0) \geqslant 0, E(0) \geqslant 0, I(0) \geqslant 0, A(0) \geqslant 0, R(0) \geqslant 0$. Here, ${}_0^{FFM}D_\ell^{\psi,\phi}$ stands for the fractal-fractional derivatives for $\psi, \phi \in (0, 1]$. For the model (2.1), the total population is categorized in five nodes. The

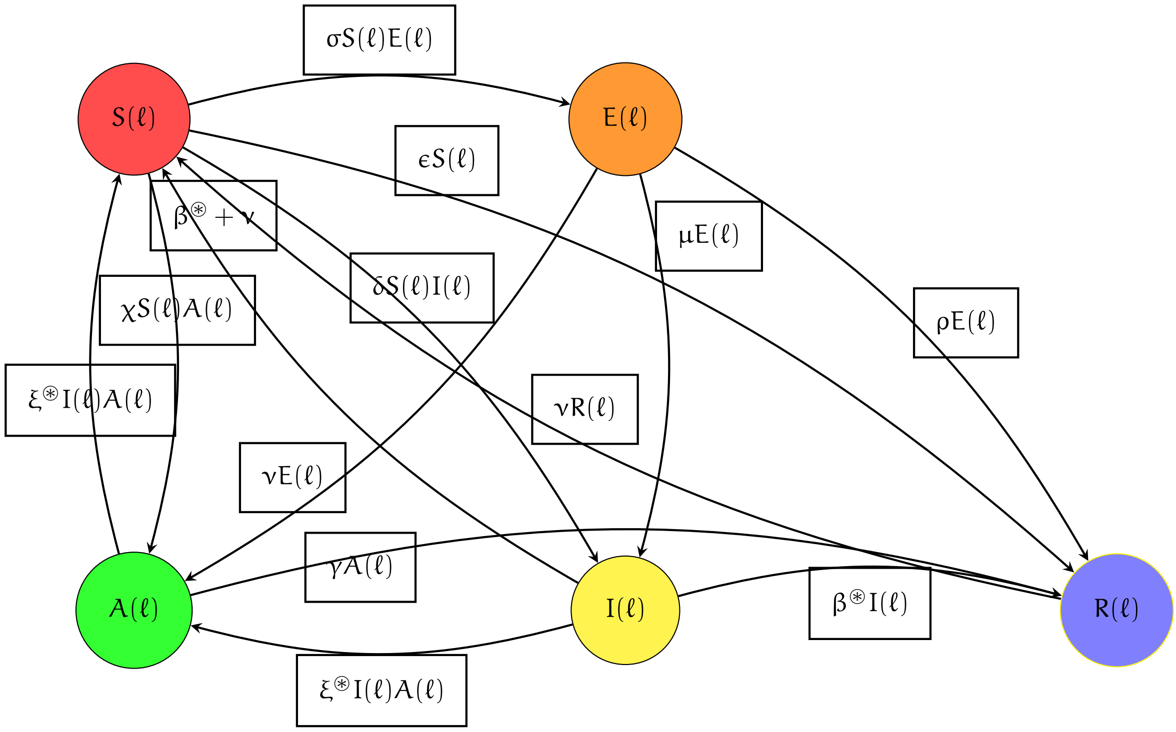

Figure 1: Flowchart of the SEIAR model with FF derivatives.

nodes $S(\ell)$, $E(\ell)$, $I(\ell)$, $A(\ell)$, $R(\ell)$ are the number of susceptible, exposed, symptomatic infected, asymptomatic infected, and recovered with time ℓ. Here, $\Pi, \delta, \sigma, \chi, \epsilon, \nu, \mu, \rho, \xi^{\circledast}, \gamma$ are all non-negative parameters, where σ is the rate of transmission of susceptible to exposed class, ρ is the rate of transmission from the exposed to the recovered, ϵ takes the susceptible to the recovered class, χ represents the conversion of susceptible to the asymptomatic infected class, μ is the rate of transmission from the exposed to the infected people, δ is the rate at which susceptible are transferred to the infected category, $\beta^{\circledast}$ is the recovery of the infected individuals, and $\xi^{\circledast}$ takes the infected to the asymptomatic class. This transmission is well expressed in the flowchart in Figure 1.

Significance and novelty

This paper introduces a new FF-model for simulating worm propagation in computer system. By using FF derivatives, the analysis provides a more significant representation of intricate network dynamics, especially those influenced by memory effects. The novel aspect of this work lies in combination of these advanced mathematical techniques with the SEIAR model, offering a sophisticated framework for analyzing network security dynamics. This approach generalizes classical approach of modeling and simulations by providing more general and realistic results.

The significance of this work is in its potential to enhance cyber-security models, particularly for worm propagation in computer networks. By incorporating FF-derivatives, the study captures the complex, memory-dependent behavior of real-world systems more effectively than existing dynamical systems. Additionally, the use of fixed-point theory ensures rigorous mathematical techniques for existence and uniqueness, while stability analysis using HU stability improves the robustness of the model (2.1). The approach provides a new way to simulate network vulnerabilities, ultimately aiding in more effective preventive protocols.

3. Preliminaries

Fractal-fractional models have revolutionized various scientific disciplines by providing a unique approach to understanding complex phenomena. Unlike traditional integer-order calculus, which operates solely on whole numbers, fractal-fractional calculus extends mathematical operations to non-integer orders. This extension enables the characterization of irregular, self-similar patterns found in natural landscapes, financial markets, and biological processes. Fractal-fractional models help in capturing phenomena at multiple scales, providing a more significant representation of data compared to classical models. By incorporating fractional derivatives, these models can effectively capture long-range dependencies and memory effects in dynamical systems. This multiscale approach enhances the accuracy and predictive capabilities of models, facilitating better decision-making in various applications. Overall, fractal-fractional models represent a significant advancement in mathematical modeling, offering enhanced accuracy, scalability, and interpretability compared to traditional approaches. Their versatility and applicability make them invaluable tools for understanding complex systems and driving innovation across diverse fields. For detail, the readers are referred to [16, 18–23, 38].

Let be assume the space $\{\Theta(\ell) \in \mathbb{C}([0,1]$ this implies $w\mathbb{R})\}$ with $\|\Theta\| = \max_{\ell\in[0,1]} |\Theta(\ell)|$, for the mathematical analysis of the model (2.1). Consider form the literature [4, 20] for the basic results.

Definition 3.1. Assume that $\Theta \in \mathbb{C}((a,b),\mathbb{R})$ is a fractionally differentiable fractal function of order $0 < \phi \leqslant 1$ for the interval (a,b). Then for $0 < \psi \leqslant 1$,

$$ {}_{0}^{FFM}D_{\ell}^{\psi,\phi}\Theta(\ell) = \frac{MN(\psi)}{1-\psi}\frac{d}{d\ell^{\phi}}\int_{0}^{\ell}\Theta(\eta^{\circledast})T_{\psi}\left[-\frac{\psi}{1-\psi}\left(\ell-\eta^{\circledast}\right)^{\psi}\right]d\eta^{\circledast}, $$

where $MN(\psi) = 1-\psi+\frac{\psi}{\Gamma(\psi)}$ and $\frac{d\Theta(\eta^{\circledast})}{d{\eta^{\circledast}}^{\phi}} = \lim_{\ell \text{ this implies } w\eta^{\circledast}} \frac{\Theta(\ell)-\Theta(\eta^{\circledast})}{\ell^{\phi}-{\eta^{\circledast}}^{\phi}}$.

Definition 3.2. For the same Θ, the Fractal-fractional integral of order $0 < \psi \leqslant 1$ is given by

$$ {}_{0}^{FFM}I_{\ell}^{\psi,\phi}\Theta(\ell) = \frac{\psi\phi}{MN(\psi)\Gamma(\psi)}\int_{0}^{\ell}{\eta^{\circledast}}^{\phi-1}\Theta(\eta^{\circledast})(\ell-\eta^{\circledast})^{\psi-1}d\eta^{\circledast} + \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)}\Theta(\ell), $$

where $MN(\psi) = 1-\psi+\frac{\psi}{\Gamma(\psi)}$.

The proof that a solution to the FF model (2.1) exists was carried out using successive iteration and derived by Definition 2.2 in [4]. One can follow the results for more details in the works [34–36]. Hence,

$$ \begin{aligned} S(\ell)-S(0) &= \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)}\left[\Pi-\delta S(\ell)I(\ell)-\sigma S(\ell)E(\ell)-\chi S(\ell)A(\ell)-(\epsilon+\nu)S(\ell)\right] \\ &\quad + \frac{\psi\phi}{MN(\psi)\Gamma(\psi)}\int_{0}^{\ell}{\eta^{\circledast}}^{\phi-1}(\ell-\eta^{\circledast})^{\psi-1}\left[\Pi-\left(\delta I(\eta^{\circledast})+\sigma E(\eta^{\circledast})+\chi A(\eta^{\circledast})\right)S(\eta^{\circledast})-(\epsilon+\nu)S(\eta^{\circledast})\right]d\eta^{\circledast}, \\ E(\ell)-E(0) &= \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)}\left[\sigma S(\ell)E(\ell)-(\mu+\rho+\nu)E(\ell)\right] \\ &\quad + \frac{\psi\phi}{MN(\psi)\Gamma(\psi)}\int_{0}^{\ell}{\eta^{\circledast}}^{\phi-1}(\ell-\eta^{\circledast})^{\psi-1}\left[\sigma S(\eta^{\circledast})E(\eta^{\circledast})-(\mu+\rho+\nu)E(\eta^{\circledast})\right]d\eta^{\circledast}, \\ I(\ell)-I(0) &= \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)}\left[\delta S(\ell)I(\ell)-\xi^{\circledast}I(\ell)A(\ell)+\mu E(\ell)-(\beta^{\circledast}+\nu)I(\ell)\right] \end{aligned} $$

$$+\frac{\psi\phi}{MN(\psi)\Gamma(\psi)}\int_0^{\ell}\eta^{\circledast\,\phi-1}(\ell-\eta^{\circledast})^{\psi-1}\left[\delta S(\eta^{\circledast})I(\eta^{\circledast})-\xi^{\circledast}I(\eta^{\circledast})A(\eta^{\circledast})+\mu E(\eta^{\circledast})-(\beta^{\circledast}+\nu)I(\eta^{\circledast})\right]d\eta^{\circledast},$$

$$A(\ell)-A(0)=\frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)}\left[\xi^{\circledast}I(\ell)A(\ell)+\chi S(\ell)A(\ell)-(\gamma+\nu)A(\ell)\right]$$
$$+\frac{\psi\phi}{MN(\psi)\Gamma(\psi)}\int_0^{\ell}\eta^{\circledast\,\phi-1}(\ell-\eta^{\circledast})^{\psi-1}\left[\xi^{\circledast}I(\eta^{\circledast})A(\eta^{\circledast})+\chi S(\eta^{\circledast})A(\eta^{\circledast})-(\gamma+\nu)A(\eta^{\circledast})\right]d\eta^{\circledast},$$

$$R(\ell)-R(0)=\frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)}\left[\epsilon S(\ell)+\rho E(\ell)+\beta^{\circledast}I(\ell)+\gamma A(\ell)-\nu R(\ell)\right]$$
$$+\frac{\psi\phi}{MN(\psi)\Gamma(\psi)}\int_0^{\ell}\eta^{\circledast\,\phi-1}(\ell-\eta^{\circledast})^{\psi-1}\left[\epsilon S(\eta^{\circledast})+\rho E(\eta^{\circledast})+\beta^{\circledast}I(\eta^{\circledast})+\gamma A(\eta^{\circledast})-\nu R(\eta^{\circledast})\right]d\eta^{\circledast}.$$

Regarding the functions $\Lambda_k^{\circledast}$ for $k=1,2,3,4,5$ or $k\in\mathbb{N}_1^5$, gives

$$\begin{aligned}\Lambda_1^{\circledast}(\ell,S)&=\Pi-\delta S(\ell)I(\ell)-\sigma S(\ell)E(\ell)-\chi S(\ell)A(\ell)-(\epsilon+\nu)S(\ell),\\ \Lambda_2^{\circledast}(\ell,E)&=\sigma S(\ell)E(\ell)-(\mu+\rho+\nu)E(\ell),\\ \Lambda_3^{\circledast}(\ell,I)&=\delta S(\ell)I(\ell)-\xi^{\circledast}I(\ell)A(\ell)+\mu E(\ell)-(\beta^{\circledast}+\nu)I(\ell),\\ \Lambda_4^{\circledast}(\ell,A)&=\xi^{\circledast}I(\ell)A(\ell)+\chi S(\ell)A(\ell)-(\gamma+\nu)A(\ell),\\ \Lambda_5^{\circledast}(\ell,R)&=\epsilon S(\ell)+\rho E(\ell)+\beta^{\circledast}I(\ell)+\gamma A(\ell)-\nu R(\ell).\end{aligned}\tag{3.1}$$

4. Existence criteria

In the present part we provide an assumption which is needed later.

(C*) All $S(\ell),S^*(\ell),E(\ell),E^*(\ell),I(\ell),I^*(\ell),A(\ell),A^*(\ell),R(\ell),R^*(\ell)\in L[0,1]$ are continuous so that $\|S\|\leqslant\pi_1,\|E\|\leqslant\pi_2,\|I\|\leqslant\pi_3,\|A\|\leqslant\pi_4$, for some positive constants $\pi_1,\pi_2,\pi_3,\pi_4>0$.

Theorem 4.1. *The kernels $\Lambda_k^{\circledast}$ fulfill LC under the assumption C^* and $\Omega_k<1$, for $k\in\mathbb{N}_1^5$.*

Proof. Let be first check $\Lambda_1^{\circledast}(\ell,S)$. Considering S and S^* yields

$$\begin{aligned}\left\|\Lambda_1^{\circledast}(S)-\Lambda_1^{\circledast}(S^*)\right\|&=\|\Pi-\sigma SE-\delta SI-\chi SA-(\epsilon+\nu)S-(\Pi-\sigma S^*E-\delta S^*I-\chi S^*A-(\epsilon+\nu)S^*)\|\\&\leqslant[\sigma\|E\|+\delta\|I\|+\chi\|A\|+\epsilon+\nu]\|S-S^*\|\\&\leqslant[\sigma\pi_2+\delta\pi_3+\chi\pi_4+\epsilon+\nu]\|S-S^*\|\leqslant\Omega_1\|S-S^*\|,\end{aligned}\tag{4.1}$$

where $\Omega_1=[\sigma\pi_2+\delta\pi_3+\chi\pi_4+\epsilon+\nu]$. Hence, $\Lambda_1^{\circledast}$ satisfies LC with Lipschitz-constant Ω_1. Next we check $\Lambda_2^{\circledast}(\ell,E)$. Considering E and E^* yields

$$\begin{aligned}\left\|\Lambda_2^{\circledast}(E)-\Lambda_2^{\circledast}(E^*)\right\|&=\|\sigma SE-(\mu+\rho+\nu)E-(\sigma SE^*-(\mu+\rho+\nu)E^*)\|\\&\leqslant[\sigma\|S\|+\mu+\rho+\nu]\|E-E^*\|\leqslant[\sigma\pi_1+\mu+\rho+\nu]\|E-E^*\|\leqslant\Omega_2\|E-E^*\|,\end{aligned}\tag{4.2}$$

where $\Omega_2=[\sigma\pi_1+\mu+\rho+\nu]$. Hence, $\Lambda_2^{\circledast}$ satisfies LC with Lipschitz-constant Ω_2. We also check $\Lambda_3^{\circledast}(\ell,I)$. Considering I and I^* yields

$$\begin{aligned}\left\|\Lambda_3^{\circledast}(I)-\Lambda_3^{\circledast}(I^*)\right\|&=\left\|\delta SI-\xi^{\circledast}IA+\mu E-(\beta^{\circledast}+\nu)I-(\delta SI^*-\xi^{\circledast}I^*A+\mu E-(\beta^{\circledast}+\nu)I^*)\right\|\\&\leqslant\left[\delta\|S\|+\xi^{\circledast}\|A\|+\beta^{\circledast}+\nu\right]\|I-I^*\|\\&\leqslant\left[\delta\pi_1+\xi^{\circledast}\pi_4+\beta^{\circledast}+\nu\right]\|I-I^*\|\leqslant\Omega_3\|I-I^*\|,\end{aligned}\tag{4.3}$$

where $\Omega_3=\left[\delta\pi_1+\xi^{\circledast}\pi_4+\beta^{\circledast}+\nu\right]$. Hence, $\Lambda_3^{\circledast}$ satisfies LC with Lipschitz-constant Ω_3. Similarly we check $\Lambda_4^{\circledast}(\ell,A)$. Considering A and A^* yields

$$\begin{aligned}\left\|\Lambda_4^{\circledast}(A)-\Lambda_4^{\circledast}(A^*)\right\|&=\left\|\xi^{\circledast}IA+\chi SA-(\gamma+\nu)A-(\xi^{\circledast}IA^*+\chi SA^*-(\gamma+\nu)A^*)\right\|\\&\leqslant\left[\xi^{\circledast}\|I\|+\chi\|S\|+\gamma+\nu\right]\|A-A^*\|\leqslant\left[\xi^{\circledast}\pi_3+\chi\pi_1+\gamma+\nu\right]\|A-A^*\|\leqslant\Omega_4\|A-A^*\|,\end{aligned}\tag{4.4}$$

where $\Omega_4 = \left[\xi^{\circledast}\pi_3 + \chi\pi_1 + \gamma + \nu\right]$. Hence, $\Lambda_4^{\circledast}$ satisfies LC with Lipschitz-constant Ω_4. Finally we check $\Lambda_5^{\circledast}(\ell, R)$. Considering R and R^* yields

$$\begin{aligned}\left\|\Lambda_5^{\circledast}(R) - \Lambda_5^{\circledast}(R^*)\right\| &= \left\|\epsilon S + \rho E + \beta^{\circledast} I + \gamma A - \nu R - (\epsilon S + \rho E + \beta^{\circledast} I + \gamma A - \nu R^*)\right\| \\ &\leqslant \nu \left\|R - R^*\right\| \leqslant \Omega_5 \left\|R - R^*\right\|,\end{aligned} \tag{4.5}$$

where $\Omega_5 = \nu$. Hence, $\Lambda_5^{\circledast}$ satisfies LC with Lipschitz-constant Ω_5. Thus, from (4.1)-(4.5), $\Lambda_k^{\circledast}$ for $k = 1, 2, 3, 4, 5$, satisfy the Lipschitz property and the result is accomplished. □

Assume

$$\begin{aligned}
S(\ell) - S(0) &= \frac{\psi\phi}{MN(\psi)\Gamma(\psi)} \int_0^{\ell} \eta^{\circledast\phi-1}(\ell - \eta^{\circledast})^{\psi-1}\Lambda_1^{\circledast}\left(\eta^{\circledast}, S(\eta^{\circledast})\right) d\eta^{\circledast} + \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)}\Lambda_1^{\circledast}\left(\ell, S(\ell)\right),\\
E(\ell) - E(0) &= \frac{\psi\phi}{MN(\psi)\Gamma(\psi)} \int_0^{\ell} \eta^{\circledast\phi-1}(\ell - \eta^{\circledast})^{\psi-1}\Lambda_2^{\circledast}\left(\eta^{\circledast}, E(\eta^{\circledast})\right) d\eta^{\circledast} + \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)}\Lambda_2^{\circledast}\left(\ell, E(\ell)\right),\\
I(\ell) - I(0) &= \frac{\psi\phi}{MN(\psi)\Gamma(\psi)} \int_0^{\ell} \eta^{\circledast\phi-1}(\ell - \eta^{\circledast})^{\psi-1}\Lambda_3^{\circledast}\left(\eta^{\circledast}, I(\eta^{\circledast})\right) d\eta^{\circledast} + \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)}\Lambda_3^{\circledast}\left(\ell, I(\ell)\right),\\
A(\ell) - A(0) &= \frac{\psi\phi}{MN(\psi)\Gamma(\psi)} \int_0^{\ell} \eta^{\circledast\phi-1}(\ell - \eta^{\circledast})^{\psi-1}\Lambda_4^{\circledast}\left(\eta^{\circledast}, A(\eta^{\circledast})\right) d\eta^{\circledast} + \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)}\Lambda_4^{\circledast}\left(\ell, A(\ell)\right),\\
R(\ell) - R(0) &= \frac{\psi\phi}{MN(\psi)\Gamma(\psi)} \int_0^{\ell} \eta^{\circledast\phi-1}(\ell - \eta^{\circledast})^{\psi-1}\Lambda_5^{\circledast}\left(\eta^{\circledast}, R(\eta^{\circledast})\right) d\eta^{\circledast} + \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)}\Lambda_5^{\circledast}\left(\ell, R(\ell)\right).
\end{aligned}$$

Theorem 4.2. *There is at least a solution of the fractal-fractional SEIAR model (2.1) if* $X = \max\left[\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5\right] < 1$.

Proof. Consider

$$\begin{aligned}
&{\xi^{\circledast}}_{1_\tau}(\ell) = S_{\tau+1} - S(\ell), && {\xi^{\circledast}}_{2_\tau}(\ell) = E_{\tau+1} - E(\ell), && {\xi^{\circledast}}_{3_\tau}(\ell) = I_{\tau+1} - I(\ell),\\
&{\xi^{\circledast}}_{4_\tau}(\ell) = A_{\tau+1} - A(\ell), && {\xi^{\circledast}}_{5_\tau}(\ell) = R_{\tau+1} - R(\ell).
\end{aligned}$$

Then,

$$\begin{aligned}
\left\|{\xi^{\circledast}}_{1_\tau}(\ell)\right\| &= \frac{\psi\phi}{MN(\psi)\Gamma(\psi)} \int_0^{\ell} \left(\ell - \eta^{\circledast}\right)^{\psi-1} \eta^{\circledast\phi-1} \left\|\Lambda_1^{\circledast}\left(\eta^{\circledast}, S_\tau(\eta^{\circledast})\right) - \Lambda_1^{\circledast}\left(\eta^{\circledast}, S(\eta^{\circledast})\right)\right\| d\eta^{\circledast}\\
&\quad + \frac{\phi(1-\psi)}{MN(\psi)}\ell^{\phi-1}\left\|\Lambda_1^{\circledast}\left(\ell, S_\tau(\ell)\right) - \Lambda_1^{\circledast}\left(\ell, S(\ell)\right)\right\|\\
&\leqslant \frac{\psi\phi}{MN(\psi)\Gamma(\psi)} \int_0^{\ell} \left(\ell - \eta^{\circledast}\right)^{\psi-1} \eta^{\circledast\phi-1}\Omega_1 \left\|S_\tau - S\right\| d\eta^{\circledast} + \frac{\phi(1-\psi)}{MN(\psi)}\ell^{\phi-1}\Omega_1\left\|S_\tau - S\right\|\\
&\leqslant \left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)} + \frac{\phi(1-\psi)}{MN(\psi)}\right]\Omega_1\left\|S_\tau - S\right\| \leqslant \left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)} + \frac{\phi(1-\psi)}{MN(\psi)}\right]^{\tau}\Omega_1^{\tau}\left\|S_1 - S\right\|,
\end{aligned}$$

in which for $\Omega_1 < 1$ and as τ this implies $w\infty$, then S_τ this implies wS. Similarly

$$\left\|{\xi^{\circledast}}_{2_\tau}(\ell)\right\| \leqslant \left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)} + \frac{\phi(1-\psi)}{MN(\psi)}\right]^{\tau}\Omega_2^{\tau}\left\|E_1 - E\right\|, \tag{4.6}$$

$$\left\|{\xi^{\circledast}}_{3_\tau}(\ell)\right\| \leqslant \left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)} + \frac{\phi(1-\psi)}{MN(\psi)}\right]^{\tau}\Omega_3^{\tau}\left\|I_1 - I\right\|, \tag{4.7}$$

$$\left\|\xi^{\circledast}{}_{4_\tau}(\ell)\right\| \leqslant \left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)} + \frac{\phi(1-\psi)}{MN(\psi)}\right]^{\tau} \Omega_4^{\tau} \left\|A_1 - A\right\|, \tag{4.8}$$

$$\left\|\xi^{\circledast}{}_{5_\tau}(\ell)\right\| \leqslant \left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)} + \frac{\phi(1-\psi)}{MN(\psi)}\right]^{\tau} \Omega_5^{\tau} \left\|R_1 - R\right\|, \tag{4.9}$$

By (4.6)-(4.9), when τ this implies $w\infty$, then $\xi^{\circledast}{}_{j_\tau}$ this implies $w0, j \in \mathbb{N}_2^5$, for $\Lambda_j^{\circledast} < 1, j = 2,3,4,5$. Ultimately, the SEIAR model (2.1) has a solution. □

5. Unique solution

In this section, the solutions of uniqueness are investigated for the model (2.1).

Theorem 5.1. *The fractal-fractional SEIR-model* (2.1) *possesses one solution exactly if* (C^*) *is satisfied and the following holds:*

$$\left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)} + \frac{\phi(1-\psi)}{MN(\psi)}\right] \Omega_k \leqslant 1, \quad k \in \mathbb{N}_1^5.$$

Proof. Suppose that the result of the theorem is not valid. Hence, another solution exists for the SEIR model (2.1). So, we presume $\tilde{S}(\ell), \tilde{E}(\ell), \tilde{I}(\ell), \tilde{A}(\ell), \tilde{R}(\ell)$ as another solution with $\tilde{S}(0), \tilde{E}(0), \tilde{I}(0), \tilde{A}(0), \tilde{R}(0)$ the initial values, such that

$$\tilde{S}(\ell) - S(0) = \frac{\psi\phi}{MN(\psi)\Gamma(\psi)} \int_0^{\ell} \eta^{\circledast\,\phi-1}(\ell-\eta^{\circledast})^{\psi-1} \Lambda_1^{\circledast}\left(\eta^{\circledast}, \tilde{S}(\eta^{\circledast})\right) d\eta^{\circledast} + \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)} \Lambda_1^{\circledast}\left(\ell, \tilde{S}(\ell)\right),$$

$$\tilde{E}(\ell) - E(0) = \frac{\psi\phi}{MN(\psi)\Gamma(\psi)} \int_0^{\ell} \eta^{\circledast\,\phi-1}(\ell-\eta^{\circledast})^{\psi-1} \Lambda_2^{\circledast}\left(\eta^{\circledast}, \tilde{E}(\eta^{\circledast})\right) d\eta^{\circledast} + \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)} \Lambda_2^{\circledast}\left(\ell, \tilde{E}(\ell)\right),$$

$$\tilde{I}(\ell) - I(0) = \frac{\psi\phi}{MN(\psi)\Gamma(\psi)} \int_0^{\ell} \eta^{\circledast\,\phi-1}(\ell-\eta^{\circledast})^{\psi-1} \Lambda_3^{\circledast}\left(\eta^{\circledast}, \tilde{I}(\eta^{\circledast})\right) d\eta^{\circledast} + \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)} \Lambda_3^{\circledast}\left(\ell, \tilde{I}(\ell)\right),$$

$$\tilde{A}(\ell) - A(0) = \frac{\psi\phi}{MN(\psi)\Gamma(\psi)} \int_0^{\ell} \eta^{\circledast\,\phi-1}(\ell-\eta^{\circledast})^{\psi-1} \Lambda_4^{\circledast}\left(\eta^{\circledast}, \tilde{A}(\eta^{\circledast})\right) d\eta^{\circledast} + \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)} \Lambda_4^{\circledast}\left(\ell, \tilde{A}(\ell)\right),$$

$$\tilde{R}(\ell) - R(0) = \frac{\psi\phi}{MN(\psi)\Gamma(\psi)} \int_0^{\ell} \eta^{\circledast\,\phi-1}(\ell-\eta^{\circledast})^{\psi-1} \Lambda_5^{\circledast}\left(\eta^{\circledast}, \tilde{R}(\eta^{\circledast})\right) d\eta^{\circledast} + \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)} \Lambda_5^{\circledast}\left(\ell, \tilde{R}(\ell)\right).$$

Now

$$\begin{aligned}
\left\|S - \tilde{S}\right\| &= \frac{\psi\phi}{MN(\psi)\Gamma(\psi)} \int_0^{\ell} \eta^{\circledast\,\phi-1}(\ell-\eta^{\circledast})^{\psi-1} \left\|\Lambda_1^{\circledast}\left(S_1(\eta^{\circledast})\right) - \Lambda_1^{\circledast}\left(\tilde{S}(\eta^{\circledast})\right)\right\| d\eta^{\circledast} + \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)} \left\|\Lambda_1^{\circledast}(S) - \Lambda_1^{\circledast}(\tilde{S})\right\|, \\
&\leqslant \frac{\psi\phi}{MN(\psi)\Gamma(\psi)} \int_0^{\ell} \eta^{\circledast\,\phi-1}(\ell-\eta^{\circledast})^{\psi-1} \Omega_1 \left\|S - \tilde{S}\right\| d\eta^{\circledast} + \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)} \Omega_1 \left\|S - \tilde{S}\right\|, \\
&\leqslant \left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)} + \frac{\phi(1-\psi)}{MN(\psi)}\right] \Omega_1 \left\|S - \tilde{S}\right\|.
\end{aligned}$$

Hence

$$\left[1 - \left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)} + \frac{\phi(1-\psi)}{MN(\psi)}\right] \Omega_1\right] \left\|S - \tilde{S}\right\| \leqslant 0. \tag{5.1}$$

Clearly, (5.1) is justified for $\left\|S - \tilde{S}\right\| = 0$, which implies that $S = \tilde{S}$. Similarly,

$$\left\|E - \tilde{E}\right\| \leqslant \left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)} + \frac{\phi(1-\psi)}{MN(\psi)}\right] \Omega_2 \left\|E - \tilde{E}\right\|, \qquad \left\|I - \tilde{I}\right\| \leqslant \left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)} + \frac{\phi(1-\psi)}{MN(\psi)}\right] \Omega_3 \left\|I - \tilde{I}\right\|,$$

$$\|A-\tilde{A}\| \leqslant \left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)}+\frac{\phi(1-\psi)}{MN(\psi)}\right]\Omega_4\|A-\tilde{A}\|, \quad \|R-\tilde{R}\| \leqslant \left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)}+\frac{\phi(1-\psi)}{MN(\psi)}\right]\Omega_5\|R-\tilde{R}\|.$$

This implies that,

$$\left[1-\left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)}+\frac{\phi(1-\psi)}{MN(\psi)}\right]\Omega_2\right]\|E-\tilde{E}\| \leqslant 0, \quad \left[1-\left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)}+\frac{\phi(1-\psi)}{MN(\psi)}\right]\Omega_3\right]\|I-\tilde{I}\| \leqslant 0,$$
$$\left[1-\left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)}+\frac{\phi(1-\psi)}{MN(\psi)}\right]\Omega_4\right]\|A-\tilde{A}\| \leqslant 0, \quad \left[1-\left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)}+\frac{\phi(1-\psi)}{MN(\psi)}\right]\Omega_5\right]\|R-\tilde{R}\| \leqslant 0.$$

Thus

$$\|E-\tilde{E}\| = 0, \text{ this implies } w \quad E=\tilde{E}, \qquad \|I-\tilde{I}\| = 0, \text{ this implies } w \quad I=\tilde{I},$$
$$\|A-\tilde{A}\| = 0, \text{ this implies } w \quad A=\tilde{A}, \qquad \|R-\tilde{R}\| = 0, \text{ this implies } w \quad R=\tilde{R}.$$

Hence, the SEIAR model (2.1) contains a unique solution. □

6. HU-stability

This section analyzes the solution of SEIAR model (2.1) through the definition of HU-stability.

Definition 6.1. The fractal-fractional SEIAR model (2.1) is termed as HU-stable if $\exists\delta_k, k \in \mathbb{N}_1^5$ provided that $\forall\upsilon \in \mathbb{N}_1^5$ and for each $(S^*, E^*, I^*, A^*, R^*)$ satisfying

$$\left|{}_0^{FFM}D_\ell^{\psi,\phi}S^*(\ell)-\Lambda_1^{\circledast}(\ell,S^*)\right| \leqslant \upsilon_1, \quad \left|{}_0^{FFM}D_\ell^{\psi,\phi}E^*(\ell)-\Lambda_1^{\circledast}(\ell,E^*)\right| \leqslant \upsilon_2, \quad \left|{}_0^{FFM}D_\ell^{\psi,\phi}I^*(\ell)-\Lambda_1^{\circledast}(\ell,I^*)\right| \leqslant \upsilon_3,$$
$$\left|{}_0^{FFM}D_\ell^{\psi,\phi}A^*(\ell)-\Lambda_1^{\circledast}(\ell,A^*)\right| \leqslant \upsilon_4, \quad \left|{}_0^{FFM}D_\ell^{\psi,\phi}R^*(\ell)-\Lambda_1^{\circledast}(\ell,R^*)\right| \leqslant \upsilon_5, \tag{6.1}$$

there exists (S, E, I, A, R) satisfying the model (2.1) and furthermore

$$\|S-S^*\| \leqslant \kappa_1\upsilon_1, \quad \|E-E^*\| \leqslant \kappa_1\upsilon_2, \quad \|I-I^*\| \leqslant \kappa_1\upsilon_3, \quad \|A-A^*\| \leqslant \kappa_1\upsilon_4,, \quad \|R-R^*\| \leqslant \kappa_1\upsilon_5,$$

where $\Lambda_k^{\circledast}, k \in \mathbb{N}_1^5$ are presented in (3.1).

Remark 6.2. Consider that the function S^* is a solution of the first inequality (6.1) iff a continuous map m_1 exists so that

(a) $|m_1(\ell)| < \upsilon_1$; and
(b) ${}_0^{FFM}D_\ell^{\psi,\phi}S^*(\ell) = \Lambda_1^{\circledast}(\ell,S^*)+m_1(\ell)$; similarly, for other nodes of the model (2.1) for some $m_k, k \in \mathbb{N}_2^5$.

Theorem 6.3. *Assume that* (A^*) *holds. Then the fractal-fractional model* (2.1) *is HU-stable if*

$$\left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)}+\frac{\phi(1-\psi)}{MN(\psi)}\right]\Omega_k \leqslant 1, \quad k \in \mathbb{N}_1^5.$$

Proof. Let $\upsilon_1 > 0$ and the function S^* be arbitrary such that

$$\left|{}_0^{FFM}D_\ell^{\psi,\phi}S^*(\ell)-\Lambda_1^{\circledast}(\ell,S^*)\right| \leqslant \upsilon_1.$$

From Remark 6.2, a function like m_1 with $|m_1(\ell)| < \upsilon_1$, satisfies

$${}_0^{FFM}D_\ell^{\psi,\phi}S^*(\ell) = \Lambda_1^{\circledast}(\ell,S^*)+m_1(\ell).$$

Accordingly,

$$S^*(\ell) = S(0)+\frac{\psi\phi}{MN(\psi)\Gamma(\psi)}\int_0^\ell \eta^{\circledast\,\phi-1}(\ell-\eta^{\circledast})^{\psi-1}\Lambda_1^{\circledast}(\eta^{\circledast},S^*(\eta^{\circledast}))\,d\eta^{\circledast}$$
$$+\frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)}\Lambda_1^{\circledast}(\ell,S^*(\ell))+\frac{\psi\phi}{MN(\psi)\Gamma(\psi)}\int_0^\ell \eta^{\circledast\,\phi-1}(\ell-\eta^{\circledast})^{\psi-1}m_1(\eta^{\circledast})d\eta^{\circledast}+\frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)}m_1(\ell).$$

Let S be the Unique solution of model (2.1). Then,

$$S(\ell) = S(0) + \frac{\psi\phi}{MN(\psi)\Gamma(\psi)} \int_0^\ell \eta^{\circledast\,\phi-1}(\ell-\eta^\circledast)^{\psi-1}\Lambda_1^\circledast\left(\eta^\circledast, S(\eta^\circledast)\right) d\eta^\circledast + \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)}\Lambda_1^\circledast\left(\ell, S(\ell)\right).$$

Hence

$$\begin{aligned}
|S^*(\ell) - S(\ell)| \leqslant{}& \frac{\psi\phi}{MN(\psi)\Gamma(\psi)} \int_0^\ell \eta^{\circledast\,\phi-1}(\ell-\eta^\circledast)^{\psi-1}\left|\Lambda_1^\circledast\left(\eta^\circledast, S^*(\eta^\circledast)\right) - \Lambda_1^\circledast\left(\eta^\circledast, S(\eta^\circledast)\right)\right| d\eta^\circledast \\
&+ \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)}\left|\Lambda_1^\circledast\left(\ell, S^*(\ell)\right) - \Lambda_1^\circledast\left(\ell, S(\ell)\right)\right| \\
&+ \frac{\psi\phi}{MN(\psi)\Gamma(\psi)} \int_0^\ell \eta^{\circledast\,\phi-1}(\ell-\eta^\circledast)^{\psi-1}\left|m_1(\eta^\circledast)\right| d\eta^\circledast + \frac{\phi(1-\psi)\ell^{\phi-1}}{MN(\psi)}|m_1(\ell)| \\
\leqslant{}& \left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)} + \frac{\phi(1-\psi)}{MN(\psi)}\right]\Omega_1 |S^*(\ell) - S(\ell)| + \left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)} + \frac{\phi(1-\psi)}{MN(\psi)}\right]\upsilon_1,
\end{aligned}$$

which implies that,

$$\|S^* - S\| \leqslant \frac{\left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)} + \frac{\phi(1-\psi)}{MN(\psi)}\right]\upsilon_1}{1 - \left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)} + \frac{\phi(1-\psi)}{MN(\psi)}\right]\Omega_1}.$$

If $\kappa_1 = \dfrac{\left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)} + \frac{\phi(1-\psi)}{MN(\psi)}\right]}{1 - \left[\frac{\psi\phi\Gamma(\phi)}{MN(\psi)\Gamma(\psi+\phi)} + \frac{\phi(1-\psi)}{MN(\psi)}\right]\Omega_1}$, then $\|S^* - S\| \leqslant \kappa_1\upsilon_1$. Similarly,

$$\|E^* - E\| \leqslant \kappa_1\upsilon_2, \quad \|I^* - I\| \leqslant \kappa_1\upsilon_3, \quad \|A^* - A\| \leqslant \kappa_1\upsilon_4, \quad \|R^* - R\| \leqslant \kappa_1\upsilon_5.$$

Thus, system (2.1) is HU-stable. □

7. Numerical simulation

In this section, Lagrange interpolation polynomials method approached to developed the numerical simulation for the model (2.1) is developed. Consider the linear general differential equation ${}_0^{FFM}D_\ell^{\psi,\phi}\mathbb{Z}(\ell) = \Lambda^\circledast(\ell, \mathbb{Z}(\ell))$. Hence, the Atangana-Baleanu fractal-fractional derivative as ${}_0^{AB}D_\ell^{\psi,\phi-1}\mathbb{Z}(\ell) = \phi\ell^{\phi-1}\Lambda^\circledast(\ell, \mathbb{Z}(\ell)) = \mathbb{T}(\ell, \mathbb{Z}(\ell))$ is obtained. Then, it is clear that

$$\mathbb{Z}(\ell) = \mathbb{Z}(0) + \frac{1-\psi}{MN(\psi)}\mathbb{T}(\ell, \mathbb{Z}(\ell)) + \frac{\psi}{MN(\psi)\Gamma(\psi)}\int_0^\ell \iota^{\phi-1}(\ell-\iota)^{\psi-1}\mathbb{T}(\iota, \mathbb{Z}(\iota))\, d\iota.$$

Replacing ℓ by ℓ_{t+1} yields

$$\mathbb{Z}(\ell_{t+1}) = \mathbb{Z}(0) + \frac{1-\psi}{MN(\psi)}\mathbb{T}(\ell_t, \mathbb{Z}(\ell_t)) + \frac{\psi}{MN(\psi)\Gamma(\psi)}\int_0^{\ell_{t+1}} \iota^{\phi-1}(\ell_{t+1}-\iota)^{\psi-1}\mathbb{T}(\iota, \mathbb{Z}(\iota))\, d\iota. \tag{7.1}$$

Considering the Lagrange polynomials we derive that

$$\begin{aligned}
\mathbb{T}(x, \mathbb{Z}(\ell)) &= \frac{(x-\ell_{n_0-1})\,\mathbb{U}\left(\ell_{n_0}, \mathbb{Z}(\ell_{n_0})\right)}{\ell_{n_0} - \ell_{n_0-1}} - \frac{(x-\ell_{n_0})\,\mathbb{U}\left(\ell_{n_0-1}, \mathbb{Z}(\ell_{n_0-1})\right)}{\ell_{n_0} - \ell_{n_0-1}} \\
&= \frac{(x-\ell_{n_0})\,\mathbb{T}\left(\ell_{n_0}, \mathbb{Z}(\ell_{n_0})\right)}{\ell_{n_0} - \ell_{n_0-1}} - \frac{(x-\ell_{n_0})\,\mathbb{T}\left(\ell_{n_0-1}, \mathbb{Z}(\ell_{n_0-1})\right)}{\ell_{n_0} - \ell_{n_0-1}} \\
&= \frac{(x-\ell_{n_0})\,\mathbb{T}\left(\ell_{n_0}, \mathbb{Z}(\ell_{n_0})\right)}{h} - \frac{(x-\ell_{n_0})\,\mathbb{T}\left(\ell_{n_0-1}, \mathbb{Z}(\ell_{n_0-1})\right)}{h}.
\end{aligned}$$

Using the Lagrange polynomial for (7.1) leads to

$$\mathbb{Z}(\ell_{t+1}) = \mathbb{Z}(0) + \frac{1-\psi}{MN(\psi)}\mathbb{T}\left(\ell_t, \mathbb{Z}(\ell_t)\right) + \frac{\psi}{MN(\psi)\Gamma(\psi)}\sum_{n_0=1}^{t}\left[\frac{\mathbb{T}\left(\ell_{n_0}, \mathbb{Z}\left(\ell_{n_0}\right)\right)}{h}\int_{\ell_{n_0}}^{\ell_{n_0+1}}\left(\iota-\ell_{n_0-1}\right)\left(\ell_{t+1}-\iota\right)^{\psi-1}d\iota\right.$$
$$\left.-\frac{\mathbb{T}\left(\ell_{n_0-1}, \mathbb{Z}\left(\ell_{n_0-1}\right)\right)}{h}\int_{\ell_{n_0}}^{\ell_{n_0+1}}\left(\iota-\ell_{n_0}\right)\left(\ell_{t+1}-\iota\right)^{\psi-1}d\iota\right].$$

Next solving the integral equation yields

$$\mathbb{Z}(\ell_{t+1}) = \mathbb{Z}(0) + \frac{1-\psi}{MN(\psi)}\mathbb{T}\left(\ell_t, \mathbb{Z}(\ell_t)\right) + \frac{\psi h^{\psi}}{MN(\psi)\Gamma(\psi+2)}\sum_{n_0=1}^{t}\left[\mathbb{T}\left(\ell_{n_0}, \mathbb{Z}\left(\ell_{n_0}\right)\right)\left((t+1-n_0)^{\psi}(t-n_0+2+\psi)\right.\right.$$
$$\left.\left.-(t-n_0)^{\psi}(t+2-n_0+2\psi)\right) - \mathbb{T}\left(\ell_{n_0-1}, \mathbb{Z}\left(\ell_{n_0-1}\right)\right)\left((t+1-n_0)^{\psi+1}-(t-n_0)^{\psi}(t+1-n_0+\psi)\right)\right].$$

Using $\mathbb{T}\left(\ell, \mathbb{Z}(\ell)\right)$ yields

$$\mathbb{Z}(\ell_{t+1}) = \mathbb{Z}(0) + \phi\ell_t^{\phi-1}\frac{1-\psi}{MN(\psi)}\Lambda^{\circledast}\left(\ell_t, \mathbb{Z}(\ell_t)\right) + \frac{\phi h^{\psi}}{MN(\psi)\Gamma(\psi+2)}$$
$$\times\sum_{n_0=1}^{t}\left[\ell_{n_0}^{\phi-1}\Lambda^{\circledast}\left(\ell_{n_0}, \mathbb{Z}(\ell_{n_0})\right)\left((t+1-n_0)^{\psi}(t-n_0+2+\psi)-(t-n_0)^{\psi}(t+2-n_0+2\psi)\right)\right.$$
$$\left.-\ell_{n_0-1}^{\phi-1}\Lambda^{\circledast}\left(\ell_{n_0-1}, \mathbb{Z}(\ell_{n_0-1})\right)\left((t+1-n_0)^{\psi+1}-(t-n_0)^{\psi}(t+1-n_0+\psi)\right)\right].$$

Thus

$$\Psi_1(t, n_0) = \left((t+1-n_0)^{\psi}(t-n_0+2+\psi)-(t-n_0)^{\psi}(t+2-n_0+2\psi)\right),$$
$$\Psi_2(t, n_0) = \left((t+1-n_0)^{\psi+1}-(t-n_0)^{\psi}(t+1-n_0+\psi)\right).$$

Hence, the numerical scheme for (2.1) is obtained as the following

$$S(\ell_{t+1}) = S(0) + \phi\ell_t^{\phi-1}\frac{1-\psi}{MN(\psi)}\Lambda_1^{\circledast}\left(\ell_t, S(\ell_t)\right) + \frac{\phi h^{\psi}}{MN(\psi)\Gamma(\psi+2)}$$
$$\times\sum_{n_0=1}^{t}\left[\ell_{n_0}^{\phi-1}\Lambda_1^{\circledast}\left(\ell_{n_0}, S(\ell_{n_0})\right)\Psi_1(t, n_0)-\ell_{n_0-1}^{\phi-1}\Lambda_1^{\circledast}\left(\ell_{n_0-1}, S(\ell_{n_0-1})\right)\Psi_2(t, n_0)\right],$$
$$E(\ell_{t+1}) = E(0) + \phi\ell_t^{\phi-1}\frac{1-\psi}{MN(\psi)}\Lambda_2^{\circledast}\left(\ell_t, E(\ell_t)\right) + \frac{\phi h^{\psi}}{MN(\psi)\Gamma(\psi+2)}$$
$$\times\sum_{n_0=1}^{t}\left[\ell_{n_0}^{\phi-1}\Lambda_2^{\circledast}\left(\ell_{n_0}, E(\ell_{n_0})\right)\Psi_1(t, n_0)-\ell_{n_0-1}^{\phi-1}\Lambda_2^{\circledast}\left(\ell_{n_0-1}, E(\ell_{n_0-1})\right)\Psi_2(t, n_0)\right],$$
$$I(\ell_{t+1}) = I(0) + \phi\ell_t^{\phi-1}\frac{1-\psi}{MN(\psi)}\Lambda_3^{\circledast}\left(\ell_t, I(\ell_t)\right) + \frac{\phi h^{\psi}}{MN(\psi)\Gamma(\psi+2)}$$
$$\times\sum_{n_0=1}^{t}\left[\ell_{n_0}^{\phi-1}\Lambda_3^{\circledast}\left(\ell_{n_0}, I(\ell_{n_0})\right)\Psi_1(t, n_0)-\ell_{n_0-1}^{\phi-1}\Lambda_3^{\circledast}\left(\ell_{n_0-1}, I(\ell_{n_0-1})\right)\Psi_2(t, n_0)\right],$$
$$A(\ell_{t+1}) = A(0) + \phi\ell_t^{\phi-1}\frac{1-\psi}{MN(\psi)}\Lambda_4^{\circledast}\left(\ell_t, A(\ell_t)\right) + \frac{\phi h^{\psi}}{MN(\psi)\Gamma(\psi+2)}$$
$$\times\sum_{n_0=1}^{t}\left[\ell_{n_0}^{\phi-1}\Lambda_4^{\circledast}\left(\ell_{n_0}, A(\ell_{n_0})\right)\Psi_1(t, n_0)-\ell_{n_0-1}^{\phi-1}\Lambda_4^{\circledast}\left(\ell_{n_0-1}, A(\ell_{n_0-1})\right)\Psi_2(t, n_0)\right],$$

$$R(\ell_{t+1}) = R(0) + \phi\ell_t^{\phi-1}\frac{1-\psi}{MN(\psi)}\Lambda_5^{\circledast}\left(\ell_t, R(\ell_t)\right) + \frac{\phi h^{\psi}}{MN(\psi)\Gamma(\psi+2)}$$
$$\times\sum_{n_0=1}^{t}\left[\ell_{n_0}^{\phi-1}\Lambda_5^{\circledast}\left(\ell_{n_0}, R(\ell_{n_0})\right)\Psi_1(t,n_0) - \ell_{n_0-1}^{\phi-1}\Lambda_5^{\circledast}\left(\ell_{n_0-1}, R(\ell_{n_0-1})\right)\Psi_2(t,n_0)\right].$$

The details of the computations of the model (2.1) with particular formulation are presented in this section. The numerical description of the model (2.1) is presented in the following figures. Figures 2 (a), 4 (a), 6 (a), 8 (a), and 10 (a) represent the time lines of the susceptible, exposed, symptomatic infected, asymptomatic infected, and recovered nodes by considering the approximate numerical values as $\Pi = 0.6$, $\delta = 0.99$, $\sigma = 1.5$, $\chi = 0.99$, $\epsilon = 0.95$, $\nu = 0.6$, $\mu = 0.24$, $\rho = 0.1$, $\xi^{\circledast} = 0.06$, $\beta^{\circledast} = 0.85$, $\gamma = 0.6$ with the initial condition $S(1) = 0.92$, $E(1) = 0.04$, $I(1) = 0.02$, $A(1) = 0.02$, $R(1) = 0$. Here the fractal-fractional is $\psi = 0.7$, $\phi = 0.9$, and $h = 0.01$. Also, when the fractal-fractional is $\psi = 0.6$, $\phi = 0.9$, and $h = 0.01$, the time lines of the susceptible, exposed, symptomatic infected, asymptomatic infected, and recovered nodes are displayed in Figures 3 (a), 5 (a), 7 (a), 9 (a), 11 (a) for the same parameter values.

Figures 2 (b), 4 (b),6 (b), 8 (b), and 10 (b) are analytical effect of the susceptible, exposed, symptomatic infected, asymptomatic infected, and recovered nodes for different fractal values $\phi = 0.6, 0.7, 0.8, 0.9, 0.99$ with fixing the fractional value $\psi = 0.7$ and the other parameter values are same. Similarly, Figures 3 (b), 5 (b), 7 (b), 9 (b), and 11 (b) are analytical effect of the susceptible, exposed, symptomatic infected, asymptomatic infected, and recovered nodes for different fractional values $\psi = 0.75, 0.8, 0.85, 0.9, 0.95$ with fixing the fractal value $\phi = 0.9$ and the other parameter values are same.

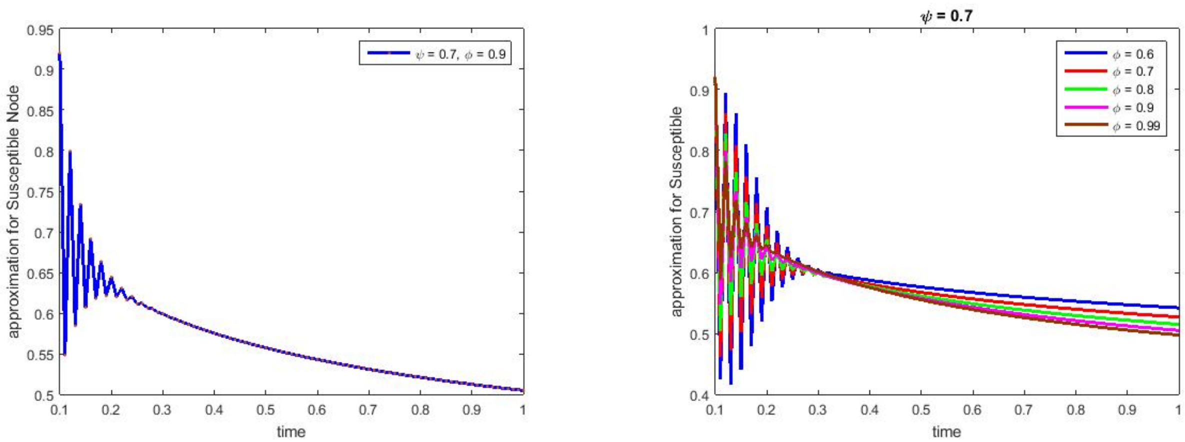

Figure 2: Analytical effect of fractal (ϕ) derivative for susceptible node of model (2.1).

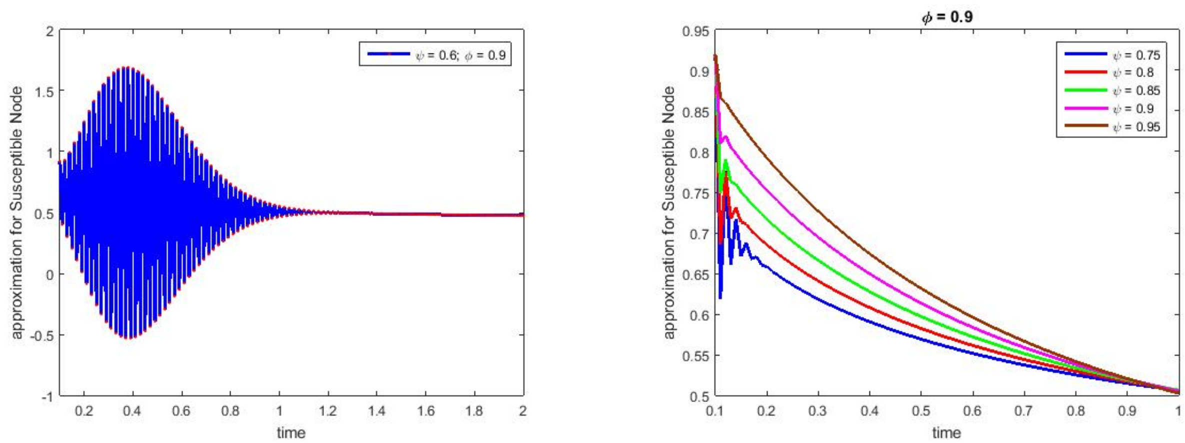

Figure 3: Analytical effect of fractional (ψ) derivative susceptible node of model (2.1).

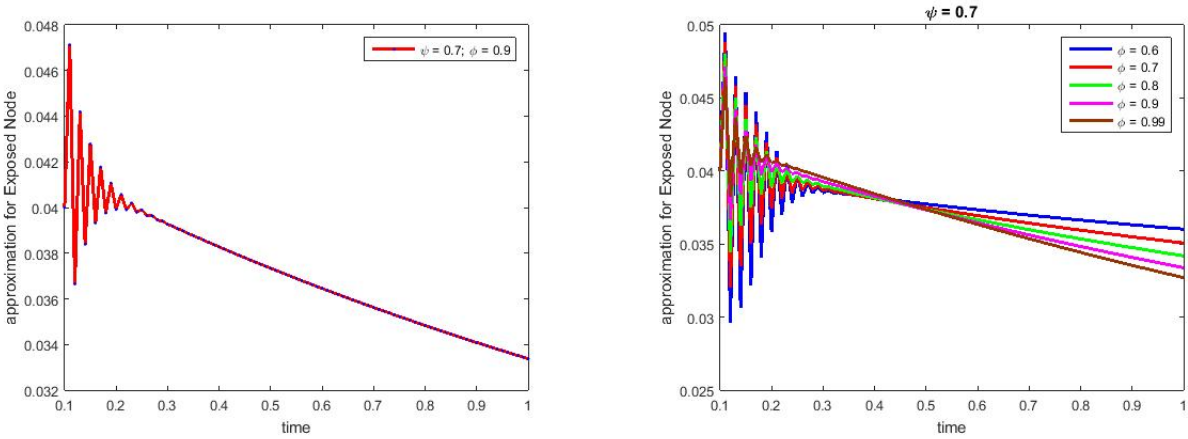

Figure 4: Analytical effect of fractal (ϕ) derivative for exposed node of model (2.1).

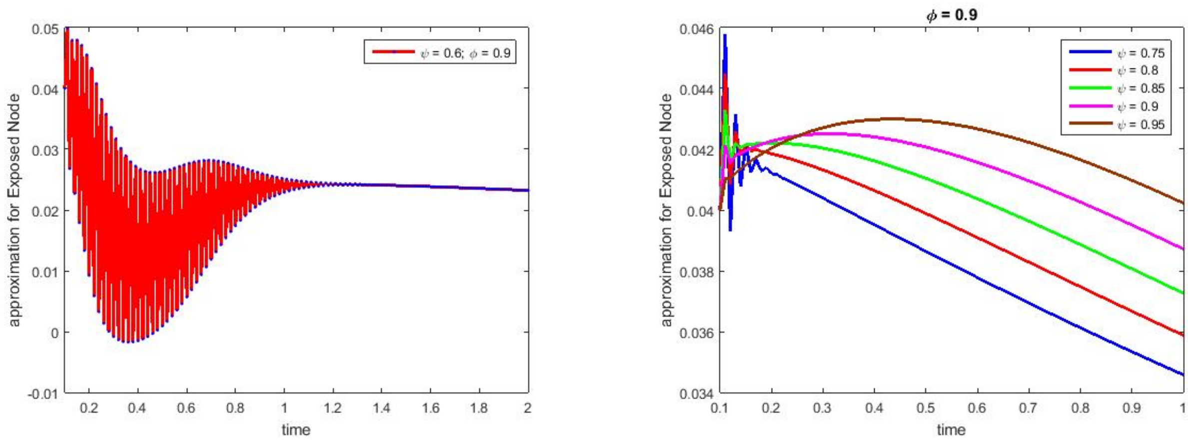

Figure 5: Analytical effect of fractional (ψ) derivative exposed node of model (2.1).

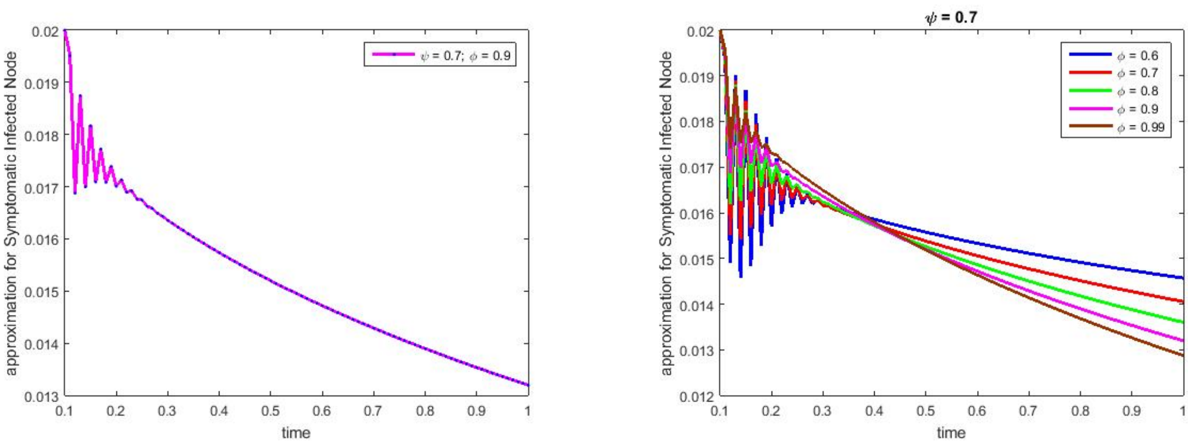

Figure 6: Analytical effect of fractal (ϕ) derivative for symptomatic infected node of model (2.1).

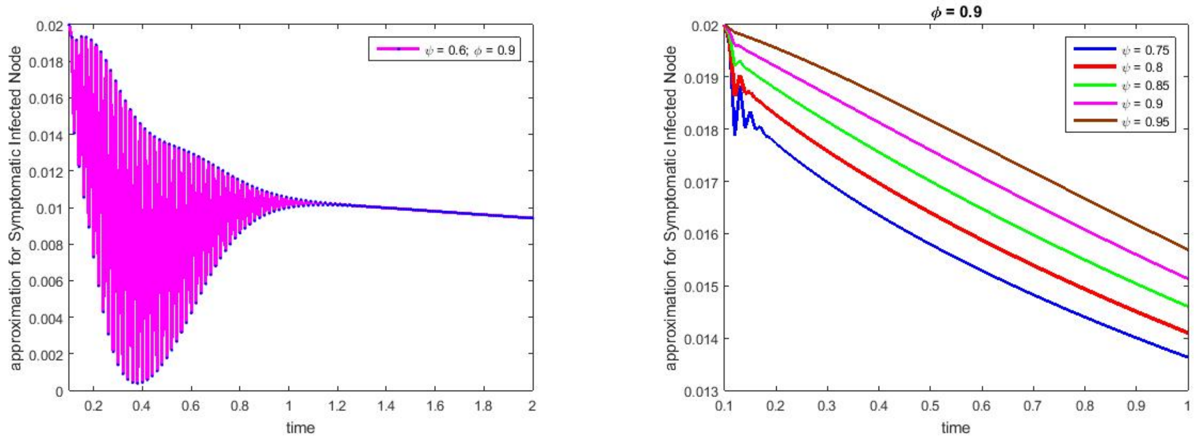

Figure 7: Analytical effect of fractional (ψ) derivative symptomatic infected node of model (2.1).

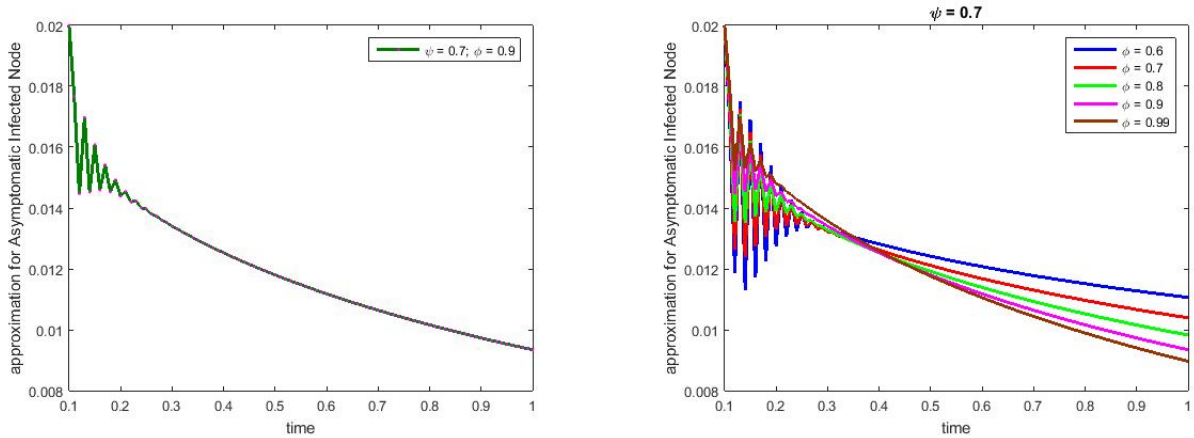

Figure 8: Analytical effect of fractal (ϕ) derivative for asymptomatic infected node of model (2.1).

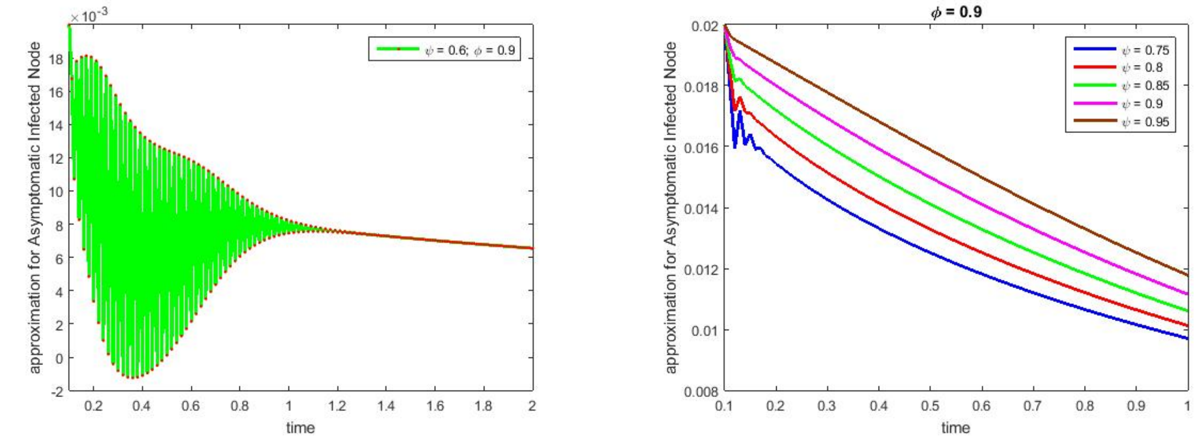

Figure 9: Analytical effect of fractional (ψ) derivative asymptomatic infected node of model (2.1).

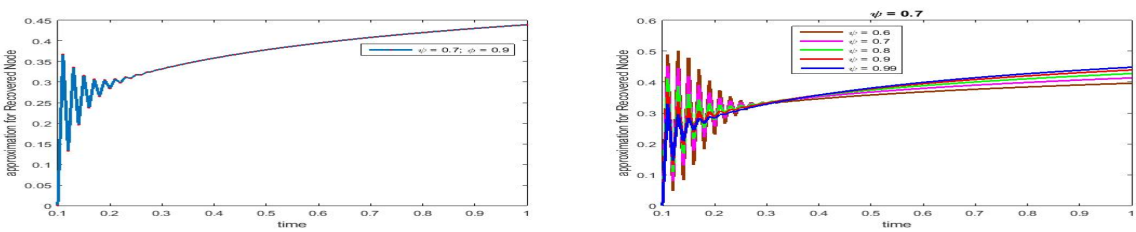

Figure 10: Analytical effect of fractal (ϕ) derivative for recovered node of model (2.1).

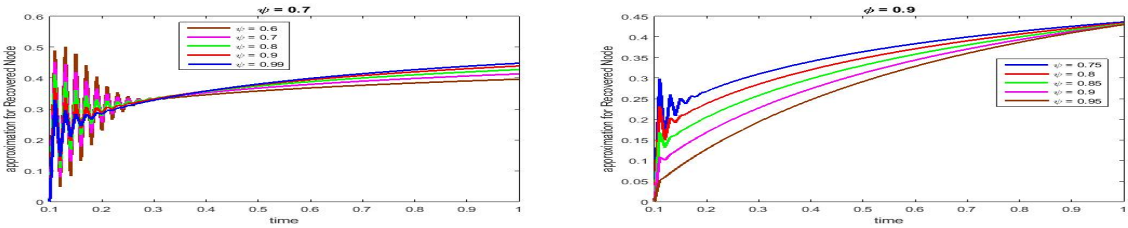

Figure 11: Analytical effect of fractional (ψ) derivative for recovered node of model (2.1).

8. Conclusion

This article introduces an advanced modeling approach for simulating worm propagation in computer networks by using the FF-derivatives. This structure, which accounts for memory effects, offers a more comprehensive and versatile model compared to classical existing models. The fixed-point theory ensures the validity of the model by affirming the existence and uniqueness of solutions, while stability is rigorously analyzed using HU-stability. The numerical results, validated with real-world data, highlight the enhanced accuracy of the model in discussing the dynamics of worm propagation. This approach shows a deeper analysis of the complex behavior in network vulnerabilities, establishing the way for improved methods in cybersecurity. The results contribute both to the theoretical development of network modeling and to applications in mitigating cyber threats, especially where memory and information capacity influence the spread of network issues.

Authors' contributions

All authors contributed equally and significantly to this paper. All authors have read and approved the final version of the manuscript.

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