A numerical algorithm for solving nonlinear fuzzy differential equations

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Abstract

In this paper, we propose a numerical algorithm based on Runge-Kutta methods to and solution of nonlinear fuzzy differential equations (FDEs) such that it solution satisffes solution found via differential inclusions. Our interpretation of the FDEs is a family of fuzzy differential inclusions. The method is illustrated by some examples.

1. Introduction

Fuzzy set theory is an approach for modeling of uncertainty which is existence in environment. This theory has application in the golden mean [1], quantum optics and gravity [2].

Particularly, the concept of differentiability of fuzzy function is important in many applications. This concept is initially introduced based on Zadeh's extension principle, then further research were done by Puri and Ralescu [3].

In many papers, using of Hakuhara derivative to find solution of FDEs. It is shown that the solution of many FDEs hasn't existed. Thereby a generalization derivative was given for fuzzy mappings enlarging

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the class of differentiable fuzzy mappings by Bede and Gal in [4,5], and Chalco-Cano and Flores use from this concept to
solve FDEs in [6,7,8]. For solving the FDE, x'(t)=f(t,x(t)) , $x(0) = x_0$, is shown that when g(x) is nondecreasing or nonincreasing the solution of FDEs is exists.
In this paper we consider this problem when g(x) is no ones. We proposed an algorhtim that tries to find the solution.
This paper has been organized as follows : Section 2 contains the basic material to be used in the rest of the article, in section 3 we shall see that this definition of fuzzy lateral H-derivative leads us to interpret a FDE in two different forms, generating new solutions for FDE, in section 4,we present a numerical procedure for solving the fuzzy initial value problem x'(t)=f(t,x(t)) , $x(0) = x_0$, when f is decreasing(increasing) or not decreasing(not increasing). Finally, numerical algorithms of calculating approximates to these solutions where designed.

2. Basic concepts

Let X be a Banach space. We denote by $K(X)$ the family of all nonempty compact subsets of X. If $A, B \in K(X)$ and $\lambda \in R$, then the operations of addition and scalar multiplication are defined as,
$A + B = \{\, a + b \,|\, a \in A, b \in B \,\}$, $\lambda A = \{\, \lambda a \,|\, a \in A \,\}$.
A fuzzy set u on a universe set X is a mapping $u: X \to [0,1]$. Then $u(x)$ is interpreted as the degree of membership of a element x in the fuzzy set u for each $x \in X$.
If X is a Banach space and u is a fuzzy set on X, We define $[u]^\alpha = \{\, x \in X \,|\, u(x) \not\leq \alpha$ the α-level of u, for $0 < \alpha \leq 1$. For $\alpha = 0$ the support of u is defined as $[u]^0 = \mathrm{supp}(u) = \overline{\{\, x \in X \,|\, u(x) > 0 \,\}}$, where $\overline{A}$ denotes the closure of $A \subset X$.
A fuzzy set u on X is called compact if $[u]^\alpha \in K(X)$, $\forall \alpha \in [0,1]$. Also, u is called convex if $[u]^\alpha$ is a convex set for all $\alpha \in [0,1]$. We will denote by $F(X)$ the space of all compact and convex fuzzy sets on X.
If $u \in F(R)$ be, then u is called a fuzzy interval and the α-level set $[u]^\alpha$ is a nonempty compact interval for all $\alpha \in [0,1]$.
If $u, v \in F(R^n)$ and $\lambda \in R$, then the following properties are true :
$[u + v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda [u]^\alpha$, $\forall \alpha \in [0,1]$.
Also, the Hausdorff metric H on $K(R^n)$ can be extended to $F(R^n)$ by the following definition
$$D(u,v) = \sup_{\alpha \in [0,1]} H\big([u]^\alpha, [v]^\alpha\big), \quad \forall u, v \in F(R^n).$$

3. Fuzzy differential equations

Definition 1. Let $u, v \in F(R^n)$ be. If there exists $w \in F(R^n)$ such that $u = v + w$, then w is called the H-difference of u and v and denoted by $u \ominus v$.

Definition 2. Let be $F: [0,T] \to F(R^n)$, and $t_0 \in [0,T]$, we say that F is differentiable at t_0 if :

(I): It exists an element $F'(t_0) \in F(R^n)$ such that, for all $h > 0$ suffciently near to 0, there are $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus \qquad F(t_0 - h)$ and the limits (in D-metric)
$$\lim_{h \to 0^+} \frac{F(t_0 + h) \theta F(t_0)}{h} = \lim_{h \to 0^+} \frac{F(t_0) \theta F(t_0 - h)}{h} = F'(t_0) \qquad (1)$$

(II): It exists an element $F'(t_0) \in F(R^n)$ such that, for all $h < 0$ suffciently near to 0, there are $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus \qquad F(t_0 - h)$ and the limits (in D-metric)

$$\lim_{h \to 0^-} \frac{F(t_0 + h)\theta F(t_0)}{h} = \lim_{h \to 0^-} \frac{F(t_0)\theta F(t_0 - h)}{h} = F'(t_0) \qquad (2)$$

Theorem 1. Let $F:[0,T] \to F(R)$ be a function and denote $[F(t)]^\alpha = [f_\alpha(t), g_\alpha(t)]$, for each $\alpha \in [0,1]$. Then,

(1) If F is differentiable in the first form (I), then f_α and g_α are differentiable functions and

$$[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)] \qquad (3)$$

(2) If F is differentiable in the second form (II), then f_α and g_α are differentiable functions and

$$[F'(t)]^\alpha = [g'_\alpha(t), f'_\alpha(t)] \qquad (4)$$

Proof. see demonstration of Theorem 5 in [6].

Let $q:[0,T] \times F(R) \to F(R)$ be a function. consider the following fuzzy differential equation:

$$\begin{cases} X'(t) = q(t, X(t)) \\ X(0) = X_0 \end{cases} \qquad (5)$$

Let $[X(t)]^\alpha = [f_\alpha(t), g_\alpha(t)]$, $[X_0]^\alpha = [f_\alpha^0, g_\alpha^0]$ and $[q(t,X(t))]^\alpha = [k_\alpha(t, f_\alpha(t), g_\alpha(t)), l_\alpha(t, f_\alpha(t), g_\alpha(t))]$.

If X(t) is differentiable in the first form (I), then from (3) we have $[X'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)]$ and the problem (5) is changed to:

$$\begin{cases} f'_\alpha(t) = k_\alpha(t, f_\alpha(t), g_\alpha(t)) & f_\alpha(0) = f_\alpha^0 \\ g'_\alpha(t) = l_\alpha(t, f_\alpha(t), g_\alpha(t)) & g_\alpha(0) = g_\alpha^0 \end{cases} \qquad (6)$$

and if X(t) is differentiable in the second form (II), then from (4) we have $[X'(t)]^\alpha = [g'_\alpha(t), f'_\alpha(t)]$ and the problem (5) is changed to:

$$\begin{cases} f'_\alpha(t) = l_\alpha(t, f_\alpha(t), g_\alpha(t)) & f_\alpha(0) = f_\alpha^0 \\ g'_\alpha(t) = k_\alpha(t, f_\alpha(t), g_\alpha(t)) & g_\alpha(0) = g_\alpha^0 \end{cases} \qquad (7)$$

4. Fuzzy differential inclusions

In this section the FDE is interpret as a fuzzy differential inclusion. In the FDE (5), $q:[0,T] \times F(R) \to F(R)$ is

obtained by Zadeh's extension principle from a differentiable function, $qs:[0,T] \times R \to R$.

We interpret the fuzzy initial value problem (5) as family of differential inclusions

$$y'_\alpha(t) \in [q(t, y_\alpha(t))]^\alpha, \quad y_\alpha(0) \in [X_0]^\alpha, \quad 0 \le \alpha \le 1 \qquad (8)$$

$$y'_\alpha(t) = qs(t, y_\alpha(t)), \quad y_\alpha(0) \in [X_0]^\alpha, \quad 0 \le \alpha \le 1 \qquad (9)$$

The attainable sets is defined as $A_\alpha(t) = \{ y_\alpha(t) | y_\alpha(t) \text{ is a solution of (9)} \}$. A more complete account can be found in [14,21,23].

Theorem 2. Let $X_0 \in F(R^n)$ and let Ω be an open subset of $R \times R^n$ containing $\{0\} \times \text{supp}(X_0)$. Suppose that $q:\Omega \to F(R^n)$ is use and compact. Let the boundedness assumption hold, for all $x_0 \in \text{supp}(X_0)$ and the inclusion

$$y'(t) \in [q(t, y(t))]^{\alpha=0}, \qquad y(0) \in \text{supp}(X_0)$$

Then, the attainable sets $A_\alpha(t)$ of the family of inclusions (9) are the level sets of a fuzzy set $A(t)$, [21,23].

Theorem 3. Let U be an open set in R^n and $X_0 \in F(R^n)$. Suppose that qs is continuous, that for each $c \in U$ there exists one unique solution $x(\cdot, c)$ of problem (9) and that $x(t, \cdot)$ is continuous in U for each $t \in [0,T]$. Then :

If qs is nondecreasing with respect to the second argument then, using the first form (I), the fuzzy solution of (5) and the solution via differential inclusions are identical.

If qs is nonincreasing with respect to the second argument then, using the derivative in the second form (II), the fuzzy solution of (5) and the solution via differential inclusions are identical.
Proof. see demonstration of Theorem 7,8 in [6].

5. Numerical Algorithm

Let f_α, g_α be as mention in Theorem 1, denote $dq(t,x) = \dfrac{\partial}{\partial x} qs(t,x)$.

In the following we proposed an algorithm which based on second order Runge-Kutta method.
Algorithm .

1. choose m, n two integer number. let $h_\alpha = \dfrac{1}{m}$, $h_t = \dfrac{T}{n}$.

2. Repeat steps $3-12$ for $i = 0,\ldots,m$:

3. $\alpha = (m-i)h_\alpha$, $y_\alpha(0) = [y_{1\alpha}(0), y_{2\alpha}(0)] = [X_0]^\alpha$.

4. Repeat steps $5-12$ for $j = 1,\ldots,n$:

5. $t = jh_t$.

6. if $dq(t, y_1(t)) > 0$, then repeat steps $7-9$, else repeat $10-12$.

7. $k_1 = h_t k_\alpha(t, y_{1\alpha}(t), y_{2\alpha}(t))$, $l_1 = h_t l_\alpha(t, y_{1\alpha}(t), y_{2\alpha}(t))$.

8. $k_2 = h_t k_\alpha(t + h_t, y_{1\alpha}(t) + k_1, y_{2\alpha}(t) + l_1)$, $l_2 = h_t l_\alpha(t + h_t, y_{1\alpha}(t) + k_1, y_{2\alpha}(t) + l_1)$.

9. $y_{1\alpha}(t + h_t) = y_{1\alpha}(t) + \dfrac{1}{2}(k_1 + k_2)$, $y_{2\alpha}(t + h_t) = y_{2\alpha}(t) + \dfrac{1}{2}(l_1 + l_2)$.

10. $k_1 = h_t l_\alpha(t, y_{1\alpha}(t), y_{2\alpha}(t))$, $l_1 = h_t k_\alpha(t, y_{1\alpha}(t), y_{2\alpha}(t))$.

11. $k_2 = h_t l_\alpha(t + h_t, y_{1\alpha}(t) + k_1, y_{2\alpha}(t) + l_1)$, $l_2 = h_t k_\alpha(t + h_t, y_{1\alpha}(t) + k_1, y_{2\alpha}(t) + l_1)$

12. $y_{1\alpha}(t + h_t) = y_{1\alpha}(t) + \dfrac{1}{2}(k_1 + k_2)$, $y_{2\alpha}(t + h_t) = y_{2\alpha}(t) + \dfrac{1}{2}(l_1 + l_2)$.

6. Numerical Example

Let us consider the nonlinear fuzzy problem,

$$\begin{cases} X'(t) = t^2 X(t) - 4tX(t) + 3X(t) & t \in [0,2] \\ X(0) = (0,1,1) \end{cases}$$

Approximation solution of Example 1, for $t = 0, 0.02, 0.04, 0.06$ are tabulated in Table 1:

Table 1: Approximation solution of Example 1

α	$t = 0$	$t = 0.02$	$t = 0.04$	$t = 0.06$
0	[-1.0 , 1.0]	[-1.0627 , 1.0627]	[-1.1292 , 1.1292]	[-1.2000 , 1.2000]
0.1	[-0.9 , 0.9]	[-0.9564 , 0.9564]	[-1.0163 , 1.0163]	[-1.0800 , 1.0800]
0.2	[-0.8 , 0.8]	[-0.8501 , 0.8501]	[-0.9034 , 0.9034]	[-0.9600 , 0.9600]
0.3	[-0.7 , 0.7]	[-0.7439 , 0.7439]	[-0.7905 , 0.7905]	[-0.8400 , 0.8400]
0.4	[-0.6 , 0.6]	[-0.6376 , 0.6376]	[-0.6775 , 0.6775]	[-0.7200 , 0.7200]
0.5	[-0.5 , 0.5]	[-0.5313 , 0.5313]	[-0.5646 , 0.5646]	[-0.6000 , 0.6000]
0.6	[-0.4 , 0.4]	[-0.4251 , 0.4251]	[-0.4517 , 0.4517]	[-0.4800 , 0.4800]
0.7	[-0.3 , 0.3]	[-0.3188 , 0.3188]	[-0.3388 , 0.3388]	[-0.3600 , 0.3600]
0.8	[-0.2 , 0.2]	[-0.2125 , 0.2125]	[-0.2258 , 0.2258]	[-0.2400 , 0.2400]
0.9	[-0.1 , 0.1]	[-0.1063 , 0.1063]	[-0.1129 , 0.1129]	[-0.1200 , 0.1200]
1	[0.0 , 0.0]	[0.0 , 0.0]	[0.0 , 0.0]	[0.0 , 0.0]

7. Conclusion

In this paper we present some results on fuzzy differential equation with generalized derivative. Also, we obtain new method for solving the differential equations that are not strictly increasing or strictly decreasing.

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