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Solving multi-order fractional differential equations using covariant contraction principle

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Abstract

This article introduces the concept of a generalized fuzzy bipolar metric space which extends the framework of fuzzy bipolar metric spaces. Within this novel setting, we establish the Ćirić quasi-contraction theorem tailored to generalized fuzzy bipolar metric spaces. To achieve these results, we introduce the notions of covariant and contravariant contractions, which represent significant advances in this field. To illustrate the applicability of the theoretical findings, detailed examples are provided. Moreover, the study delves into the well-posedness of the fuzzy fixed point problem, demonstrating the existence and uniqueness of solutions for multi-order fractional differential equations.

Keywords: Generalized fuzzy bipolar metric space, fuzzy fixed point results, well-posedness of fixed point problem, multi-order fractional differential equations, integral boundary conditions.

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1. Introduction

Significant progress has been made in the field of fuzzy metric spaces (FMSs), beginning with the pioneering work of Schweizer and Sklar [23] and continuing through the contributions of George and Veermani [5], along with other researchers (see [4, 11, 17, 25, 26]). In 2020, Bartwal et al. introduced a novel generalization of FMSs called the fuzzy bipolar metric space (FBMS), designed to measure the distance between points from two distinct sets [3]. Many of these spaces have been further developed within the broader context of FBMSs, see [14, 15, 22]. Ashraf et al. [1] proposed the concept of generalized fuzzy metric spaces (GFMSs). GFMSs extend several topological spaces, including FMSs, fuzzy b-metric spaces, and dislocated fuzzy metric spaces.

In 2021, Roy and Saha [21] proposed the sequential bipolar metric space (SBMS), a generalization of both bipolar metric spaces [18] and bipolar b-metric spaces [20]. In this paper, we introduce a fuzzy interpretation of their work, referred to as the generalized fuzzy bipolar metric space (GFBMS). This concept extends the FBMS framework, as well as the fuzzy triple-controlled bipolar metric spaces (FTCBMSs)

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introduced in [14]. An example is provided to clarify the definition of a GFBMS, demonstrating that the category of GFBMSs encompasses both FBMSs and FTCBMSs.

Moreover, several fixed-point results are established within the framework of GFBMS. For a deeper understanding of these findings, we recommend consulting [7, 8, 16, 19]. The paper also investigates the well-posedness of the fuzzy fixed-point problem (FFP problem). An application of the main results is presented to prove the existence and uniqueness of solutions to multi-order fractional differential equations (M-OFDEs) with integral boundary conditions (IBCs).

To provide context, we now review some fundamental concepts that will be useful later in this paper.

Definition 1.1 ([14])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty sets. Let $\lambda, \alpha, a : \mathcal{A} \times \mathcal{B} \to [1, +\infty)$ be three noncomparable functions. A quadruple $(\mathcal{A}, \mathcal{B}, \mathcal{M}, *)$ is called a FTCBMS, where $*$ is a continuous t-norm (ctn) and $\mathcal{M}$ is a fuzzy set on $\mathcal{A} \times \mathcal{B} \times (0, +\infty)$, if for all $\rho, \gamma, \iota > 0$:

($\mathcal{M}$1) $\mathcal{M}(\varpi, \nu, \iota) > 0$ for all $(\varpi, \nu) \in \mathcal{A} \times \mathcal{B}$;
($\mathcal{M}$2) $\mathcal{M}(\varpi, \nu, \iota) = 1$ if and only if $\varpi = \nu$ for $\varpi \in \mathcal{A}$ and $\nu \in \mathcal{B}$;
($\mathcal{M}$3) $\mathcal{M}(\varpi, \nu, \iota) = \mathcal{M}(\nu, \varpi, \iota)$ for all $\sigma, \nu \in \mathcal{A} \cap \mathcal{B}$;
($\mathcal{M}$4) $\mathcal{M}(\varpi, \nu, \rho+\gamma+\iota) \geqslant \mathcal{M}(\varpi, \nu_1, \frac{\rho}{\lambda(\varpi,\nu_1)}) * \mathcal{M}(\sigma_2, \nu_1, \frac{\gamma}{\alpha(\sigma_2,\nu_1)}) * \mathcal{M}(\sigma_2, \nu_2, \frac{\iota}{a(\sigma_2,\nu)})$ for all $(\varpi, \nu_1), (\nu, \sigma_2) \in \mathcal{A} \times \mathcal{B}$;
($\mathcal{M}$5) $\mathcal{M}(\varpi, \nu, .) : [0, +\infty) \to [0, 1]$ is left continuous;
($\mathcal{M}$6) $\mathcal{M}(\varpi, \nu, .)$ is non-decreasing for all $\varpi \in \mathcal{A}$ and $\nu \in \mathcal{B}$.

Remark 1.2. If $\lambda = \alpha = a$ in Definition 1.1, we arrive at the definition of a FTCBMS as presented by Mani et al. [13]. Also, if $\lambda = \alpha = a = 1$ we obtain the definition of FBMS, see [3].

Definition 1.3 ([21])**.** Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty sets and $\mathcal{D} : \mathcal{A} \times \mathcal{B} \to [0, +\infty]$ be a function. Define the following sets:

$$C_L(\mathcal{A}, \mathcal{D}, \nu) = \{\{\varpi_n\} \subset \mathcal{A} : \lim_{n\to+\infty} \mathcal{D}(\varpi_n, \nu) = 0\}, \quad C_R(\mathcal{B}, \mathcal{D}, \varpi) = \{\{\nu_n\} \subset \mathcal{B} : \lim_{n\to+\infty} \mathcal{D}(\nu_n, \varpi) = 0\},$$

where $\mathcal{D}$ satisfies the following conditions:

($\mathcal{D}$1) $\mathcal{D}(\varpi, \nu) = 0$ implies $\varpi = \nu \in \mathcal{A} \cap \mathcal{B}$;
($\mathcal{D}$2) $\mathcal{D}(\varpi, \nu) = \mathcal{D}(\nu, \varpi)$ for all $\varpi \in \mathcal{A}$ and $\nu \in \mathcal{B}$;
($\mathcal{D}$3) there exists some $k > 0$ such that for all $(\varpi_1, \nu_1), (\varpi_2, \nu_2) \in \mathcal{A} \times \mathcal{B}$ we have

$$\mathcal{D}(\varpi_1, \nu_2) \leqslant k \limsup_{n\to+\infty} [\mathcal{D}(\varpi_1, \nu_1) + \mathcal{D}(\varpi_n, \nu_1)], \forall \{\varpi_n\} \in C_L(\mathcal{A}, \mathcal{D}, \nu_2);$$
$$\mathcal{D}(\varpi_1, \nu_2) \leqslant k \limsup_{n\to+\infty} [\mathcal{D}(\varpi_2, \nu_2) + \mathcal{D}(\varpi_2, \nu_n)], \forall \{\nu_n\} \in C_R(\mathcal{B}, \mathcal{D}, \varpi_1).$$

Then $\mathcal{D}$ is called a SBM and the triplet $(\mathcal{A}, \mathcal{B}, \mathcal{D})$ is called a SBMS.

2. GFMS

In this part, we introduce the notion of GFMSs and prove various fixed point theorems in these domains. Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty sets and let $\mathcal{F} : \mathcal{A} \times \mathcal{B} \times (0, +\infty) \to [0, 1]$ be a function. For $(\varpi, \nu) \in \mathcal{A} \times \mathcal{B}, t > 0$ let us establish the following sets:

$$C_L(\mathcal{A}, \mathcal{F}, \nu) = \{\{\varpi_n\} \subset \mathcal{A} : \lim_{n\to+\infty} \mathcal{F}(\varpi_n, \nu, t) = 1\}, \quad C_R(\mathcal{B}, \mathcal{F}, \varpi) = \{\{\nu_n\} \subset \mathcal{B} : \lim_{n\to+\infty} \mathcal{F}(\nu_n, \varpi, t) = 1\}.$$

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two not empty sets and $*$ is a ctn. A mapping $\mathcal{F} : \mathcal{A} \times \mathcal{B} \times (0, +\infty) \to [0, 1]$ is called a GFBM if it satisfies the following conditions:

($\mathcal{F}$1) $\mathcal{F}(\varpi, \nu, t) > 0$;

($\mathcal{F}2$) $\mathcal{F}(\varpi, \nu, t) = 1$ implies $\varpi = \nu$ for all $t > 0$;

($\mathcal{F}3$) $\mathcal{F}(\varpi, \nu, t) = \mathcal{F}(\nu, \varpi, t)$ for all $\varpi, \nu \in \mathcal{A} \cap \mathcal{B}$;

($\mathcal{F}3$) There exists a constant $a \geqslant 1$ such that for $(\varpi_1, \nu_1), (\varpi_2, \nu_2) \in \mathcal{A} \times \mathcal{B}$, the following condition is satisfied:

$$\mathcal{F}(\varpi_1, \nu_2, t) \geqslant \limsup_{n\to+\infty}[\mathcal{F}(\varpi_1, \nu_1, \frac{t}{a}) * \mathcal{F}(\varpi_n, \nu_1, \frac{t}{a})], \quad \forall\{\varpi_n\} \in C_L(\mathcal{A}, \mathcal{F}, \nu_2),$$

$$\mathcal{F}(\varpi_1, \nu_2, t) \geqslant \limsup_{n\to+\infty}[\mathcal{F}(\varpi_2, \nu_2, \frac{t}{a}) * \mathcal{F}(\varpi_2, \nu_n, \frac{t}{a})], \quad \forall\{\nu_n\} \in C_R(\mathcal{B}, \mathcal{F}, \varpi_1);$$

($\mathcal{F}4$) $\mathcal{F}(\varpi, \nu, .) : (0, +\infty) \to [0, 1]$ is continuous and $\lim_{t\to+\infty} \mathcal{F}(\varpi, \nu, t) = 1$.

Then$(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ is called a GFBMS.

Example 2.2. Consider a SBMS $(\mathcal{A}, \mathcal{B}, \mathcal{D})$. Define a mapping $\mathcal{F} : \mathcal{A} \times \mathcal{B} \times (0, +\infty) \to [0, 1]$ as follows:

$$\mathcal{F}(\varpi, \nu, t) = \exp \frac{-\mathcal{D}(\varpi, \nu)}{t}, (\varpi, \nu) \in \mathcal{A} \times \mathcal{B}, \tag{2.1}$$

and any $t > 0$, define the following sets

$$C_L(\mathcal{A}, \mathcal{F}, \nu) = \{\{\varpi_n\} \subset \mathcal{A} : \lim_{n\to+\infty} \mathcal{F}(\varpi_n, \nu, t) = 1\}, \quad C_R(\mathcal{B}, \mathcal{F}, \varpi) = \{\{\nu_n\} \subset \mathcal{B} : \lim_{n\to+\infty} \mathcal{F}(\nu_n, \varpi, t) = 1\}.$$

Then $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ is a GFBMS, where "$*$" is considered as product ctn, i.e., $a * b = a \cdot b$.

Proof. The conditions ($\mathcal{F}1$), ($\mathcal{F}2$), ($\mathcal{F}3$), and ($\mathcal{F}5$) specified in Definition 2.1 seem to be satisfied. The verification is achieved by establishing the validity of ($\mathcal{F}4$). Let $(\varpi_1, \nu_1), (\varpi_2, \nu_2) \in \mathcal{A} \times \mathcal{B}$, $\{\varpi_n\} \in C_L(\mathcal{A}, \mathcal{F}, \nu_2)$, and $\{\nu_n\} \in C_R(\mathcal{B}, \mathcal{F}, \varpi_1)$. From equation (2.1) it follows that $\{\varpi_n\} \in C_L(\mathcal{A}, \mathcal{D}, \nu_2)$ and $\{\nu_n\} \in C_R(\mathcal{B}, \mathcal{D}sb, \varpi_1)$. Given that $(\mathcal{D})$ is a SBMS, by considering condition ($\mathcal{D}3$) of Definition 1.3 and equation (2.1), we obtain the following:

$$\begin{aligned}
\mathcal{F}(\varpi_1, \nu_2, t) &= \exp \frac{-\mathcal{D}(\varpi_1, \nu_2)}{t} \\
&\geqslant \exp - \frac{k \limsup_{n\to+\infty}[\mathcal{D}(\varpi_1, \nu_1) + \mathcal{D}(\varpi_n, \nu_1)]}{t}, k > 0 \\
&\geqslant \exp - \frac{a \limsup_{n\to+\infty}[\mathcal{D}(\varpi_1, \nu_1) + \mathcal{D}(\varpi_n, \nu_1)]}{t}, a \geqslant 1 \text{ and } a \geqslant k > 0 \\
&\geqslant \limsup_{n\to+\infty} \exp \frac{-[\mathcal{D}(\varpi_1, \nu_1) + \mathcal{D}(\varpi_n, \nu_1)]}{\frac{t}{a}} \geqslant \limsup_{n\to+\infty} \mathcal{F}(\varpi_1, \nu_1, \frac{t}{a}) * \mathcal{F}(\varpi_n, \nu_1, \frac{t}{a}).
\end{aligned}$$

And,

$$\begin{aligned}
\mathcal{F}(\varpi_1, \nu_2, t) &= \exp \frac{-\mathcal{D}(\varpi_1, \nu_2)}{t} \\
&\geqslant \exp - \frac{k \limsup_{n\to+\infty}[\mathcal{D}(\varpi_2, \nu_n) + \mathcal{D}(\varpi_2, \nu_2)]}{t}, k > 0 \\
&\geqslant \exp - \frac{a \limsup_{n\to+\infty}[\mathcal{D}(\varpi_2, \nu_n) + \mathcal{D}(\varpi_2, \nu_2)]}{t}, a \geqslant 1 \text{ and } a \geqslant k > 0 \\
&\geqslant \limsup_{n\to+\infty} \exp \frac{-[\mathcal{D}(\varpi_2, \nu_n) + \mathcal{D}(\varpi_2, \nu_2)]}{\frac{t}{a}} \geqslant \limsup_{n\to+\infty} \mathcal{F}(\varpi_2, \nu_n, \frac{t}{a}) * \mathcal{F}(\varpi_2, \nu_2, \frac{t}{a}).
\end{aligned}$$

□

Proposition 2.3. *A GFBMS is an extension of the FBMS $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$.*

Proof. It is evident that conditions ($\mathcal{F}1$), ($\mathcal{F}2$), ($\mathcal{F}3$), and ($\mathcal{F}5$) in Definition 2.1 are met. Let $\varpi_1, \varpi_2 \in \mathcal{A}, \nu_1, \nu_2 \in \mathcal{B}, \{\varpi_n\} \in C_L(\mathcal{A}, \mathcal{F}, \nu_2)$, and $\{\nu_n\} \in C_R(\mathcal{B}, \mathcal{F}, \varpi_1)$, then

$$\mathcal{F}(\varpi_1, \nu_2, t) \geqslant \mathcal{F}(\varpi_1, \nu_1, \frac{t}{3}) * \mathcal{F}(\varpi_n, \nu_1, \frac{t}{3}) * \mathcal{F}(\varpi_n, \nu_2, \frac{t}{3}),$$
$$\mathcal{F}(\varpi_1, \nu_2, t) \geqslant \mathcal{F}(\varpi_1, \nu_n, \frac{t}{3}) * \mathcal{F}(\varpi_2, \nu_n, \frac{t}{3}) * \mathcal{F}(\varpi_2, \nu_2, \frac{t}{3}), \text{ for all } n \geqslant 1, t > 0.$$

By taking $n \to +\infty$, it is seen that ($\mathcal{F}4$) in Definition 2.1 is satisfied for $a = 3$. □

Remark 2.4. A GFBMS may not always satisfy the conditions of being a FBM. The subsequent example supports our argument.

Example 2.5. Let $\mathcal{A} = \{0, 1, 2, \ldots\}, \mathcal{B} = \{0, 1, -2, \ldots\}$, and $\mathcal{F} : \mathcal{A} \times \mathcal{B} \times (0, +\infty) \to [0, 1]$ be defined by

$$\mathcal{F}(\varpi, \nu, t) = \begin{cases} 1, & \varpi = \nu = 0, \\ \exp^{\frac{-1}{t(-\nu+1)}}, & \varpi = 0, \nu \neq 0 \\ \exp\frac{-1}{t\varpi}, & \varpi \neq 0, \nu = 0, \\ \exp\frac{-20}{t}, & \varpi \neq 0, \nu \neq 0, \end{cases}$$

for every $\varpi \in \mathcal{A}, \nu \in \mathcal{B}$, and $t > 0$, where "$*$" is taken as product ctn. Thus, $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ constitutes a GFBMS; however, it does not qualify as a FBM.

Proof. The conditions ($\mathcal{F}1$), ($\mathcal{F}2$), ($\mathcal{F}3$), and ($\mathcal{F}5$) in Definition 2.1 are directly hold. Let $(\varpi_1, \nu_1), (, \varpi_2, \nu_2) \in \mathcal{A} \times \mathcal{B}, \{\varpi_n\} \in C_L(\mathcal{A}, \mathcal{F}, \nu_2)$ and $\{\nu_n\} \in C_R(\mathcal{B}, \mathcal{F}, \varpi_1)$. Now for condition ($\mathcal{F}4$) we have four cases.

Case I. If $\varpi_1 = \nu_2 = 0$, then the condition ($\mathcal{F}4$) is evidently content with any$\varpi_2 \in \mathcal{A}, \nu_1 \in \mathcal{B}$.

Case II. Let $\varpi_1 = 0, \nu_2 \neq 0$ and ν_1 be arbitrary . Let $\{\nu_n\} \in C_R(\mathcal{B}, \mathcal{F}, 0)$ such that $\nu_n \neq 0$ for all $n \in \mathbb{N}$. Then two subcases arise.

Subcase I. Let $\varpi_2 = 0$. Next, it can be observed that

$$\begin{aligned}&\limsup_{n\to+\infty}[\mathcal{F}(\varpi_2, \nu_n, t) * \mathcal{F}(\varpi_2, \nu_2, t)] \\ &\quad = \limsup_{n\to+\infty}[\mathcal{F}(0, \nu_n, t) * \mathcal{F}(0, \nu_2, t)] = 1 * \mathcal{F}(0, \nu_2, t) = \exp\frac{-1}{t(-\nu_2+1)} = \mathcal{F}(\varpi_1, \nu_2, at), \text{ so } a = 1.\end{aligned}$$

Subcase II. $\varpi_2 \neq 0, \varpi_2 \neq \nu_2$. Then we see that

$$\begin{aligned}&\limsup_{n\to+\infty}[\mathcal{F}(\varpi_2, \nu_n, t) * \mathcal{F}(\varpi_2, \nu_2, t)] \\ &\quad \leqslant \exp\frac{-20}{t} \cdot \exp\frac{-20}{t} \leqslant \exp\frac{-40}{t} \leqslant \exp\frac{-1}{t(-\nu_2+1)} = \mathcal{F}(\varpi_1, \nu_2, \tilde{a}t), \text{ where } a = 1.\end{aligned}$$

Case III. Let $\varpi_1 \neq 0, \nu_2 = 0$ and $\varpi_2 \in \mathcal{A}$ be arbitrary. Let $\varpi_n \in C_L(\mathcal{A}, \mathcal{F}, 0)$ such that $\varpi_n \neq 0$ for all $n \in \mathbb{N}$. Then we have two subcases arise.

Subcase I. If $\nu_1 = 0$, then

$$\begin{aligned}&\limsup_{n\to+\infty}[\mathcal{F}(\varpi_1, \nu_1 t) * \mathcal{F}(\varpi_n, \nu_1, t)] \\ &\quad = \limsup_{n\to+\infty}[\mathcal{F}(\varpi_1, 0, t) * \mathcal{F}(\varpi_n, 0, t)] \\ &\quad = \limsup_{n\to+\infty}[\mathcal{F}(\varpi_1, 0, t) * 1 = \limsup_{n\to+\infty}\mathcal{F}(\varpi_1, 0, t) = \exp\frac{-1}{t\varpi_1} = \mathcal{F}(\varpi_1, \nu_2, at), \text{ where } a = 1.\end{aligned}$$

Subcase II. If $\nu_1 \neq 0, \nu_1 \neq \varpi_1$,

$$\limsup_{n\to+\infty}[\mathcal{F}(\varpi_1,\nu_1,t) * \mathcal{F}(\varpi_n,\nu_1,t) \\ = \exp\frac{-20}{t}\cdot\exp\frac{-20}{t} = \exp\frac{-40}{t} \leqslant \exp\frac{-20}{t} \leqslant \exp\frac{-1}{t\varpi_1} \leqslant \mathcal{F}(\varpi_1,\nu_2,t), \text{ where } a = 1.$$

Case IV. If $\varpi_1, \nu_2 \neq 0$, then it is clear that condition $(\mathcal{F}4)$ is satisfied at a point $(\varpi_2, \nu_1) \in \mathcal{A}\times\mathcal{B}$. Since $C_L(\mathcal{A},\mathcal{F},\nu_2) = C_R(\mathcal{B},\mathcal{F},\varpi_1) = \phi$. Therefore, it can be deduced that $(\mathcal{A},\mathcal{B},\mathcal{F},*)$ forms a GFBMS. However, it does not qualify as a FBM. Clearly if we choose $\varpi_1,\nu_2 \neq 0, \varpi_1 \neq \nu_2$ such that $\varpi_1 = n, \nu_2 = -m, m, n \in \mathbb{N}, m \neq n$, and $\varpi_2 = \nu_1 = 0$,

$$\mathcal{F}(\varpi_1,\nu_1,\frac{t}{3}) * \mathcal{F}(\varpi_2,\nu_1,\frac{t}{3}) * \mathcal{F}(\varpi_2,\nu_2,\frac{t}{3}) \\ = \mathcal{F}(\varpi_1,\nu_1,\frac{t}{3}) * 1 * \mathcal{F}(\varpi_2,\nu_2,\frac{t}{3}) \\ = \exp\frac{-3}{t\varpi_1} * \exp\frac{-3}{t(-\nu_2+1)} = \exp\frac{-3}{tn}\cdot\exp\frac{-3}{t(m+1)} \to 1 \text{ as } m,n\to+\infty.$$

Also, $\mathcal{F}(\varpi_1,\nu_2,t) = \exp\frac{-20}{t} \leqslant 1$. Therefore, $(\mathcal{A},\mathcal{B},\mathcal{F},*)$ does not qualify as a FBMS. □

Proposition 2.6. *A FTCBMS $(\mathcal{A},\mathcal{B},\Upsilon,*)$ is also a FFBMS.*

Proof. Clearly conditions $(\mathcal{F}1)$, $(\mathcal{F}2)$, $(\mathcal{F}3)$, and $(\mathcal{F}5)$ in Definition 2.1 are satisfied. Let $\varpi_1,\varpi_2 \in \mathcal{A}, \varpi_1,\nu_2 \in \mathcal{B}$, $\{\varpi_n\} \in C_L(\mathcal{A},\mathcal{F},\nu_2)$ and $\{\nu_n\} \in C_R(\mathcal{B},\mathcal{F},\varpi_1)$, for all $t > 0$ we have by (M4) of Definition 1.1,

$$\mathcal{F}(\varpi_1,\nu_2,t) \geqslant \mathcal{F}(\varpi_1,\nu_1,\frac{t}{3\gamma(\varpi_1,\nu_1)}) * \mathcal{F}(\varpi_n,\nu_1,\frac{t}{3\alpha(\varpi_n,\nu_1)}) * \mathcal{F}(\varpi_n,\nu_2,\frac{t}{3\gamma(\varpi_n,\nu_2)}) \\ \geqslant \mathcal{F}(\varpi_1,\nu_1,\frac{t}{a}) * \mathcal{F}(\varpi_n,\nu_1,\frac{t}{a}) * \mathcal{F}(\varpi_n,\nu_2,\frac{t}{a})$$

and

$$\mathcal{F}(\varpi_1,\nu_2,t) \geqslant \mathcal{F}(\varpi_1,\nu_n,\frac{t}{3\gamma(\varpi_1,\nu_n)}) * \mathcal{F}(\varpi_2,\nu_n,\frac{t}{3\alpha(\varpi_2,\nu_n)}) * \mathcal{F}(\varpi_2,\nu_2,\frac{t}{3\gamma(\varpi_2,\nu_2)}) \\ \geqslant \mathcal{F}(\varpi_1,\nu_n,\frac{t}{a}) * \mathcal{F}(\varpi_2,\nu_n,\frac{t}{a}) * \mathcal{F}(\varpi_2,\nu_2,\frac{t}{a}), \text{ for all } n \geqslant 1, t > 0,$$

where,

$$a = \max\{\sup_{\varpi\in\mathfrak{A},\nu\in\mathcal{B}} 3\lambda(\varpi,\nu), \sup_{\varpi\in\mathcal{A},\nu\in\mathcal{B}} 3\alpha(\varpi,\nu), \sup_{\varpi\in\mathcal{A},\nu\in\mathcal{B}} 3\gamma(\varpi,\nu)\}.$$

According to $(\mathcal{M}6)$ in Definition 1.1, as $n \to +\infty$, it becomes evident that condition $(\mathcal{F}4)$ in Definition 2.1 is satisfied. □

Definition 2.7. Let $(\mathcal{A}_1,\mathcal{B}_1)$ and $(\mathcal{A}_2,\mathcal{B}_2)$ be two pairs of sets.

(i) The function $\Upsilon : (\mathcal{A}_1,\mathcal{B}_1) \rightrightarrows (\mathcal{A}_2,\mathcal{B}_2)$ is called a covariant mapping if $\Upsilon(\mathcal{A}_1) \subset \mathcal{A}_2$ and $\Upsilon(\mathcal{B}_1) \subset \mathcal{B}_2$.
(ii) The function $\Upsilon : (\mathcal{A}_1,\mathcal{B}_1) \rightleftharpoons (\mathcal{A}_2,\mathcal{B}_2)$ is called a contravariant mapping if $\Upsilon(\mathcal{A}_1) \subset \mathcal{B}_2$ and $\Upsilon(\mathcal{B}_1) \subset \mathcal{A}_2$.

Definition 2.8. Let $(\mathcal{A},\mathcal{B},\mathcal{F},*)$ be a GFBMS.

(i) A point $\varpi \in \mathcal{A}\cup\mathcal{B}$ is classified as a left point when $\varpi \in \mathcal{A}$, as a right point when $\varpi \in \mathcal{B}$, and as a central point when both conditions are met.
(ii) A sequence $\{\varpi_n\} \subseteq \mathcal{A}$ is called a left sequence and a sequence $\{\nu_n\} \subseteq \mathcal{B}$ is called a right sequence.

(iii) A sequence $\{\nu_n\} \subset \mathcal{A} \cup \mathcal{B}$ is said to converge to a point ν if and only if $\{\nu_n\}$ is a left sequence, ν is a right point and for every $\epsilon \in (0,1)$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mathcal{F}(\nu_n, \nu, t) > 1 - \epsilon$ for every $n \geqslant n_0$, i.e., $\mathcal{F}(\nu_n, \nu, t) \to 1$ as $n \to +\infty$ or $\{\nu_n\}$ is a right sequence, ν is a left point and for every $\epsilon \in (0,1)$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mathcal{F}(\nu, \nu_n, t) > 1 - \epsilon$ for every $n \geqslant n_0$, i.e., $\mathcal{F}(\nu, \nu_n, t) \to 1$ as $n \to +\infty$.

Definition 2.9. Let $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ be a GFBMS, then

(i) A sequence $\{(\varpi_n, \nu_n)\}$ is referred to as a bisequence.
(ii) A sequence $(\varpi_n, \nu_n) \subseteq \mathcal{A} \times \mathcal{B}$ is considered convergent if both $\{\varpi_n\}$ and $\{\nu_n\}$ converge to the same central point. In this case, the sequence $\{(\varpi_n, \nu_n)\}$ is termed a biconvergent sequence.
(iii) A bisequence $\{(\varpi_n, \nu_n)\}$ be defined as a Cauchy sequenceIf for every $\epsilon \in (0,1)$ and $t > 0$, there exists a $n_0 \in \mathbb{N}$ such that $\mathcal{F}(\varpi_n, \nu_m, t) > 1 - \epsilon$ for all $n, m \geqslant n_0$. In other words, a bisequence $\{(\varpi_n, \nu_n)\}$ is considered a Cauchy bisequence if $\mathcal{F}(\varpi_n, \nu_m, t) \to 1$ as $n, m \to +\infty$.

Definition 2.10. A GFBMS is considered complete when every Cauchy bisequence converges.

Proposition 2.11. *Consider A GFBMS $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$. If $\mathcal{F}(\varpi, \varpi, t) = 1$ for $t > 0$, then the sequence $\{(\varpi_n, \nu_n)\}$ converges to a unique limit ϖ, which is a central point.*

Proof. Let $\{\varpi_n\}$ be a left sequence in $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ which converges to some $\varpi \in \mathcal{A} \cap \mathcal{B}$ with $\mathcal{F}(\varpi, \varpi, t) = 1$ for all $t > 0$. If $\nu \in \mathcal{B}$ is a limit of this sequence, then we get

$$\mathcal{F}(\varpi, \nu, t) \geqslant \limsup_{n \to +\infty} (\mathcal{F}(\varpi, \varpi, \frac{t}{a}) * \mathcal{F}(\varpi, \varpi_n, \frac{t}{a})), \quad a \geqslant 1 = 1 * 1 = 1. \tag{2.2}$$

Consequently, (2.2) demonstrates that $\mathcal{F}(\varpi, \nu, t)$, meaning that $\varpi = \nu$. Therefore, ϖ is the unique limit of the sequence $\{\varpi_n\}$. Likewise, if $\{\nu_n\}$ denotes a right sequence in $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$. □

Proposition 2.12. *Every Cauchy bisequence in a GFBMS $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ that converges also biconverges.*

Proof. Let $\{(\varpi_n, \nu_n)\}$ be a Cauchy sequence that converges to $(\varpi, \nu) \in \mathcal{A} \times \mathcal{B}$. Then, we have

$$\mathcal{F}(\varpi, \nu, t) \geqslant \limsup_{m \to +\infty} [\mathcal{F}(\varpi, \nu_m, \frac{t}{a}) * \mathcal{F}(\varpi_m, \nu_m, \frac{t}{a})], \quad a \geqslant 1 \text{ and } t > 0 = 1 * 1 = 1.$$

Thus, we obtain $\mathcal{F}(\varpi, \nu, t) = 1$, which implies $\varpi \in \mathcal{A} \cap \mathcal{B}$. □

Remark 2.13. From Proposition 2.12, we can deduce that if a Cauchy bisequence biconverges to some $\varpi \in \mathcal{A} \times \mathcal{B}$, then $\mathcal{F}(\varpi, \varpi, t) = 1$.

3. FFPT results

This section presents the proof of several fuzzy fixed point theorems within the framework of a GFBMS.

Definition 3.1. Let $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ be a GFBMS. A mapping $\Upsilon : (\mathcal{A}, \mathcal{B}, \mathcal{F}, *) \rightrightarrows (\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ is a generalized fuzzy bipolar (GFB) α-contraction of type-I if for all $(\varpi, \nu) \in \mathcal{A} \times \mathcal{B}, 0 < \alpha < 1$, the following inequality holds for all $t > 0$

$$\mathcal{F}(\Upsilon\varpi, \Upsilon\nu, \alpha t) \geqslant \mathcal{F}(\varpi, \nu, t). \tag{3.1}$$

Theorem 3.2. *Let $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ be a complete GFBMS and $\Upsilon : \mathcal{A} \cup \mathcal{B} \rightrightarrows \mathcal{A} \cup \mathcal{B}$ be a GFB α-contraction of type-I. If there exists $\omega = (\varpi_0, \nu_0)$ in $\mathcal{A} \times \mathcal{B}$, for all $t > 0$, $\sigma_t(\mathcal{F}, \Upsilon, \omega) = \inf_{s,r \in \mathbb{N}} (\Upsilon^r \varpi_0, \Upsilon^s \nu_0, t)\} > 0$. Then $\mathrm{Fix}_\Upsilon = \{\varpi\}$, $\varpi \in \mathcal{A} \cap \mathcal{B}$ (Fix_Υ is defined as the set of fixed points of Υ).*

Proof. Let $\omega = (\varpi_0, \nu_0)$ be any point in $\mathcal{A} \times \mathcal{B}$ that satisfies for any $t > 0$,

$$\sigma_t(\mathcal{F}, \Upsilon, \omega) = \inf_{s,r\in\mathbb{N}} \{\mathcal{F}(\Upsilon^r \varpi_0, \Upsilon^s \nu_0, t)\} > 0.$$

Since Υ is a covariant mapping, we have $\varpi_n = \Upsilon^n \varpi_0, \nu_n = \Upsilon^n \nu_0$ and

$$\sigma_t(\mathcal{F}, \Upsilon^{\kappa+1}, \omega) = \inf_{s,r\in\mathbb{N}} \{\mathcal{F}(\Upsilon^{\kappa+r} \varpi_0, \Upsilon^{\kappa+s} \nu_0, t)\},$$

for every constant $\kappa = 0, 1, 2 \dots$, we get that

$$\{\mathcal{F}(\Upsilon^{\kappa+r} \varpi_0, \Upsilon^{\kappa+r} \nu_0, t) : r, s \geqslant 1\} \subseteq \{\mathcal{F}(\Upsilon^r \varpi_0, \Upsilon^s \nu_0, t) : r, s \geqslant 1\},$$

which implies that

$$\sigma_t(\mathcal{F}, \Upsilon^{\kappa+1}, (\varpi_0, \nu_0)) \geqslant \sigma_t(\mathcal{F}, \Upsilon, (\varpi_0, \nu_0)) > 0. \tag{3.2}$$

For every $r, s, n \in \mathbb{N}$,

$$\mathcal{F}(\Upsilon^{n+r} \varpi_0, \Upsilon^{n+s} \nu_0, t) \geqslant \mathcal{F}(\Upsilon^{n+r-1} \varpi_{\bar{0},} \Upsilon^{n+s-1} \nu_0, \frac{t}{\alpha}) \geqslant \sigma_{\frac{t}{\alpha}}(\mathcal{F}, \Upsilon^n, \omega). \tag{3.3}$$

It can be deduced from (3.2) and (3.3) that

$$\mathcal{F}(\Upsilon^{n+r} \varpi_0, \Upsilon^{n+s} \nu_0, t) \geqslant \sigma_t(\mathcal{F}, \Upsilon^{n+1}, \omega) \geqslant \sigma_{\frac{t}{\alpha}}(\mathcal{F}, \Upsilon^n, \omega) \geqslant \sigma_{\frac{t}{\alpha^2}}(\mathcal{F}, \Upsilon^{n-1}, \omega) \geqslant \dots \geqslant \sigma_{\frac{t}{\alpha^n}}(\mathcal{F}, \Upsilon, \omega). \tag{3.4}$$

Therefore, for every $1 \leqslant n < m$ we use (3.4) to obtain,

$$\mathcal{F}(\varpi_n, \nu_m, t) = \mathcal{F}(\Upsilon^n \varpi_0, \Upsilon^m \nu_0, t) \geqslant \sigma_{\frac{t}{\alpha^n}}(\mathcal{F}, \Upsilon^n, \omega) \to 1 \text{ as } n \to +\infty.$$

Because of the condition $\sigma_t(\mathcal{F}, \Upsilon^n, \omega) > 0$ for all $t > 0$ and $\alpha \in (0, 1)$, as outlined in ($\mathcal{F}$5) within Definition 2.1. Therefore, we obtain $\lim_{n\to+\infty} \mathcal{F}(\varpi_n, \nu_m, t) = 1$. Hence, $\{(\varpi_n, \nu_m)\}$ forms a Cauchy sequence. Given that $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ is complete, this sequence converges. Consequently, according to Remark 2.13, it biconverges to some $\varpi \in \mathcal{A} \cap \mathcal{B}$ such that $\mathcal{F}(\varpi, \varpi, t) = 1$. Now

$$\mathcal{F}(\varpi_{n+1}, \Upsilon\varpi, t) \geqslant \mathcal{F}(\varpi_n, \varpi, \frac{t}{\alpha}) \to 1 \text{ as } n \to +\infty.$$

Since $\{\varpi_n\}$ converges to the central limit $\varpi \in \mathcal{A} \cap \mathcal{B}$ with $\mathcal{F}(\varpi, \varpi, t) = 1$, then by Proposition 2.11 we get $\Upsilon\varpi = \varpi$. Now, suppose that $\vartheta \in \mathcal{A}$ is an additional fixed point of Υ with $\mathcal{F}(\varpi, \vartheta, t) > 0$. By (3.1), it can be observed that

$$\mathcal{F}(\vartheta, \varpi, t) \geqslant \mathcal{F}(\Upsilon\vartheta, \Upsilon\varpi, \frac{t}{\alpha}) = \mathcal{F}(\vartheta, \varpi, \frac{t}{\alpha}) \geqslant \mathcal{F}(\Upsilon\vartheta, \Upsilon\varpi, \frac{t}{\alpha^2}) \geqslant \dots \geqslant \mathcal{F}(\Upsilon\vartheta, \Upsilon\varpi, \frac{t}{\alpha^n}) \to 1 \text{ as } n \to +\infty.$$

Then, $\vartheta = \varpi$. The same result applies when ϑ belongs to the set $\mathcal{B}$. □

Example 3.3. Let $\mathcal{A} = \{(\varpi, 0) \in \mathbb{R}^2 : 0 \leqslant \varpi \leqslant 1\}$ and $\mathcal{B} = \{(0, \nu) \in \mathbb{R}^2 : 0 \leqslant \nu \leqslant 1\}$. Define

$$\mathcal{F}((\varpi, 0), (0, \nu), t) = \frac{t}{t + |\varpi + \nu|},$$

for all $t > 0, (\varpi, 0) \in \mathcal{A}, (0, \nu) \in \mathcal{B}$. Clearly, $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ is a complete GFBMS, where $*$ is a product ctn. Let $\Upsilon : \mathcal{A} \cup \mathcal{B} \rightrightarrows \mathcal{A} \cup \mathcal{B}$ be defined by

$$\begin{cases} \Upsilon(\varpi, 0) = (\frac{\varpi}{2}, 0), \\ \Upsilon(0, \nu) = (0, \frac{\nu}{2}), \end{cases}$$

for all $(\varpi, 0) \in \mathcal{A}$ and $(0, \nu) \in \mathcal{B}$.

Proof. Υ is a GFB α-contraction of type-I since $\Upsilon(\mathcal{A}) \subseteq \mathcal{A}, \Upsilon(\mathcal{B}) \subseteq \mathcal{B}$ so it is a covariant mapping and contraction, where $\alpha = \frac{1}{2} \in (0,1)$. To clear it, suppose that

$$\begin{aligned}\mathcal{F}(\Upsilon(\varpi,0),\Upsilon(0,\nu),\alpha t) &= \mathcal{F}(\frac{\varpi}{2},0),(0,\frac{\nu}{2}),\alpha t)\\ &= \frac{\alpha t}{\alpha t + |\frac{\varpi}{2}+\frac{\nu}{2}|} = \frac{\alpha t}{\alpha t + \frac{1}{2}|\varpi+\nu|} = \frac{t}{t+|\varpi+\nu|},\quad \alpha=\frac{1}{2} \geqslant \mathcal{F}(\varpi,\nu,t),\quad \forall t>0,\end{aligned}$$

by constructing the bisequences $\varpi_{n+1} = \varpi_n$ and $\nu_{n+1} = \nu_n$ for all $n \in \mathbb{N} \cup \{0\}$ by taking some $(0,0) \neq (\varpi_0,0) \in \mathcal{A}, (0,0) \neq (0,\nu_0) \in \mathcal{B}$, a non-trivial bisequence is derived as

$$(\varpi_n,\nu_n) = \{((\varpi_0,0),(0,\nu_0)),((\frac{\varpi_0}{2},0),(0,\frac{\nu_0}{2})),((\frac{\varpi_0}{2^2},0),(\frac{\nu_0}{2^2},0)),\ldots\},$$

which implies that for any $t > 0$,

$$\sigma(\mathcal{F},\Upsilon,(\varpi_0,\nu_0))_t = \inf_{s,r\in\mathbb{N}}\{\mathcal{F}(\Upsilon^s\varpi_0,\Upsilon^r\nu_0,t)\} = \inf_{s,r\in\mathbb{N}}\{\frac{t}{t+|\frac{\varpi_0}{2^{s-1}}+\frac{\nu_0}{2^{r-1}}|},t>0\} > 0.$$

By Theorem 3.2, we get $\text{Fix}_\Upsilon = \{(0,0)\}$. □

Definition 3.4. Let $(\mathcal{A},\mathcal{B},\mathcal{F},*)$ be a GFBMS. A contravariant mapping $\Upsilon : (\mathcal{A},\mathcal{B},\mathcal{V},*) \rightleftharpoons (\mathcal{A},\mathcal{B},\mathcal{F},*)$ is a GFB α-contraction of type-II if for all $(\varpi,\nu) \in \mathcal{A}\times\mathcal{B}$ and for some $\alpha \in (0,1)$ and for all $t > 0$, we have

$$\mathcal{F}(\Upsilon\varpi,\Upsilon\nu,\alpha t) \geqslant \mathcal{F}(\nu,\varpi,t).$$

Theorem 3.5. *Let $(\mathcal{A},\mathcal{B},\mathcal{F},*)$ be a complete GFBMS and Υ be a GFB α-contraction of type-II. If there exists $(\varpi_0,\Upsilon\varpi_0) \in \mathcal{A}\times\mathcal{B}$ such that for all $t > 0$,*

$$\sigma_t(\mathcal{F},\Upsilon,(\varpi_0,\Upsilon\varpi_0)) = \inf_{s,r\in\mathbb{N}}\{\mathcal{F}(\varpi_r,\varpi_s,t)\} > 0,$$

where $\nu_n = \Upsilon\varpi_n, \varpi_{n+1} = \Upsilon\nu_n, n \geqslant 0$, then $\{(\varpi_n,\nu_n)\}$ biconverges to some $\varpi \in \mathcal{A}\cap\mathcal{B}$ with $\mathcal{F}(\varpi,\varpi,t) = 1$ and $\text{Fix}_\Upsilon = \{\varpi\}$, $\varpi \in \mathcal{A}\cap\mathcal{B}$.

Proof. Let us denote

$$\sigma_t(\mathcal{F},\Upsilon^{\kappa+1},(\varpi_0,\Upsilon\varpi_0)) = \inf_{s,r\in\mathbb{N}}\{\mathcal{F}(\varpi_{s+\kappa},\nu_{r+\kappa},t) : t>0\}$$

for all $\kappa = 0,1,2,\ldots$. Let $(\varpi_0,\Upsilon\varpi_0) \in \mathcal{A}\times\mathcal{B}$ such that for any $t > 0$ it holds that

$$\sigma_t(\mathcal{F},\Upsilon,(\varpi_0,\Upsilon\varpi_0)) = \inf_{r,s,\in\mathbb{N}}\{\mathcal{F}(\Upsilon^s\varpi_0,\Upsilon^r(\Upsilon\varpi_0),t)\} > 0.$$

Given that Υ is a contravariant function, we can conclude that $\nu_n = \Upsilon\varpi_n$, $\varpi_{n+1} = \Upsilon\nu_n$,

$$\{\mathcal{F}(\Upsilon^{\kappa+r}\varpi_0,\Upsilon^{\kappa+r}(\Upsilon\varpi_0),t) : r,s \geqslant 1\} \subseteq \{\mathcal{F}(\Upsilon^r\varpi_0,\Upsilon^s(\Upsilon\varpi_0),t) : r,s \geqslant 1\},$$

for all $t > 0$, which implies that

$$\sigma_t(\mathcal{F},\Upsilon^{\kappa+1},(\varpi_0,\Upsilon\varpi_0)) \geqslant \sigma_t(\mathcal{F},\Upsilon,(\varpi_0,\Upsilon\varpi_0)) > 0. \tag{3.5}$$

Now, for all $s,r \in \mathbb{N}$,

$$\begin{aligned}\mathcal{F}(\varpi_{n+s},\nu_{n+r},t) &= \mathcal{F}(\Upsilon\nu_{n+s-1},\Upsilon\varpi_{n+r},t)\\ &\geqslant \mathcal{F}(\varpi_{n+s},\nu_{n+r-1},\frac{t}{\alpha}) \geqslant \sigma_{\frac{t}{\alpha}}(\mathcal{F},\Upsilon^n,(\varpi_0,\Upsilon\varpi_0)) \text{ for all } n \geqslant 1.\end{aligned} \tag{3.6}$$

From (3.5) and (3.6) it follows that for any $n \geqslant 1, t > 0$,

$$\begin{aligned} \mathcal{F}(\Upsilon^{n+s}(\varpi_0), \Upsilon^{n+r}(\Upsilon\varpi_0), t) &\geqslant \sigma_t(\mathcal{F}, \Upsilon^{n+1}, (\varpi_0, \Upsilon\varpi_0)) \\ &\geqslant \sigma_{\frac{t}{\alpha}}(\mathcal{F}, \Upsilon^{n}, (\varpi_0, \Upsilon\varpi_0)) \\ &\geqslant \sigma_{\frac{t}{\alpha^n}}(\mathcal{F}, \Upsilon^{n-1}, (\varpi_0, \Upsilon\varpi_0)) \geqslant \cdots \geqslant \sigma_{\frac{t}{\alpha^n}}(\mathcal{F}, \Upsilon, (\varpi_0, \Upsilon\varpi_0)). \end{aligned} \tag{3.7}$$

Therefore, for every $1 \leqslant n < m$, we use (3.7) to obtain

$$\mathcal{F}(\varpi_n, \nu_m, t) = \mathcal{F}(\Upsilon^n\varpi_0, \Upsilon^m\varpi_0, t) \geqslant \sigma_t(\mathcal{F}, \Upsilon^n, (\varpi_0, \Upsilon\varpi_0)).$$

Since $\sigma_t(\mathcal{F}, \Upsilon^n, (\varpi_0, \Upsilon\varpi_0)) > 0,$ for all $t > 0, \alpha \in (0, 1)$, by ($\mathcal{F}$5) in Definition 2.1 we get

$$\lim_{n\to+\infty} \mathcal{F}(\varpi_n, \nu_m, t) = 1.$$

Therefore, $\{(\varpi_n, \nu_m)\}$ is a Cauchy bisequence. Since $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ be a complete, this sequence converges and thus by Proposition 2.12, biconverges to some $\varpi \in \mathcal{A} \cap \mathcal{B}$ such that $\mathcal{F}(\varpi, \varpi, t) = 1$. Now $\mathcal{F}(\varpi_{n+1}, \Upsilon\varpi, t) = \mathcal{F}(\Upsilon\nu_n, \Upsilon\varpi, t) \geqslant \mathcal{F}(\varpi, \nu_n, \frac{t}{\alpha}) \to 1$ as $n \to +\infty$. Since $\{\varpi_n\}$, and $\{\nu_n\}$ converge to the central limit $\varpi \in \mathcal{A} \cap \mathcal{B}$ with $\mathcal{F}(\varpi, \varpi, t) = 1$, then by Proposition 2.11 we get $\Upsilon\varpi = \varpi$ and ϖ is a fixed point of Υ.

Let $\theta \in \mathcal{A}$ represent another fixed point of Υ such that $\mathcal{F}(\varpi, \theta, t) > 0$. Consequently, according to the contraction condition (3.1), it can be observed that

$$\begin{aligned} \mathcal{F}(\theta, \varpi, t) = \mathcal{F}(\Upsilon\theta, \Upsilon\varpi, t) \geqslant \mathcal{F}(\varpi, \theta, \frac{t}{\alpha}) = \mathcal{F}(\Upsilon\varpi, \Upsilon\theta, \frac{t}{\alpha}) \\ \geqslant \mathcal{F}(\theta, \varpi, \frac{t}{\alpha^2}) \geqslant \cdots \geqslant \mathcal{F}(\Upsilon\theta, \Upsilon\varpi, \frac{t}{\alpha^n}) \to 1 \text{ as } n \to +\infty. \end{aligned}$$

Then $\theta = \varpi$. The same conclusion applies for any θ belonging to $\mathcal{B}$. □

Example 3.6. Let $\mathcal{A} = \{(\varpi, \nu) \in \mathbb{R}^2 : \varpi \geqslant 0 \text{ and } \nu \geqslant 0\} \cup \{(-2, -2)\}, \mathcal{B} = \{(\varpi, \nu) \in \mathbb{R}^2 : \varpi \leqslant 0 \text{ and } \nu \leqslant 0\}$. Define for any $t > 0, (\varpi_1, \nu_1) \in \mathcal{A}, (\varpi_2, \nu_2) \in \mathcal{B}$,

$$\mathcal{F}((\varpi_1, \nu_1), (\varpi_2, \nu_2), t) = \frac{t}{t + |\varpi_1 - \varpi_2| + |\nu_1 - \nu_2|}.$$

Clearly, $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ is a complete GFBMS, where $*$ is a product ctn. Let $\Upsilon : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be defined by $\Upsilon(\varpi, \nu) = (\frac{-\varpi}{2}, \frac{-\nu}{2})$, for all $(\varpi, \nu) \in \mathcal{A} \times \mathcal{B}$.

Proof. Υ is a GFB α-contraction of type-II since $\Upsilon(\mathcal{A}) \subseteq \mathcal{B}, \Upsilon(\mathcal{B}) \subseteq \mathcal{A}$, so it is a contravariant mapping and contraction, where $\alpha = \frac{1}{2} \in (0, 1)$. To clear it, suppose that, with $t > 0$,

$$\begin{aligned} \mathcal{F}(\Upsilon(\varpi_2, \nu_2), \Upsilon(\varpi_1, \nu_1), \alpha t) &= \mathcal{F}((\frac{-\varpi_2}{2}, \frac{-\nu_2}{2}), (\frac{-\varpi_1}{2}, -\frac{\nu_1}{2}), \alpha t) \\ &= \frac{\alpha t}{\alpha t + \frac{1}{2}(|-\varpi_2 + \varpi_1| + |-\nu_2 + \nu_1|)} \\ &= \frac{\alpha t}{\alpha t + \frac{1}{2}(|\varpi_1 - \varpi_2| + |\nu_1 - \nu_2|)} \\ &= \frac{t}{t + (|\varpi_1 - \varpi_2| + |\nu_1 - \nu_2|)} \geqslant \mathcal{F}((\varpi_2, \nu_2), (\varpi_1, \nu_1), t). \end{aligned}$$

Now, non-trivial bisequences can be constructed as follows.

$\varpi_{n+1} = \varpi_n$ and $\nu_{n+1} = \nu_n$ for all $n \in \mathbb{N} \cup \{0\}$ by taking some $(0, 0) \neq (\varpi_0, \Upsilon\varpi_0) \in \mathcal{A} \times \mathcal{B}$, i.e.,

$$(\varpi_n, \nu_n) = \{((\varpi_0, \nu_0), (\frac{-\varpi_0}{2}, \frac{-\nu_0}{2})), ((\frac{\varpi_0}{2^2}, \frac{\nu_0}{2^2}), (\frac{-\varpi_0}{2^3}, \frac{-\nu_0}{2^3})), \ldots\},$$

which implies that for any fixed $t > 0$,

$$\sigma_t(\mathcal{F}, \Upsilon, (\varpi_0, \nu_0)) = \varpi\{\mathcal{F}(\Upsilon^s\varpi_0, \Upsilon^r\nu_0, t)\} = \inf_{s,r\in\mathbb{N}}\{\frac{t}{t + |\frac{(-1)^{s+1}\varpi_0}{2^{s-1}} - \frac{(-1)^r\varpi_0}{2^r}| + |\frac{(-1)^{s+1}\nu_0}{2^{s-1}} - \frac{(-1)^r\nu_0}{2^r}|}\} > 0.$$

According to Theorem 3.2, it follows that $\mathrm{Fix}_\Upsilon = \{(0, 0\}$. □

Remark 3.7. The fixed point theorems for α-contractions of type-I and type-II can be derived from Theorem 3.2 and Theorem 3.5, similar to the observation made in Proposition 2.3. These fixed-point theorems can be utilized within the framework of FBMSs.

Definition 3.8. Let $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ be a GFBMS. A covariant mapping $\Upsilon : (\mathcal{A}, \mathcal{B}, \mathcal{F}, *) \rightrightarrows (\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ is a GFB α-quasi contraction of type-I for every $(\varpi, \nu) \in \mathcal{A} \times \mathcal{B}$, for any $t > 0$, and for a certain $\alpha \in (0, 1)$, it holds true that

$$\mathcal{F}(\Upsilon\varpi, \Upsilon\nu, \alpha t) \geqslant \min\{\mathcal{F}(\varpi, \nu, t), \mathcal{F}(\varpi, \Upsilon\nu, t), \mathcal{F}(\nu, \Upsilon\varpi, t)\}. \tag{3.8}$$

Theorem 3.9. *Let $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ be a complete GFBMS and $\Upsilon : (\mathcal{A}, \mathcal{B}, \mathcal{F}, *) \rightrightarrows (\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ be a GFB α-quasi contraction of type-I. If there exists $(\varpi_0, \nu_0) \in \mathcal{A} \times \mathcal{B}, \forall t > 0$, $\sigma_t(\mathcal{F}, \Upsilon, (\varpi_0, \nu_0)) = \inf_{s,r\in\mathbb{N}}\{\mathcal{F}(\Upsilon^s\varpi_0, \Upsilon^r\nu_0, t)\} > 0$, then $\mathrm{Fix}_\Upsilon = \{\varpi\}$, $\varpi \in \mathcal{A} \cap \mathcal{B}$.*

Proof. Let $(\varpi_0, \nu_0) \in \mathcal{A} \times \mathcal{B}$ satisfies for any $t > 0$,

$$\sigma_t(\mathcal{F}, \Upsilon, (\varpi_0, \nu_0)) = \inf_{s,r\in\mathbb{N}}\{\mathcal{F}(\Upsilon^s\varpi_0, \Upsilon^r\nu_0, t)\} > 0.$$

Since Υ is a covariant mapping, so we have $\varpi_n = \Upsilon^n\varpi_0$,$\nu_n\Upsilon = \Upsilon^n\nu_0$ and

$$\sigma_t(\mathcal{F}, \Upsilon^{\kappa+1}, (\varpi_0, \nu_0)) = \inf_{s,r\in\mathbb{N}}\{\mathcal{F}(\Upsilon^{\kappa+s}\varpi_0, \Upsilon^{\kappa+r}\nu_0, t), t > 0\},$$

for every constant $\kappa = 0, 1, 2, \ldots$, we obtain

$$\{\mathcal{F}(\Upsilon^{\kappa+s}\varpi_0, \Upsilon^{\kappa+r}\nu_0, t) : s, r \geqslant 1, t > 0\} \subseteq \{\mathcal{F}(\Upsilon^s\varpi_0, \Upsilon^r\nu_0, t) : s, r \in \mathbb{N}, t > 0\},$$

which implies that

$$\sigma_t(\mathcal{F}, \Upsilon^{\kappa+1}, (\varpi_0, \nu_0)) \geqslant \sigma_t(\mathcal{F}, \Upsilon, (\varpi_0, \nu_0)) > 0. \tag{3.9}$$

Now, for all $s, r \in \mathbb{N}$,

$$\begin{aligned}\mathcal{F}(\Upsilon^{n+s}\varpi_0, \Upsilon^{n+r}\nu_0, t)\} \geqslant \min\{&\mathcal{F}(\Upsilon^{n+s-1}\varpi_0, \Upsilon^{n+r-1}\nu_0, \frac{t}{\alpha}), \mathcal{F}(\Upsilon^{n+s-1}\varpi_0, \Upsilon^{n+r}\nu_0, \frac{t}{\alpha}),\\ &\mathcal{F}(\Upsilon^{n+s-1}\nu_0, \Upsilon^{n+r}\varpi_0, \frac{t}{\alpha})\}.\end{aligned}$$

For every $s, r \geqslant 1$, it holds that

$$\begin{aligned}&\min\{\mathcal{F}(\Upsilon^{n+s}\varpi_0, \Upsilon^{n+r}\nu_0, t)\}\\ &\quad\geqslant \min\{\min\{\mathcal{F}(\Upsilon^{n+s-1}\varpi_0, \Upsilon^{n+r-1}\nu_0, \frac{t}{\alpha}), \mathcal{F}(\Upsilon^{n+s-1}\varpi_0, \Upsilon^{n+r}\nu_0, \frac{t}{\alpha}), \mathcal{F}(\Upsilon^{n+s-1}\nu_0, \Upsilon^{n+r}\varpi_0, \frac{t}{\alpha})\}\}\\ &\Rightarrow \sigma_t(\mathcal{F}, \Upsilon^{n+1}, (\varpi_0, \nu_0)) \geqslant \min\{\sigma_{\frac{t}{\alpha}}(\mathcal{F}, \Upsilon^n, (\varpi_0, \nu_0))\sigma_{\frac{t}{\alpha}}(\mathcal{F}, \Upsilon^n, (\varpi_0, \Upsilon\nu_0))\sigma_{\frac{t}{\alpha}}(\mathcal{F}, \Upsilon^n, (\nu_0, \Upsilon\varpi_0))\}.\end{aligned} \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\sigma_t(\mathcal{F}, \Upsilon^{n+1}, (\varpi_0, \nu_0)) \geqslant \sigma_{\frac{t}{\alpha}}(\mathcal{F}, \Upsilon^n, (\varpi_0, \nu_0)).$$

It follows that for all $n \geqslant 1, t > 0$,

$$\begin{aligned}\mathcal{F}(\Upsilon^{n+s}\varpi_0, \Upsilon^{n+r}\nu_0, t) &\geqslant \sigma_t(\mathcal{F}, \Upsilon^{n+1}, (\varpi_0, \nu_0))\\ &\geqslant \sigma_{\frac{t}{\alpha}}(\mathcal{F}, \Upsilon^n, (\varpi_0, \nu_0)) \geqslant \sigma_{\frac{t}{\alpha^2}}(\mathcal{F}, \Upsilon^{n-1}, (\varpi_0, \nu_0)) \geqslant \cdots \geqslant \sigma_{\frac{t}{\alpha^n}}(\mathcal{F}, \Upsilon, (\varpi_0, \nu_0)).\end{aligned} \tag{3.11}$$

Therefore, for every $1 \leqslant n < m$ we use (3.11) to obtain

$$\mathcal{F}(\varpi_n, \nu_m, t) = \mathcal{F}\Upsilon^n\varpi_0, \Upsilon^m\nu_0, t) \geqslant \sigma_t(\mathcal{F}, \Upsilon^n, (\varpi_0, \nu_0)) \to 1 \text{ as } n \to +\infty,$$

since $\sigma_t(\mathcal{F}, \Upsilon^n, (\varpi_0, \nu_0)) > 0,$ for all $t > 0, \alpha \in (0,1)$, by ($\mathcal{F}$4) in Definition 2.1 we get

$$\lim_{n\to+\infty} \mathcal{F}(\varpi_n, \nu_m, t) = 1.$$

Hence, the sequence $\{(\varpi_n, \nu_m)\}$ is a Cauchy bisequence. Given that $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ is complete, this sequence converges. Therefore, according to Proposition 2.12, it biconverges to a certain $\zeta \in \mathcal{A} \cap \mathcal{B}$ such that $\mathcal{F}(\zeta, \zeta, t) = 1$. Now

$$\begin{aligned}
\mathcal{F}(\varpi_{n+1}, \Upsilon\zeta, t) &\geqslant \min\{\mathcal{F}(\varpi_n, \zeta, \frac{t}{\alpha}), \mathcal{F}(\varpi_n, \Upsilon\zeta, \frac{t}{\alpha}), \mathcal{F}(\zeta, \Upsilon\varpi_n, \frac{t}{\alpha})\} \\
&= \min\{\mathcal{F}(\varpi_n, \zeta, \frac{t}{\alpha}), \mathcal{F}(\varpi_n, \Upsilon\zeta, \frac{t}{\alpha}), \mathcal{F}(\zeta, \varpi_{n+1}, \frac{t}{\alpha})\} \\
&= \min\{1, \mathcal{F}(\varpi_n, \Upsilon\zeta, \frac{t}{\alpha}), 1)\} \\
&= \mathcal{F}(\varpi_n, \Upsilon\zeta, \frac{t}{\alpha}) \\
&\geqslant \cdots \geqslant \min\{\mathcal{F}(\varpi_0, \zeta, \frac{t}{\alpha^{n+1}}), \mathcal{F}(\varpi_0, \Upsilon\zeta, \frac{t}{\alpha^{n+1}}), \mathcal{F}(\zeta, \Upsilon\varpi_0, \frac{t}{\alpha^{n+1}})\} \to 1 \text{ as } n \to +\infty.
\end{aligned}$$

Also we have by ($\mathcal{F}$4) of Definition 2.1,

$$\mathcal{F}(\zeta, \Upsilon\zeta, t) \geqslant \mathcal{F}(\varpi_{m+1}, \Upsilon\zeta, \frac{t}{a}) * \mathcal{F}(\varpi_{m+1}, \nu_n, \frac{t}{a}), \ a \geqslant 1.$$

Letting $n \to +\infty$ we get $\mathcal{F}(\zeta, \Upsilon\zeta, t) \geqslant 1 * 1$. So by ($\mathcal{F}$2) of Definition 2.1 we have $\zeta = \Upsilon\zeta$. Let $\theta \in \mathcal{B}$ satisfies $\Upsilon\theta = \theta$ such that $\mathcal{F}(\zeta, \theta, t) > 0$. Then, due to the (3.8) it is evident that

$$\begin{aligned}
\mathcal{F}(\theta, \zeta, t) &\geqslant \min\{\mathcal{F}(\theta, \zeta, \frac{t}{\alpha}), \mathcal{F}(\Upsilon\theta, \zeta, \frac{t}{\alpha}), \mathcal{F}(\theta, \Upsilon\zeta, \frac{t}{\alpha})\} \\
&= \min\{\mathcal{F}(\theta, \zeta, \frac{t}{\alpha}) \geqslant \cdots \geqslant \mathcal{F}(\theta, \zeta, \frac{t}{\alpha^n}) \to 1 \text{ as } n \to +\infty.
\end{aligned}$$

That is $\theta = \zeta$. The same conclusion applies whenever $\theta \in \mathcal{A}$. □

Example 3.10. Let $\mathcal{A} = \{U = (u_1, u_2, \ldots, u_n) \in [0,1]^n\}$ and $\mathcal{B} = \{V = (v_1, v_2, \ldots, v_n) \in ([-1,0] \cup \{1\})^n\}$. Let $\mathcal{F} : \mathcal{A} \times \mathcal{B} \times (0, +\infty) \to [0,1]$, defined by

$$\mathcal{F}(U, V, t) = \exp \frac{-\sum_{s=1}^{n} (|u_s| + |v_s|)^2}{t}$$

for all $U \in \mathcal{A}, V \in \mathcal{B}$. The conditions ($\mathcal{F}$1), ($\mathcal{F}$2), ($\mathcal{F}$3), and ($\mathcal{F}$5) in Definition 2.1 directly hold. Let $U, W \in \mathcal{A}, V, D \in \mathcal{B}$ if $D \neq (0, 0, \ldots, 0)_{1\times n}$, then $C_L(\mathcal{A}, \mathcal{F}, D) = \phi$. If $D = (0, 0, \ldots, 0)_{1\times n}$ and $\{U_k = (u_s^{(k)})_{s=1}^n\} \in C_L(\mathcal{A}, \mathcal{F}, V)$, then

$$\mathcal{F}(U, V, t) * \mathcal{F}(U_k, V, t) = \exp \frac{-\sum_{s=1}^{n} [(|u_s| + |v_s|)^2}{t} \cdot \exp \frac{-\sum_{s=1}^{n} (|u_s^{(k)}| + |v_s|)^2}{t}.$$

Thus,

$$\mathcal{F}(U, D, t) = \exp \frac{-\sum_{s=1}^{n} (|u_s||)^2}{t} \geqslant \limsup_{k\to+\infty} \exp \frac{-\sum_{s=1}^{n} [(|u_s| + |v_s|)^2}{t} \cdot \exp \frac{-\sum_{s=1}^{n} [(|u_s^{(k)}| + |v_s|)^2]}{t}.$$

Also, if $U \neq (0,0,\dots,0)$, then $C_R(\mathcal{B},\mathcal{F},U) = \phi$ and $\{V_k = (V_s^{(k)})_{s=1}^n\} \in C_R(\mathcal{B},\mathcal{F},U)$,

$$\mathcal{F}(U,D,t) \geqslant \limsup_{k\to+\infty} \mathcal{F}(W,D,t) * \mathcal{F}(W,V_k,t),$$

$(\mathcal{A},\mathcal{B},\mathcal{F},*)$ is a GFBMS, where "$*$" is a product ctn. Clearly, $(\mathcal{A},\mathcal{B},\mathcal{F},*)$ is a complete GFBMS, where $*$ is a product ctn. Let $\Upsilon : \mathcal{A}\cup\mathcal{B} \to \mathcal{A}\cup\mathcal{B}$ be defined by

$$\begin{cases} \Upsilon((u_s)_{s=1}^n) = (\frac{u_s}{2}+\frac{1}{2})_{s=1}^n, & \text{if } (u_s)_{s=1}^n \in \mathcal{A}, \\ \Upsilon((v_s)_{s=1}^n) = (\underbrace{0,0,\dots,0}_{n \text{ times}}), & \text{if } (v_s)_{s=1}^n \in \mathcal{B}. \end{cases}$$

$\Upsilon : (\mathcal{A},\mathcal{B},\mathcal{F},*) \rightrightarrows (\mathcal{A},\mathcal{B},\mathcal{F},*)$ is a GFB α-quasi contraction of type-I, since $\Upsilon(\mathcal{A}) \subseteq \mathcal{A}, \Upsilon(\mathcal{B}) \subseteq \mathcal{B}$ so it is covariant and quasi contraction, where $\alpha = \frac{1}{4}$. To clear it, let $U = (u_s) \in \mathcal{A}$ and $V = (v_s) \in \mathcal{B}$, then

$$\begin{aligned}\mathcal{F}(\Upsilon U,\Upsilon V,\alpha t) &= \exp \frac{-\sum_{s=1}^n |\frac{u_s}{2}+\frac{1}{2}|^2}{\alpha t} \\ &= \exp \frac{-\sum_{s=1}^n |u_s+1|^2}{4\alpha t}, \alpha = \frac{1}{4} \\ &\geqslant \min\{\exp \frac{-\sum_{s=1}^n [u_s+|v_s|]^2}{t}, \exp \frac{-\sum_{s=1}^n [|u_s|]^2}{t}, \exp \frac{-\sum_{s=1}^n [|\frac{u_s}{2}+\frac{1}{2}|^2|+|v_s|]^2}{t}\}.\end{aligned}$$

Now, we can construct the bisequences $\varpi_{n+1} = \varpi_n$ and $v_{n+1} = v_n$ for all $n \in \mathbb{N}\cup\{0\}$ by taking some $A = (a_s)_{s=1}^n \in \mathcal{A}, B = (b_s)_{s=1}^n \in \mathcal{B}$, we obtain a non-trivial sequence as

$$(\varpi_n, v_n) = \{((a_s),(b_s))_{s=1}^n, ((\frac{a_s}{2}+\frac{1}{2}),(0_s)_{s=1}^n, ((\frac{a_s}{2^2}+1),(0_s))_{s=1}^n, ((\frac{a_s}{2^3}+\frac{3}{2}),(0_s))_{s=1}^n,\dots\},$$

which implies that for any fixed $t > 0$,

$$\sigma_t(\mathcal{F},\Upsilon,(A,B)) = \inf_{s,r\in\mathbb{N}}\{\mathcal{F}(\Upsilon^s A, \Upsilon^r B, t), t>0\} = \inf_{s,r\in\mathbb{N}}\{\exp \frac{-\sum_{s=1}^n [|\frac{a_s}{2^{s-1}}+\frac{s}{2}|}{t}\} > 0.$$

Thus, Υ satisfies all the requirements stated in Theorem 3.9 and $\mathrm{Fix}_\Upsilon = \{(\underbrace{1,1,\dots,1}_{n \text{ times}})\}$.

Definition 3.11. Let $(\mathcal{A},\mathcal{B},\mathcal{F},*)$ be a GFBMS. A contravariant mapping $\Upsilon : (\mathcal{A},\mathcal{B},\mathcal{F},*) \rightleftharpoons (\mathcal{A},\mathcal{B},\mathcal{F},*)$ is a GFB α-a-quasi contraction of type-II if for all $(\varpi,v) \in \mathcal{A}\times\mathcal{B}$, for all $t>0$ and for some $\alpha < \frac{1}{a}$, where $a \geqslant 1$ is a constant in Definition 2.1, we have

$$\mathcal{F}(\Upsilon\varpi,\Upsilon v,\alpha t) \geqslant \min\{\mathcal{F}(\varpi,v,t), \mathcal{F}(\varpi,\Upsilon\varpi,t), \mathcal{F}(v,\Upsilon v,t)\}. \tag{3.12}$$

Theorem 3.12. *Let $(\mathcal{A},\mathcal{B},\mathcal{F},*)$ be a complete GFBMS and Υ be a GFB α-a-quasi contraction of type-II. If there exists $(\varpi_0,\Upsilon\varpi_0) \in \mathcal{A}\times\mathcal{B}$, $\sigma_t(\mathcal{F},\Upsilon,(\varpi_0,\Upsilon\varpi_0)) = \inf_{s,r\in\mathbb{N}}\{\mathcal{F}(\varpi_s,v_r,t) : v_n = \Upsilon\varpi_n$ and $\varpi_{n+1} = \Upsilon v_n, n \geqslant 0\} > 0, \forall t>0$, then (ϖ_n,v_n) biconverges to some $\varpi \in \mathcal{A}\cap\mathcal{B}$ with $\mathcal{F}(\varpi,\varpi,t) = 1$ and $\mathrm{Fix}_\Upsilon = \{\varpi\}$.*

Proof. Let $\varpi_0 \in \mathcal{A}$ satisfies

$$\sigma_t(\mathcal{F},\Upsilon,(\varpi_0,\Upsilon\varpi_0)) = \inf_{s,r\in\mathbb{N}}\{\mathcal{F}(\Upsilon^s\varpi_0,\Upsilon^r v_0,t)\} > 0, \forall t>0.$$

The bisequence is constructed by (ϖ_n,v_n), where $\varpi_{n+1} = \Upsilon^n\varpi_0, v_n = \Upsilon^n v_0, n \geqslant 0$, and $\forall t>0$,

$$\sigma_t(\mathcal{F},\Upsilon^{\kappa+1},(\varpi_0,\Upsilon\varpi_0)) = \inf_{s,r\in\mathbb{N}}\{\mathcal{F}(\Upsilon^{\kappa+s}\varpi_0,\Upsilon^{\kappa+r}v_0,t)\},$$

for every fixed $\kappa = 0, 1, 2 \dots$, it holds that

$$\{\mathcal{F}(\Upsilon^{\kappa+s}\varpi_0, \Upsilon^{\kappa+r}\nu_0, t); s, r \geqslant 1\} \subseteq \{\mathcal{F}(\Upsilon^s\varpi_0, \Upsilon^r\nu_0, t), s, r \in \mathbb{N}\},$$

which implies that

$$\sigma_t(\mathcal{F}, \Upsilon^{\kappa+1}, (\varpi_0, \Upsilon\varpi_0), t) \geqslant \sigma_t(\mathcal{F}, \Upsilon, (\varpi_0, \Upsilon\varpi_0)) > 0. \tag{3.13}$$

Now, for all $s, r \in \mathbb{N}$,

$$\begin{aligned}\mathcal{F}(\Upsilon^{n+s}\varpi_0, \Upsilon^{n+r}\nu_0, t) &= \mathcal{F}(\Upsilon\nu_{n+s-1}, \Upsilon\varpi_{n+r}, t)\}\\ &\geqslant \min\{\mathcal{F}(\varpi_{n+r}, \Upsilon\nu_{n+r-1}, \frac{t}{\alpha}), \mathcal{F}(\varpi_{n+r-1}, \Upsilon\varpi_{n+r-1}, \frac{t}{\alpha}), \mathcal{F}(\nu_{n+s-1}, \Upsilon\nu_{n+s-1}, \frac{t}{\alpha})\}.\end{aligned}$$

For every $s, r \geqslant 1$, it holds that

$$\begin{aligned}&\min\{\mathcal{F}(\Upsilon^{n+s}\varpi_0, \Upsilon^{n+r}\nu_0, t)\}\\ &\quad\geqslant \min\{\min\{\mathcal{F}(\varpi_{n+r}, \Upsilon\nu_{n+r-1}, \frac{t}{\alpha}), \mathcal{F}(\varpi_{n+r-1}, \Upsilon\varpi_{n+r-1}, \frac{t}{\alpha}), \mathcal{F}(\nu_{n+s-1}, \Upsilon\nu_{n+s-1}, \frac{t}{\alpha})\}\}\\ &\Rightarrow \sigma_t(\mathcal{F}, \Upsilon^{n+1}, (\varpi_0, \Upsilon\varpi_0))\\ &\quad\geqslant \min\{\sigma_{\frac{t}{\alpha}}(\mathcal{F}, \Upsilon^n, (\varpi_0, \nu_0))\sigma_{\frac{t}{\alpha}}(\mathcal{F}, \Upsilon^n, (\varpi_0, \Upsilon\varpi_0))\sigma_{\frac{t}{\alpha}}(\mathcal{F}, \Upsilon^n, (\varpi_0, \Upsilon\varpi_0))\}.\end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14), it can be deduced that

$$\sigma_t(\mathcal{F}, \Upsilon^{n+1}, (\varpi_0, \nu_0), t) \geqslant \sigma_{\frac{t}{\alpha}}(\mathcal{F}, \Upsilon^n, (\varpi_0, \nu_0)),$$

so for all $n \geqslant 1, t > 0$, we have

$$\begin{aligned}\mathcal{F}(\Upsilon^{n+s}\varpi_0, \Upsilon^{n+r}\nu_0, t) \geqslant &\sigma_t(\mathcal{F}, \Upsilon^{n+1}, (\varpi_0, \Upsilon\varpi_0))\\ &\geqslant \sigma_{\frac{t}{\alpha}}(\mathcal{F}, \Upsilon^n, (\varpi_0, \Upsilon\varpi_0))\\ &\geqslant \sigma_{\frac{t}{\alpha^2}}(\mathcal{F}, \Upsilon^{n-1}(\varpi_0, \Upsilon\varpi_0)) \geqslant \cdots \geqslant \sigma_{\frac{t}{\alpha^n}}(\mathcal{F}, \Upsilon, (\varpi_0, \Upsilon\varpi_0)).\end{aligned} \tag{3.15}$$

Therefore, for every $1 \leqslant n < m$ we use (3.15) to obtain

$$\mathcal{F}(\varpi_n, \nu_m, t) = \mathcal{F}(\Upsilon^n\varpi_0, \Upsilon^m\nu_0, t) \geqslant \sigma_t(\mathcal{F}, \Upsilon^n, (\varpi_0, \Upsilon\varpi_0)),$$

since $\sigma_t(\mathcal{F}, \Upsilon^n, (\varpi_0, \Upsilon\varpi_0)) > 0,$ for all $t > 0, \alpha \in (0, 1)$, by ($\mathcal{F}5$) in Definition 2.1 we get

$$\lim_{n\to+\infty} \mathcal{F}(\varpi_n, \nu_m, t) = 1.$$

Therefore, $\{(\varpi_n, \nu_m)\}$ is a Cauchy bisequence. As result, by completeness of $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ and Proposition 2.12, biconverges to some $\varpi \in \mathcal{A} \cap \mathcal{B}$ such that $\mathcal{F}(\varpi, \varpi, t) = 1$. Suppose that $\varpi \neq \Upsilon\varpi$ which implies that $\mathcal{F}(\varpi, \Upsilon\varpi, t) < 1$. Now we have

$$\begin{aligned}\mathcal{F}(\varpi_{n+1}, \Upsilon\varpi, \alpha t) = \mathcal{F}(\Upsilon\nu_n, \Upsilon\varpi, t) &\geqslant \min\{\mathcal{F}(\nu_n, \varpi, t), \mathcal{F}(\nu_n, \Upsilon\nu_n, t), \mathcal{F}(\varpi, \Upsilon\varpi, t\}\\ &= \min\{\mathcal{F}(\nu_n, \varpi, t), \mathcal{F}(\nu_n, \varpi_{n+1}, t), \mathcal{F}(\varpi, \Upsilon\varpi, t)\}, \alpha \in (0, 1).\end{aligned} \tag{3.16}$$

Taking $n \to +\infty$ in (3.16) we get

$$\limsup_{n\to+\infty} \mathcal{F}(\varpi_{n+1}, \Upsilon\varpi, \alpha t) \geqslant \min\{1, 1, \mathcal{F}(\varpi, \Upsilon\varpi, t)\} = \mathcal{F}(\varpi, \Upsilon\varpi, t), t > 0. \tag{3.17}$$

Also we have

$$\mathcal{F}(\varpi, \Upsilon\varpi, t) \geqslant \mathcal{F}(\varpi_{m+1}, \nu_n, \frac{t}{a}) * \mathcal{F}(\varpi_{m+1}, \Upsilon\varpi, \frac{t}{a}), a \geqslant 1, \tag{3.18}$$

letting $n \to +\infty$ in (3.18) and by (3.17) we get

$$\mathcal{F}(\varpi, \Upsilon\varpi, t) \geqslant 1 * \mathcal{F}(\varpi, \Upsilon\varpi, \frac{t}{\alpha a}), a \geqslant 1.$$

Since $\mathcal{F}(\varpi, \Upsilon\varpi, t) < 1$ and $\mathcal{F}(\varpi, \nu, .)$ is non-decreasing and $\alpha < \frac{1}{a}, a \geqslant 1$ we get a contradiction. Hence $\varpi = \Upsilon\varpi$. Now let $\theta \in \mathcal{A}$ satisfy $\Upsilon\theta = \theta$ such that $\mathcal{F}(\varpi, \theta, t) > 0$. Then by (3.12) we see that

$$\begin{aligned}\mathcal{F}(\theta, \varpi, t) &\geqslant \min\{\mathcal{F}(\theta, \varpi, \frac{t}{\alpha}), \mathcal{F}(\Upsilon\theta, \varpi, \frac{t}{\alpha}), \mathcal{F}(\theta, \Upsilon\varpi, \frac{t}{\alpha})\} \\ &= \mathcal{F}(\theta, \varpi, \frac{t}{\alpha}) \geqslant \cdots \geqslant \mathcal{F}(\theta, \varpi, \frac{t}{\alpha^n}) \to 1 \text{ as } n \to +\infty.\end{aligned}$$

That is $\theta = \varpi$. A similar conclusion holds whenever $\nu \in \mathcal{B}$. □

Example 3.13. In Example 3.10, let $\Upsilon : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be defined by

$$\Upsilon(U) = \begin{cases} (\frac{\tilde{u}_s}{2}), & \text{if } U = (\tilde{u}_s) \in \mathcal{A}, \\ (\frac{\tilde{u}_s}{2}), & \text{if } U = (\tilde{u}_s) \in \mathcal{B}. \end{cases}$$

Proof. We find that all conditions of Theorem 3.12 are satisfied with $\alpha = \frac{1}{4}$ and the fixed point of $\text{Fix}_\Upsilon = \{(\underbrace{0, 0, \ldots, 0}_{n \text{ times}})\}$. □

Remark 3.14. As consequences of Theorems 3.9 and 3.12, as before Proposition 2.3, fixed point theorems for contractions for Ćirić quasi- can be derived within the framework of FBMSs.

4. Well-posedness of FFP problem in GFBMS

The investigation of the well-posedness of the FPP is an intriguing area of research in fixed point theory [9, 10, 19, 27].

In the setting of a GFBMS , the upcoming section presents the definitions of well-posedness concerning FFP problems.

Definition 4.1. Let $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ be a GFBMS and $\Upsilon : (\mathcal{A}, \mathcal{B}, \mathcal{F}) \rightrightarrows (\mathcal{A}, \mathcal{B}, \mathcal{F})$ be a mapping. The FFP problem of Υ is said to be well-posed if

(i) Υ has a unique fixed point $\varpi \in \mathcal{A} \cap \mathcal{B}$;
(ii) for any sequence $\{(\varpi_n, \nu_n)\}$ in $\mathcal{A} \times \mathcal{B}$ with $\mathcal{F}(\varpi_n, \Upsilon\nu_n, t) \to 1$ and $\mathcal{F}(\Upsilon\varpi_n, \nu_n, t) \to 1$ for all $t > 0$, as $n \to +\infty$, we have, for all $t > 0$, $\mathcal{F}(\varpi_n, \varpi, t) \to 1$ and $\mathcal{F}(\varpi, \nu_n, t) \to 1$ as $n \to +\infty$.

Definition 4.2. Let $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ be a GFBMS and $\Upsilon : (\mathcal{A}, \mathcal{B}, \mathcal{F}) \rightleftharpoons (\mathcal{A}, \mathcal{B}, \mathcal{F})$ be a mapping. The FFP problem of Υ is said to be well-posed if

(i) Υ has a unique fixed point $\varpi \in \mathcal{A} \cap \mathcal{B}$;
(ii) for any sequence $\{(\varpi_n, \nu_n)\}$ in $\mathcal{A} \times \mathcal{B}$ with $\mathcal{F}(\varpi_n, \Upsilon\varpi_n, t) \to 1$ and for all $t > 0$, $\mathcal{F}(\Upsilon\nu_n, \nu_n, t) \to 1$ as $n \to +\infty$, we have $\mathcal{F}(\varpi_n, \varpi, t) \to 1$ and $\mathcal{F}(\varpi, \nu_n, t) \to 1$ as $n \to +\infty$.

Theorem 4.3. *Let $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ be a complete GFBMS and Υ be a GFB α-contraction of type-I. If for some $(\varpi_0, \nu_0) \in \mathcal{A} \times \mathcal{B}$, $\sigma_t(\mathcal{F}, \Upsilon, (\varpi_0, \nu_0)) = \inf_{s,r \in \mathbb{N}}\{\mathcal{F}(\Upsilon^s\varpi_0, \Upsilon^r\nu_0, t), t > 0\} > 0$, then the function $\Upsilon : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ has a fixed point $\varpi \in \mathcal{A} \cap \mathcal{B}$. Furthermore, if $\theta \in \mathcal{A}$ such that $\mathcal{F}(\theta, \varpi, t) > 0$ for all $t > 0$, or $\theta \in \mathcal{B}$ such that $\mathcal{F}(\varpi, \theta, t) > 0$ for all $t > 0$, then the fixed point problem of Υ is well-posed.*

Proof. From Theorem 3.2 it follows that Υ has a unique fixed point $\varpi \in \mathcal{A} \cap \mathcal{B}$. Let $\{(\varpi_n, \nu_n)\}$ in $\mathcal{A} \times \mathcal{B}$ with $\mathcal{F}(\varpi_n, \Upsilon\nu_n, t) \to 1$ and $\mathcal{F}(\Upsilon\varpi_n, \nu_n, t) \to 1$ as $n \to +\infty$, for all $t > 0$. Then

$$\begin{aligned}\mathcal{F}(\varpi_n, \varpi, t) &\geqslant \mathcal{F}(\varpi_n, \Upsilon\nu_n, t) * \mathcal{F}(\varpi, \nu_n, t) * \mathcal{F}(\varpi, \varpi, t)\\ &= \mathcal{F}(\varpi_n, \Upsilon\nu_n, t) * \mathcal{F}(\Upsilon\varpi, \Upsilon\nu_n, t) * 1 \geqslant \mathcal{F}(\varpi_n, \Upsilon\nu_n, t) * \mathcal{F}(\varpi, \nu_n, \frac{t}{\alpha}), \alpha \in (0,1).\end{aligned} \tag{4.1}$$

Also,

$$\begin{aligned}\mathcal{F}(\varpi, \nu_n, t) &\geqslant \mathcal{F}(\varpi, \varpi, t) * \mathcal{F}(\Upsilon\varpi_n, \varpi, t) * \mathcal{F}(\Upsilon\varpi_n, \nu_n, t)\\ &= 1 * \mathcal{F}(\Upsilon\varpi_n, \Upsilon\varpi, t) * \mathcal{F}(\Upsilon\varpi_n, \nu_n, t) \geqslant \mathcal{F}(\varpi_n, \varpi, \frac{t}{\alpha}) * \mathcal{F}(\Upsilon\varpi_n, \nu_n, t), \alpha \in (0,1).\end{aligned} \tag{4.2}$$

Therefore, from (4.1) and (4.2) we get

$$\mathcal{F}(\varpi_n, \varpi, t) \geqslant \mathcal{F}(\varpi_n, \Upsilon\nu_n, t) * \mathcal{F}(\varpi, \nu_n, \frac{t}{\alpha}) \geqslant \mathcal{F}(\varpi_n, \Upsilon\nu_n, t) * \mathcal{F}(\varpi_n, \varpi, \frac{t}{\alpha^2}) * \mathcal{F}(\Upsilon\varpi_n, \nu_n, \frac{t}{\alpha}), \alpha \in (0,1).$$

Continuing the above steps we get

$$\lim_{n\to+\infty} \mathcal{F}(\varpi_n, \varpi, t) \geqslant \lim_{n\to+\infty} \mathcal{F}(\varpi_n, \varpi, \frac{t}{\alpha^n}) = 1,$$

it follows that $\lim_{n\to+\infty} \mathcal{F}(\varpi_n, \varpi, t) = 1$ for all $t > 0$. Hence due to Definition 4.1 we see that the fixed point problem of Υ is well-posed. □

Remark 4.4. Theorem 4.3 is applicable when Υ is a GFB α-contraction of type-II.

Theorem 4.5. *Let $(\mathcal{A}, \mathcal{B}, \mathcal{F}, *)$ be a complete GFBMS and Υ be a GFB α-quasi contraction of type-II. If for some $(\varpi_0, \Upsilon\varpi_0) \in \mathcal{A} \times \mathcal{B}$, for all $t > 0$,*

$$\sigma_t(\mathcal{F}, \Upsilon, (\varpi_0, \Upsilon\varpi_0)) = \inf_{s,r\in\mathbb{N}} \{\mathcal{F}(\varpi_s, \nu_r, t) : \nu_n = \Upsilon\varpi_n \text{ and } \varpi_{n+1} = \Upsilon\nu_n, n \geqslant 0\} > 0,$$

then (ϖ_n, ν_n) biconverges to some $\varpi \in \mathcal{A} \cap \mathcal{B}$ with $\mathcal{F}(\varpi, \varpi, t) = 1$. Then ϖ will be a fixed point of υ. Moreover, if $\theta \in \mathcal{A}$ with $\mathcal{F}(\theta, \varpi, t) > 0$ for all $t > 0$ or $\theta \in \mathcal{B}$ with $\mathcal{F}(\varpi, \theta, t) > 0$ for all $t > 0$, then the fixed point problem is well-posed.

Proof. From Theorem 3.12 it follows that Υ has a unique fixed point $\varpi \in \mathcal{A} \cap \mathcal{B}$. Let $\{(\Upsilon\nu_n, \nu_n)\}$ in $(\mathcal{A}, \mathcal{B})$ with $\mathcal{F}(\varpi_n, \Upsilon\varpi_n, t) \to 1$ and $\mathcal{F}(\Upsilon\varpi_n, \nu_n, t) \to 1$ as $n \to +\infty$, for all $t > 0$. Then

$$\begin{aligned}\mathcal{F}(\varpi_n, \varpi, t) &\geqslant \mathcal{F}(\varpi_n, \Upsilon\varpi_n, t) * \mathcal{F}(\varpi, \Upsilon\varpi_n, t)\\ &= \mathcal{F}(\varpi_n, \Upsilon\varpi_n, t) * \mathcal{F}(\Upsilon\varpi, \Upsilon\varpi_n, t)\\ &\geqslant \mathcal{F}(\varpi_n, \Upsilon\varpi_n, t) * \min\{\mathcal{F}(\varpi, \varpi_n, \frac{t}{\alpha}), \mathcal{F}(\varpi_n, \Upsilon\varpi_n, \frac{t}{\alpha}), \mathcal{F}(\varpi, \Upsilon\varpi, \frac{t}{\alpha})\}\\ &= \mathcal{F}(\varpi_n, \Upsilon\varpi_n, t) * \mathcal{F}(\varpi, \varpi_n, \frac{t}{\alpha}), \alpha \in (0,1).\end{aligned} \tag{4.3}$$

(4.3) shows after n-steps $\mathcal{F}(\varpi_n, \varpi, t) \geqslant \mathcal{F}(\varpi, \varpi_n, \frac{t}{\alpha^n}), \alpha \in (0,1)$. So, $\mathcal{F}(\varpi_n, \varpi, t) \to 1$ as $n \to +\infty$. Also,

$$\begin{aligned}\mathcal{F}(\varpi, \nu_n, t) &\geqslant \mathcal{F}(\Upsilon\nu_n, \varpi, t) * \mathcal{F}(\Upsilon\nu_n, \nu_n, t)\\ &= \mathcal{F}(\Upsilon\nu_n, \Upsilon\Upsilon\varpi, t) * \mathcal{F}(\Upsilon\varpi_n, \nu_n, t)\\ &\geqslant \min\{\mathcal{F}(\nu_n, \varpi, \frac{t}{\alpha}), \mathcal{F}(\nu_n, \Upsilon\nu_n, \frac{t}{\alpha}), \mathcal{F}(\varpi, \Upsilon\varpi, \frac{t}{\alpha})\} * \mathcal{F}(\Upsilon\nu_n, \nu_n, t)\\ &\geqslant \mathcal{F}(\nu_n, \varpi, \frac{t}{\alpha}) * \mathcal{F}(\Upsilon\nu_n, \nu_n, t), \alpha \in (0,1).\end{aligned}$$

Continuing the above steps we get

$$\lim_{n\to+\infty}\mathcal{F}(\varpi,\nu_n,t)\geqslant\lim_{n\to+\infty}\mathcal{F}(\nu_n,\varpi,\frac{t}{\alpha^n})=1,$$

it follows that $\lim_{n\to+\infty}\mathcal{F}(\varpi,\nu_n,t)=1$ for all $t>0$. Hence due to Definition 4.2 we see that the fixed point problem of Υ is well-posed. □

Remark 4.6. Theorem 4.5 is applicable when Υ is a GFB α-quasi contraction of type-I.

5. Application to M-OFDEs with IBCs

Initially, we revisit certain definitions and lemmas pertaining to the Riemann-Liouville fractional integral (RLFIs) and fractional derivatives (RLFds), as well as the Caputo fractional derivatives, within a limited interval on the real number line.

Definition 5.1 ([12])**.** The RLFIs $I^{\mu}_{0_+}$ and $I^{\mu}_{1_-}$ of order μ on $[0,1]$ are specified as

$$(I^{\mu}_{0_+}u)(\tau):=\frac{1}{\Gamma(\mu)}\int_0^{\tau}\frac{u(s)ds}{(\tau-s)^{1-\mu}}\quad\text{and}\quad(I^{\mu}_{1_-}u)(\tau):=\frac{1}{\Gamma(\mu)}\int_t^{1}\frac{u(s)ds}{(s-\tau)^{1-\mu}},$$

respectively, where $\Gamma(\mu)=\int_0^{+\infty}s^{\mu-1}e^{-s}ds$.

Definition 5.2 ([10])**.** The RLFDs $D^{\mu}_{0_+}$ and $D^{\mu}_{1_-}$ of order μ on $[0,1]$ are defined by

$$(D^{\mu}_{0_+}u)(\tau):=\Big(\frac{d}{dt}\Big)^n(I^{n-\mu}_{0_+}u)(\tau)=\frac{1}{\Gamma(n-\mu)}\Big(\frac{d}{dt}\Big)^n\int_0^{\tau}\frac{u(s)ds}{(\tau-s)^{\mu-n+1}}$$

and

$$(D^{\mu}_{1_-}u)(\tau):=\Big(-\frac{d}{d\tau}\Big)^n(I^{n-\mu}_{1_-}u)(\tau)=\frac{1}{\Gamma(n-\mu)}\Big(-\frac{d}{d\tau}\Big)^n\int_t^{1}\frac{u(s)ds}{(s-\tau)^{\mu-n+1}},$$

respectively, where $n=[\mu]+1$, where $[\mu]$ is the integer part of the number μ and Γ is the Euler gamma function.

We use the classical space $C[0,1]$ with the norm $\|\tilde{x}\|_{+\infty}=\max_{\tau\in[0,1]}\tilde{x}(\tau)$. Given $\mu>0$ and $N=[\mu]+1$, a linear space can be defined as $C^{\mu}[0,1]:=\{u(\tau):u(\tau)=I^{\mu}_{0+1}\tilde{x}(\tau)+\sum_{s=1}^{N-1}c_s\tau^{\mu-s},\tau\in[0,1]\}$, where $\tilde{x}\in C[0,1]$ and $c_s\in\mathbb{R}$, $s=1,\dots,N-1$. By means of the linear functional analysis theory, we can prove that with the norm $\|u\|_{C^{\mu}}=\|D^{\mu}_{0+}\|_{+\infty}+\cdots+\|D^{\mu-(N-1)}_{0+}\|_{+\infty}+\|u\|_{+\infty}$, $C^{\mu}[0,1]$ is a Banach space.

Consider the following M-OFDEs with IBCs (for more details see [2, 6, 24])

$$\begin{cases}({}^{C}D^{\vartheta}_{0^+}u)(\tau)+\sum_{s=1}^{m}\hat{\lambda}_s(\tau)({}^{C}D^{\vartheta_s}_{0^+}u)(\tau)+\sum_{r=1}^{n}\hat{\mu_r}(\tau)({}^{C}D^{\varpi_r}_{0^+}u)(\tau))\\+\sum_{k=1}^{\kappa}\hat{\varpi}_k(\tau)({}^{C}D^{\varphi_k}_{0^+}u)(\tau)\sum_{l=1}^{q}\hat{\omega}_l(\tau)({}^{C}D^{\sigma_l}_{0^+}u)(\tau)+\hat{\sigma}(\tau)u(\tau)+f(\tau,u(\tau))=0,\tau\in[0,1],\\ u''(0)=u'''(0)=0,\ u'(0)=\nu_1\int_0^1u(s)ds,\ u(1)=\nu_2\int_0^1u(s)ds.\end{cases}\tag{5.1}$$

Throughout this paper, we always assume that $0<\sigma_1<\sigma_2<\cdots<\sigma_q<1<\varphi_1<\varphi_2<\cdots<\varphi_\kappa<2<\varpi_1<\varpi_2<\cdots<\varpi_n<3<\vartheta_1<\vartheta_2<\cdots<\vartheta_m<\vartheta<4,\nu_1+2(1-\nu_2)\neq0,\hat{\lambda}_s,\hat{\mu}_r,\hat{\varpi}_k,\hat{\omega}_l,\hat{\sigma}:[0,1]\to\mathbb{R}$ and $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ are continuous functions. Let $\mathcal{A}=\{x\in C[0,1]:\|a\|\leqslant r,r>0\}$ and $\mathcal{B}=\{\tilde{y}\in C[0,1]:\|\tilde{y}\|>0\}$. Consider $\mathcal{F}:\mathcal{A}\times\mathcal{B}\times[0,+\infty)\to[0,1]$ defined by

$$\mathcal{F}(\tilde{x},\tilde{y},t)=\exp-\frac{\|\tilde{x}-\tilde{y}\|}{t}.$$

Then $(\mathcal{A},\mathcal{B},\mathcal{F},*)$ is a complete GFBMS, $*$ is product t-norm.

Lemma 5.3 ([24]). *If $\tilde{y} \in C[0,1]$ is a solution of the equation*

$$\begin{aligned}\tilde{y}(\tau) = f\Big(\tau, \int_0^1 \mathbb{G}(\tau,s)\tilde{y}(t)(s)ds\Big) - \sum_{s}^{m} \hat{\lambda}_s(\tau)(I_{0^+}^{\vartheta-\vartheta_s}\tilde{y})(\tau) - \sum_{r=1}^{n} \hat{\mu_r}(t)(I_{0^+}^{\vartheta-\varphi_r}\tilde{y})(\tau)\\ - \sum_{k=1}^{\kappa} \hat{\varpi}_k(\tau)(I_{0^+}^{\vartheta-\varpi_k}\tilde{y})(\tau) - \sum_{l=1}^{q} \hat{\omega}_l(\tau)\Big\{(I_{0^+}^{\varphi-\sigma_l}\tilde{y})(\tau) - \frac{2\nu_1}{[\nu_1+2(1-\nu_2)]\Gamma(2-\sigma_l)}\\ \cdot [I_{0^+}^{\varphi}\tilde{y})(1) - I_{0^+}^{\varphi+1}\tilde{y})(1)]\tau^{1-\sigma_l}\Big\} + \hat{\sigma}(\tau)\int_0^1 \mathbb{G}(\tau,s)\tilde{y}(s)ds, \tau \in [0,1],\end{aligned}$$

then $u(\tau) := \int_0^1 \mathbb{G}(\tau,s)\tilde{y}(s)ds, \tau \in [0,1]$ is a solution of the BVP (5.1) in $C^{\vartheta}[0,1]$, where $\mathbb{G}(\tau,s)$ be defined by

$$\mathbb{G}(\tau,s) = \frac{[(2\tau-1)\nu_1+2]\vartheta(1-s)^{\vartheta-1} - 2[(\tau-1)\nu_1+\nu_2](1-s)^{\vartheta}}{[\nu_1+2(1-\nu_2)]\Gamma(\vartheta+1)} - \begin{cases}\frac{(\tau-s)^{\vartheta-1}}{\Gamma(\vartheta)}, & 0 \leqslant s \leqslant \tau \leqslant 1,\\ 0, & 0 \leqslant \tau \leqslant s \leqslant 1.\end{cases}$$

Theorem 5.4. *Consider the M-OFDEs with IBCs (5.1). Suppose the following assertions are satisfied.*

1. *$\mathbb{G}(\tau,s)$ is continuous on $[0,1]^2$;*
2. *$\max_{(t,\tilde{x})\in[0,1]\times[-\tilde{M}\tilde{r},\tilde{M}\tilde{r}]} |f(\tau,x)| \leqslant (1-\varpi)r$, where*

$$\begin{aligned}\varpi = \frac{\|\hat{\lambda}_s(\tau)\|}{\Gamma(\vartheta-\vartheta_s+1)} + \sum_{r=1}^{n} \frac{\|\hat{\mu_r}(\tau)\|}{\Gamma(\vartheta-\varphi_r+1)} + \sum_{k=1}^{\kappa} \frac{\|\hat{\varpi}_k(\tau)\|}{\vartheta-\varpi_k+1}\\ + \sum_{l=1}^{q}\|\hat{\omega}_l(\tau)\|\{\frac{1}{\Gamma(\varphi-\sigma_l+1)} + \frac{1}{\Gamma(2-\sigma_l)}\cdot\frac{\varphi+2}{\Gamma(\varphi+2)}\} + \|\hat{\sigma}(\tau)\|\tilde{M};\end{aligned}$$

3. *for all $t \in [0,1], \tilde{x}_1, \tilde{x}_2 \in \mathbb{R}$, there exists a constant $0 < \tilde{\mathcal{L}} < \frac{1-\varpi}{\tilde{M}}$ such that $|f(t,\tilde{x}_1) - f(t,\tilde{x}_2)| \leqslant \tilde{\mathcal{L}}|\tilde{x}_1 - \tilde{x}_2|$.*

Then, the M-OFDE with INBCs (5.1) has a unique solution in $\mathcal{A} \cap \mathcal{B}$.

Proof. Define a covariant operator $\Upsilon : \mathcal{A} \cup \mathcal{B} \rightrightarrows \mathcal{A} \cup \mathfrak{B}$ by

$$\begin{aligned}(\Upsilon\tilde{y})(\tau) = \tilde{y}(\tau) = f\Big(t, \int_0^1 \mathbb{G}(\tau,s)\tilde{y}(s)ds\Big) - \sum_{s}^{m} \hat{\lambda}_s(\tau)(I_{0^+}^{\vartheta-\vartheta_s}\tilde{y})(t) - \sum_{r=1}^{n} \hat{\mu_r}(\tau)(I_{0^+}^{\vartheta-\varphi_r}\tilde{y})(\tau)\\ - \sum_{k=1}^{\kappa} \hat{\varpi}_k(\tau)(I_{0^+}^{\vartheta-\varpi_k}\tilde{y})(\tau) - \sum_{l=1}^{q} \hat{\omega}_l(\tau)\Big\{(I_{0^+}^{\varphi-\sigma_l}\tilde{y})(\tau) - \frac{2\nu_1}{[\nu_1+2(1-\nu_2)]\Gamma(2-\sigma_l)}\\ \cdot [I_{0^+}^{\varphi}\tilde{y})(1) - I_{0^+}^{\varphi+1}\tilde{y})(1)]\tau^{1-\sigma_l}\Big\} + \hat{\sigma}(\tau)\int_0^1 \mathbb{G}(\tau,s)\tilde{y}(s)ds, \tau \in [0,1].\end{aligned}$$

Note that for some $(\tilde{y}_1, \tilde{y}_2) \in \mathcal{A} \times \mathcal{B}$, we have for any fixed $t > 0$,

$$\sigma_t(\mathcal{F}, \Upsilon, (\tilde{y}_1, \tilde{y}_2)) = \inf_{s,r\in\mathbb{N}}\{\mathcal{F}(\Upsilon^s\tilde{y}_1, \Upsilon^r\tilde{y}_2, t)\} = \inf_{s,r\in\mathbb{N}}\{\exp - \frac{\|\Upsilon^s\tilde{y}_1 - \Upsilon^r\tilde{y}_2\|}{t}\} > 0.$$

Now, we only have to show that Υ is a GFB α-contraction of type-I. For any $\tilde{y}_1 \in \mathcal{A}$ and $\tilde{y}_2 \in \mathcal{B}$ we have

$$\begin{aligned}\Big|(\Upsilon\tilde{y}_1)(\tau) - (\Upsilon\tilde{y}_2)(\tau)\Big| \leqslant \Big|f\Big(t, \int_0^1 \mathbb{G}(\tau,s)\tilde{y}_1(s)ds\Big) - f\Big(t, \int_0^1 \mathbb{G}(\tau,s)\tilde{y}_2(s)ds\Big|\\ + \Big|\sum_{s}^{m} \hat{\lambda}_s(\tau)(I_{0^+}^{\vartheta-\vartheta_s}(\tilde{y}_1-\tilde{y}_2))(\tau)\Big|\Big|\sum_{r=1}^{n} \hat{\mu_r}(\tau)(I_{0^+}^{\vartheta-\varphi_r}(\tilde{y}_1-\tilde{y}_2))(t)\Big|\end{aligned}$$

$$
\begin{aligned}
&+\Big|\sum_{k=1}^{\kappa}\hat{\varpi}_k(\tau)(I_{0^+}^{\vartheta-\varpi_k}(\tilde{y}_1-\tilde{y}_2))(\tau)\Big|+\Big|\sum_{l=1}^{q}\hat{\omega}_l(\tau)\Big\{(I_{0^+}^{\varphi-\sigma_l}(\tilde{y}_1-\tilde{y}_2))(\tau)\\
&-\frac{2\nu_1}{[\nu_1+2(1-\nu_2)]\Gamma(2-\sigma_l)}\cdot[I_{0^+}^{\varphi}(\tilde{y}_1-\tilde{y}_2))(1)+I_{0^+}^{\varphi+1}(\tilde{y}_1-\tilde{y}_2))(1)]t^{1-\sigma_l}\Big\}\Big|\\
&+\Big|\hat{\sigma}(\tau)\int_0^1 \mathbb{G}(t,s)(\tilde{y}_1-\tilde{y}_2)(s)ds\Big|\\
\leqslant{}&\tilde{\mathcal{L}}\Big|\int_0^1 \mathbb{G}(\tau,s)(\tilde{y}_1(s)-\tilde{y}_2(s))ds\Big|+\sum_{s}^{m}|\hat{\lambda}_s(\tau)|(I_{0^+}^{\vartheta-\vartheta_s}|\tilde{y}_1-\tilde{y}_2|)(\tau)\\
&+\sum_{r=1}^{n}|\hat{\mu}_r(\tau)|(I_{0^+}^{\vartheta-\varphi_r}|\tilde{y}_1-\tilde{y}_2|)(\tau)+\sum_{k=1}^{\kappa}|\hat{\varpi}_k(\tau)|(I_{0^+}^{\vartheta-\varpi_k}|\tilde{y}_1-\tilde{y}_2|)(\tau)\\
&+\sum_{l=1}^{q}|\hat{\omega}_l(\tau)|\{(I_{0^+}^{\varphi-\sigma_l}\Big|\tilde{y}_1-\tilde{y}_2\Big|)(\tau)+\Big|\frac{2\nu_1}{[\nu_1+2(1-\nu_2)]}\Big|\frac{1}{\Gamma(2-\sigma_l)}\\
&\cdot[I_{0^+}^{\varphi}|\tilde{y}_1-\tilde{y}_2|)(1)+I_{0^+}^{\varphi+1}|\tilde{y}_1-\tilde{y}_2|)(1)]\Big\}\Big|+|\hat{\sigma}(\tau)|\Big|\int_0^1 \mathbb{G}(\tau,s)(\tilde{y}_1-\tilde{y}_2)(s)ds\Big|\\
\leqslant{}&\Big\{\tilde{\mathcal{L}}\tilde{M}+\sum_{s}^{m}\frac{\|\hat{\lambda}_s(\tau)\|}{\Gamma(\vartheta-\vartheta_s+1)}+\sum_{r=1}^{n}\frac{\|\hat{\mu}_r(\tau)\|}{\Gamma(\vartheta-\varphi_r+1)}+\sum_{k=1}^{\kappa}\frac{\|\hat{\varpi}_k(\tau)\|}{\vartheta-\varpi_k+1}\\
&+\sum_{l=1}^{q}\|\hat{\omega}_l(\tau)\|\{\frac{1}{\Gamma(\varphi-\sigma_l+1)}+\frac{1}{\Gamma(2-\sigma_l)}\cdot\frac{\varphi+2}{\Gamma(\varphi+2)}\}+\|\hat{\sigma}(\tau)\|\tilde{M}\Big\}\|\tilde{y}_1-\tilde{y}_2\|\\
\leqslant{}&(\tilde{\mathcal{L}}\tilde{M}+\varpi)\|\tilde{y}_1-\tilde{y}_2\|\leqslant\alpha\|\tilde{y}_1-\tilde{y}_2\|,\ \text{where } \alpha=\tilde{\mathcal{L}}\tilde{M}+\varpi<1.
\end{aligned}
$$

This implies

$$\mathcal{F}(\Upsilon(\tilde{y}_1),\Upsilon(\tilde{y}_2),\alpha t)=\exp-\frac{\|\Upsilon(\tilde{y}_1)-\Upsilon(\tilde{y}_2)\|}{\alpha t}\geqslant\exp\frac{-\|\tilde{y}_1-\tilde{y}_2\|}{t}.$$

Therefore, all the conditions stated in Theorem 3.2 are satisfied. Hence, (5.1) possesses a unique solution within $\mathcal{A}\cap\mathcal{B}$. □

6. Conclusions

Building upon the recent findings by Ashraf et al. [1], we have introduced the concept of a GFBMS. Within this framework, several fixed point results have been established, which significantly expand and generalize the existing results in the literature. Additionally, we have applied these results to prove the existence of a unique solution for M-OFDEs with IBCs. The examples provided illustrate and validate the theoretical findings. Looking ahead, we are eager to explore the potential of extending this research to more generalized spaces, such as rectangular fuzzy metric spaces and ultrametric metric spaces, which may open new avenues for further investigation.

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