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Hahn-Banach Theorem for functionals on hypervector spaces

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Abstract. In this paper we prove Hahn-Banach Theorem for functionals on hypervector spaces. In this regard, we introduce a new category of hypervector spaces and prove some results for them.

Keywords: hypervector space, normed hypervector space, functional.

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1 Introduction

The concept of hyperstructure was first introduced by Marty [7] in 1934 and has attracted attention of many authors in last decades and has constructed some other structures such as hyperrings, hypergroups, hypermodules, hyperfields, and hypervector spaces. These constructions have been applied to many disciplines such as geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, and etc. A wealth of applications of this concepts is given in $[1-6]$ and $[8-15]$.

In 1988 the concept of hypervector space was first introduced by Scafati-Tallini. She studied more properties of this new structure in [12] and [13]. We considered this generalization of vector space in viewpoint of analysis and proved important results in this field. See $[9-11]$. In this paper we prove Hahn-Banach Theorem and some its results on hypervector spaces. This paper is arranged as follows. In section 2 we define the preliminary concepts and then in section 3 we prove the Hahn-Banach Theorem for functionals on normal real and complex hypervector spaces and some its results.

We denote the set of all complex numbers by C and real numbers by R. Throughout of paper the field F is as C or R.

2 Preliminaries

Definition 2.1 *([13]) A weak or weakly distributive hypervectorspace over a field F is a quadruple (X,+,o,F) such that $(X,+)$ is an abelian group and $o: F \times X \to P_*(X)$ is a multivalued product times a scalar such that:*

(1) $$\forall a \in F, \forall x, y \in X, [ao(x+y)] \cap [aox + aoy] \neq \emptyset,$$

(2) $$\forall a, b \in F, \forall x \in X, [(a+b)ox] \cap [aox + box] \neq \emptyset,$$

(3) $$\forall a, b \in F, \forall x \in X, ao(box) = (ab)ox,$$

(4) $$\forall a \in F, \forall x \in X, ao(-x) = (-a)ox = -(aox),$$

(5) $$\forall x \in X, x \in 1ox.$$

We call (1) and (2) $weak\ right$ and $left\ distributive$ laws, respectively. Note that the set $ao(box)$ in (3) is of the form $\cup_{y \in box} aoy$.

Definition 2.2 *([12]) Let $(X,+)$ be an abelian group and F be a field. Then a hypervector space is a quadruple (X,+,o,F) where o is a mapping $o: F \times X \to P^*(X)$, such that the following conditions are satisfied:*

(1) $$\forall a \in F, \forall x, y \in X, ao(x+y) \subseteq aox + aoy,$$

(2) $$\forall a, b \in F, \forall x \in X, (a+b)ox \subseteq aox + box,$$

(3) $$\forall a, b \in F, \forall x \in X, ao(box) = (ab)ox,$$

(4) $$\forall a \in F, \forall x \in X, ao(-x) = (-a)ox,$$

(5) $$\forall x \in X, x \in 1ox.$$

We call (1) and (2) $right$ and $left\ distributive$ laws, respectively. Note that the set

$ao(box)$ in (3) is of the form $\cup_{y \in box} aoy$.

Example 2.3 *Suppose $0 \neq a \in R$ and $0 \neq z \in C$. C with usual sum and following product is a hypervector space on R:*

$$aoz = \{re^{i\theta}; 0 < r \leq |a||z|, \theta = arg(az)\},$$

if $a = 0$ or $z = 0$, then we define $aoz = 0$.

Example 2.4 *Suppose $a \in R$ and $z \in C$. C with usual sum and following product is a hypervector space on R:*

$$a.z = \{re^{i\theta}; 0 \leq r \leq |a||z|, 0 \leq \theta \leq 2\pi\}.$$

Definition 2.5 *([12]) Let $(X, +, o, F)$ be a hypervector space over a field F. We define a pseudonorm in X as being a mapping $\|.\|: X \to R$, of X into the reals such that:*

(i) $\| 0 \| = 0$,

$(ii) \forall x, y \in X, \| x + y \| \leq \| x \| + \| y \|$,

$(iii) \forall a \in F, \forall x \in X, \sup \| aox \| = |a| \| x \|$.

A pseudonorm in X is called norm if:

(iv) $\| x \| = 0 \Leftrightarrow x = 0$.

3 Main results

Definition 3.1 *([9]) If X is a weak hypervector space over F, $a \in F$ and $x \in X$, then z_{aox} for $0 \neq a$ is that element of aox such that $x \in a^{-1} o z_{aox}$ and for $a = 0$, we define $z_{aox} = 0$.*

Remark 3.2 *Note that z_{aox} in a weak hypervector space is the element that its norm is equal to $|a| \| x \|$.*

As the descriptions in [4], z_{aox} is not unique, necessarily. So the set of all these elements denoted by Z_{aox}. In the mentioned paper we introduced a certain category of weak hypervector spaces that Z_{aox} is singleton in them. These weak hypervector spaces have been called "normal". In [9], the following lemma stated a criterion for normality of a weak hypervector space.

Lemma 3.3 ([9]) *Let X be a weak hypervector space over F. X is normal if and only if*

$$z_{a_1 ox} + z_{a_2 ox} = z_{(a_1 + a_2) ox}, \forall x \in X, \forall a_1, a_2 \in F,$$

$$z_{ao x_1} + z_{ao x_2} = z_{ao(x_1 + x_2)}, \forall x_1, x_2 \in X, \forall a \in F.$$

Example 3.4 *The defined hypervector space in Example 2.3 is normal.*

Lemma 3.5 ([9]) *If X is a weak hypervector space over F, $a \in F$, $0 \neq b \in F$ and also $x \in X$, then the following properties hold:*

(a) $x \in Z_{1ox}$
(b) $aoz_{box} = abox$

(c) $Z_{-aox} = -Z_{aox}$
Furthermore, if X is normal, then
(d) Z_{aox} is singleton.

Definition 3.6 ([9]) *Let X be a weak hypervector space over F. A nonempty subset M of X is called a weak subhypervector space of X, when M satisfies the following properties:*
$(i) x + y \in M, \forall x, y \in M$,
$(ii) z_{aox} \in M, \forall a \in F, \forall x \in M$.

Definition 3.7 *Let X be a weak hypervector space over F. A map $f: X \to F$ is called weak linear functional if and only if f is additive and satisfies $f(z_{aox}) = af(x)$, for all $a \in F$ and $x \in X$.*

If X is normed hypervector space over F, we denote the set of all bounded (see [10]) weak linear functionals on X by X_w^*.

Now we are prepare to prove Hahn-Banach Theorem for weak linear functionals on normal real hypervector spaces.

Theorem 3.8 *Suppose*
(a) M is a weak subhypervector space of a normal real hypervector space X.
(b) $p: M \to R$ satisfies

$$p(x + y) \leq p(x) + p(y),$$

$$supp(tox) = tp(x) = p(z_{tox}),$$

if $x \in X, y \in X, t \geq 0$,
(c) $f: M \to R$ is a weak linear functional and $f(x) \leq p(x)$ on M.
Then there exists a weak linear functional $\Lambda: X \to R$ such that

$$\Lambda(x) = f(x) \quad (x \in M),$$

and

$$\Lambda(x) \leq p(x) \quad (x \in X).$$

Proof. If $M \neq X$, choose $x_1 \in X \backslash M$, and define

$$M_1 = \{x + z_{tox_1}; x \in M, t \in R\}.$$

We show that M_1 is a weak subspace of X. Let $x', y' \in M_1$. So $x' = x + z_{tox_1}$ and $y' = y + z_{sox_1}$ where $x, y \in M$ and $t, s \in R$. Since X is normal, we have

$$x' + y' = (x + z_{tox_1}) + (y + z_{sox_1}) = (x + y) + z_{(t+s)ox_1}$$

so $x' + y' \in M_1$, because $x + y \in M$. Now let $\lambda \in R$. Again by normality of X and Lemma 3.4(b) we have

$$z_{\lambda ox'} = z_{\lambda o(x + z_{tox_1})} = z_{\lambda ox} + z_{\lambda o z_{tox_1}} = z_{\lambda ox} + z_{\lambda tox_1}.$$

So it is clear that $z_{\lambda ox'} \in M_1$, because $z_{\lambda ox} \in M$. Since

$$f(x) + f(y) = f(x + y) \leq p(x + y) \leq p(x - x_1) + p(x_1 + y),$$

we have

$$f(x)-p(x-x_1)\leq p(x_1+y)-f(y)\quad (x,y\in M).\,(1)$$

Since the right side of (1) is independent of x, the left side has least upper bound, as x ranges over M. Let a be the least upper bound of the left side of (1). Then

$$f(x)-a\leq p(x-x_1)\quad (x\in M)(2)$$

and

$$f(y)+a\leq p(y+x_1)\quad (y\in M).\,(3)$$

Define f_1 on M_1 by

$$f_1(x+z_{tox_1})=f(x)+ta\quad (x\in M,t\in R).$$

It is clear that $f_1=f$ on M. Let $x'=x+z_{tox_1}$ and $y'=y+z_{sox_1}$ where $x,y\in M$ and $t,s\in R$ and also let $\lambda\in R$. So by normality of X we have

$$\begin{aligned}
&f_1(x'+y')=f_1((x+y)+z_{(t+s)ox_1})\\
&=f(x+y)+(t+s)a\\
&=f(x)+ta+f(y)+sa\\
&=f_1(x')+f_1(y')
\end{aligned}$$

and

$$\begin{aligned}
&f_1(z_{\lambda ox'})=f_1(z_{\lambda o(x+z_{tox_1})})=f_1(z_{\lambda ox}+z_{\lambda oz_{tox_1}})\\
&=f_1(z_{\lambda ox}+z_{\lambda tox_1})=f(z_{\lambda ox})+\lambda ta\\
&=\lambda f(x)+\lambda ta=\lambda(f(x)+ta)\\
&=\lambda f_1(x').
\end{aligned}$$

So f_1 is a weak linear functional. Now we show $f_1\leq p$ on M_1.

By taking $t>0$ and replacing x by $z_{t^{-1}ox}$ in (2) and y by $z_{t^{-1}oy}$ in (3) and using Lemma 3.4 we obtain

$$\begin{aligned}
&f(z_{t^{-1}ox})-a\leq p(z_{t^{-1}ox}-x_1)\\
&=p(z_{t^{-1}ox}-z_{t^{-1}oz_{tox_1}})\\
&=p(z_{t^{-1}o(x-z_{tox_1})})
\end{aligned}$$

and

$$\begin{aligned}
&f(z_{t^{-1}oy})+a\leq p(z_{t^{-1}oy}+x_1)\\
&=p(z_{t^{-1}o(y-z_{tox_1})}).
\end{aligned}$$

So by multiplying the above inequalities by t, we obtain

$$f(x)-ta\leq p(x-z_{tox_1}),$$

$$f(y)+ta\leq p(y+z_{tox_1}).$$

These inequalities and Lemma 3.4(c) imply

$$f_1(x+z_{-tox_1})\leq p(x+z_{-tox_1}),$$

$$f_1(y+z_{tox_1})\leq p(y+z_{tox_1}).$$

Therefore $f_1(x)\leq p(x)$ for all $x\in M_1$.

Let Γ be the collection of all ordered pairs (M',f'), where M' is a weak subspace of X that contains M and f' is a weak linear functional on M' that extends f. Partially order Γ by declaring $(M',f')^\circ(M'',f'')$ to mean that $M'\subseteq M''$ and $f''=f'$ on

M'. By Hausdorff's Maximality Theorem there exists a maximal totally ordered subcollection Ω of Γ. Let Φ be the collection of all M' such that $(M', f') \in \Omega$. Then Φ is totally ordered by set inclusion, and the union $\widetilde{M}$ of all members of Φ is therefore a weak subspace of X. If $x \in \widetilde{M}$ then $x \in M'$ for some $M' \in \Phi$. Define $\Lambda(x) = f'(x)$, where f' is the function which occurs in the pair $(M', f') \in \Omega$.

It is not difficult to check that Λ is well-defined, weak linear and $\Lambda \leq p$ on $\widetilde{M}$.

If $\widetilde{M}$ is a proper subhypervector space of X, the first part of the proof would give a further extension of Λ, and this would contradict the maximality of Ω. Thus $\widetilde{M} = X$.

Lemma 3.9 *Suppose X is a normal hypervector space over C.*

(a) If u is the real part of a complex weak linear functional f, then u is real weak linear and

$$f(x) = u(x) - iu(z_{iox}) \quad (x \in X). (*)$$

(b) If u is a real weak linear functional on X and f to be defined by $(*)$, then f is a complex weak linear functional on X.

(c) If X is a normed normal hypervector space and f to be defined by $(*)$, then $\| f \| = \| u \|$.

Proof. (a) It is clear that u is real weak linear. Now let v be the imaginary part of f. So

$$f(x) = u(x) + iv(x) \quad (x \in X).$$

Since $f(z_{iox}) = if(x)$, we have

$$u(z_{iox}) + iv(z_{iox}) = iu(x) - v(x).$$

This implies $v(x) = -Reif(x) = -u(z_{iox})$ and hence

$$f(x) = u(x) - iu(z_{iox}).$$

(b) By assumptions, it's clear that f is additive and $f(z_{aox}) = af(x)$ for all $a \in R$. We have

$$\begin{aligned} f(z_{iox}) &= u(z_{iox}) - iu(z_{io\, z_{iox}}) \\ &= u(z_{iox}) - iu(-x) \\ &= u(z_{iox}) + iu(x) \\ &= if(x). \end{aligned}$$

This fact together with the normality of X implies $f(z_{\alpha ox}) = \alpha f(x)$ for all $\alpha \in C$, because if $\alpha = a + bi$ where $a, b \in R$, then we have

$$\begin{aligned} f(z_{\alpha ox}) &= f(z_{(a+bi)ox}) = f(z_{aox} + z_{biox}) \\ &= f(z_{aox}) + f(z_{biox}) = af(x) + f(z_{bo\, z_{iox}}) \\ &= af(x) + bf(z_{iox}) = af(x) + bif(x) \\ &= \alpha f(x). \end{aligned}$$

(c) Since $|u(x)| \leq |f(x)|$, we obtain $\| u \| \leq \| f \|$. On the other hand, for any $x \in X$ there exists an $a \in C$ such that $|a| = 1$ and $af(x) = |f(x)|$. So

$$\begin{aligned} |f(x)| &= f(z_{aox}) = u(z_{aox}) \leq \| u \| . \| z_{aox} \| \\ &= \| u \| . |a| \, \| x \| \\ &= \| u \| . \| x \|, \end{aligned}$$

that implies $\| f \| \leq \| u \|$. Thus we obtain $\| f \| = \| u \|$.

Theorem 3.10 *Suppose M is a weak subhypervector space of a normed normal hypervector space X over F and p is a seminorm on X. Also suppose f is a weak linear functional on M and $|f(x)| \leq p(x)$ for all $x \in M$. Then f extends to a weak linear functional Λ on X that satisfies*

$$|\Lambda(x)| \leq p(x) \quad (x \in X).$$

Proof. If the scalar field is R, this is contained in Theorem 3.7, because p is a seminorm and hence $supp(tox) = |t|p(x) = p(z_{tox})$ so p has the properties of the defined map in Theorem 3.7(b).

Assume that the scalar field is C. Put $u = Ref$. By Lemma 3.8(a) u is real weak linear functional. Since $|u(x)| \leq |f(x)|$, so $|u(x)| \leq p(x)$ for all $x \in M$. Thus by Theorem 3.7 there is a real weak linear functional U on X such that $U = u$ on M and also $U \leq p$ on X. Let Λ be a complex weak linear functional on X whose real part is U. Theorem 3.8(a) implies that $\Lambda = f$ on M. Finally, to every $x \in X$ corresponds an $a \in C$, $|a| = 1$, such that $a\Lambda(x) = |\Lambda(x)|$. Hence

$$|\Lambda(x)| = \Lambda(z_{aox}) = U(z_{aox}) \leq p(z_{aox}) = |a|p(x) = p(x).$$

An immediate result of the above theorem is the following theorem.

Theorem 3.11 *Let f be a bounded weak linear functional on a weak hypervector space M of a normed normal hypervector space X over F. Then there exists a bounded weak linear functional Λ on X which is an extension of f to X and has the same norm,*

$$\| \Lambda \|_X = \| f \|_M.$$

Proof. If $M = \{0\}$, then $f = 0$, and the extension is $\Lambda = 0$. Let $M \neq \{0\}$. Set $p(x) = \| f \|_M \| x \|$. It is clear that p is a seminorm. Since f is a bounded weak linear (See [10]), so we have

$$|f(x)| \leq \| f \|_M \| x \| = p(x) \quad (x \in M).$$

Hence we can now apply Theorem 3.9 and conclude that there exists a weak linear functional Λ on X which is an extension of f and

$$|\Lambda(x)| \leq p(x) = \| f \|_M \| x \|.$$

Taking the supremum over all $x \in X$ of norm less than 1, we obtain the inequality

$$\| \Lambda \|_X = sup\{|\Lambda(x)|; x \in X, \| x \| \leq 1\} \leq \| f \|_M.$$

Since under an extension the norm cannot decrease, we also have $\| \Lambda \|_X \geq \| f \|_M$. Together we obtain $\| \Lambda \|_X = \| f \|_M$ and the theorem is proved.

Corollary 3.12 *Suppose X is a normed normal hypervector space over F. If $x_0 \in X$, then there exists a bounded weak linear functional Λ on X such that*

$$\Lambda(x_0) = \| x_0 \|.$$

Furthermore, if $x_0 \neq 0$, then

$$\| \Lambda \| = 1.$$

Proof. If $x_0 = 0$, take $\Lambda = 0$. If $x_0 \neq 0$, apply Theorem 3.10, with $M = \{z_{to\,x_0}; t \in F\}$ and $f(z_{to\,x_0}) = t \parallel x_0 \parallel$. It is not difficult to check that M is a weak hypervector space of X and f is weak linear on M. f is bounded and has norm $\parallel f \parallel = 1$, because if $x \in M$

$$|f(x)| = |f(z_{to\,x_0})| = |t| \parallel x_0 \parallel = \parallel z_{to\,x_0} \parallel = \parallel x \parallel.$$

So f has a bounded weak linear extension Λ from M to X, of norm $\parallel \Lambda \parallel = \parallel f \parallel = 1$.

Corollary 3.13 *Suppose X is a normed normal hypervector space over F. Then X_w^* separates points on X.*

Proof. It is obvious by Corollary 3.11.

Corollary 3.14 *Let x be in a normed normal hypervector space X over F. Then we have*

$$\parallel x \parallel = sup\{\frac{|f(x)|}{\parallel f \parallel}; f \in X_w^*, f \neq 0\}.$$

Hence if x_0 is such that $f(x_0) = 0$ for all $f \in X_w^*$, then $x_0 = 0$.

Proof. Let $x \neq 0$. So from Corollary 3.11 we have, writing x for x_0,

$$sup\{\frac{|f(x)|}{\parallel f \parallel}; f \in X_w^*, f \neq 0\} \geq \frac{|\Lambda(x)|}{\parallel \Lambda \parallel} = \frac{\parallel x \parallel}{1} = \parallel x \parallel,$$

and from $|f(x)| \leq \parallel f \parallel \parallel x \parallel$ for bounded weak linear functionals, we obtain

$$sup\{\frac{|f(x)|}{\parallel f \parallel}; f \in X_w^*, f \neq 0\} \leq \parallel x \parallel.$$

Thus for $x \neq 0$ we obtain $\parallel x \parallel = sup\{\frac{|f(x)|}{\parallel f \parallel}; f \in X_w^*, f \neq 0\}$. It is clear that this equality is true for $x = 0$, too.

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