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New results on delay-dependent stability conditions for uncertain linear systems with two additive time-varying delays

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Abstract

In this study, we explore a novel delay-dependent criterion for asymptotic stability in linear systems with two additive time-varying delays and nonlinear perturbations. By utilizing the Newton-Leibniz formula, the extended Jensen's double integral inequality, the extended Wirtinger's integral inequality, a novel Lyapunov-Krasovskii functional and the application of zero equations, we derive sufficient conditions for the system's asymptotic stability in the form of linear matrix inequalities. Additionally, we introduce new delay-dependent stability conditions specifically for these linear systems with two additive time-varying delays. Numerical examples are provided to illustrate the feasibility and effectiveness of the theorems.

Keywords: Delay-dependent asymptotic stability, two additive time-varying delays, extended Jensen's double integral inequality, extended Wirtinger's integral inequality, linear matrix inequality.

2020 MSC: 34K25, 93C10, 93D09, 34D20.

1. Introduction

In recent years, researchers [1–14] have studied the concept of stability as a fundamental aspect of the mathematics field, particularly in the study of differential equations and dynamical systems. Stability analysis provides crucial insights into the behavior of systems, equilibria remain consistent over time under small perturbations. A stable system is generally one in which small deviations from a state lead to behaviors that remain bounded or return to the original state, which is particularly important in real-world applications such as engineering, economics, and the natural sciences.

Differential equations are frequently used to describe continuous models. However, erroneous estimates can arise from theoretical models that neglect to account for time delays, which commonly occur across various applications. It is widely recognized that such delays can significantly impact system performance, potentially leading to degradation and instability. This understanding has spurred substantial research interest in the stability analysis of systems with time delays over recent years, including areas such as differential equations [4, 12], linear systems [2, 5, 8, 11], systems with two additive time-varying delays [3, 7, 10, 11], and nonlinear systems [1, 3, 14]. In the context of stability, linear systems provide a straightforward framework for evaluating a system's response to perturbations. The stability of a linear

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system is typically assessed by examining the eigenvalues of the system matrix, derived from linear differential equations that describe the system's dynamics [12]. A system is deemed stable if all its eigenvalues have negative real parts, ensuring that any perturbations diminish over time and the system returns to equilibrium [2].

In 2007, it is notable that Lam et al. [7] were pioneers who specifically focused their research on systems that incorporate two time-varying delay components additively. These systems are found in numerous real-world scenarios, such as communication networks and process control. Their groundbreaking work established the foundation for understanding and analyzing the stability of systems with these complex delay components.

When considering nonlinear perturbations alongside these delays, the complexity of the stability analysis increases. Nonlinear perturbations can arise from various factors in real-world systems, such as environmental influences or system parameter variations [4]. These perturbations can cause deviations from assumed linear models, necessitating robust methods for stability analysis.

This paper was motivated by the above statement. It examines the stability criterion for linear systems with two additive time-varying delays and nonlinear perturbations and introduces a novel criterion for assessing the impact of delays on system stability. Furthermore, we have obtained stability conditions for a linear system with two additive time-varying delays. The primary achievements of this paper are encapsulated as follows.

- Based on the decomposition technique of coefficient constant, Newton-Leibniz formula, the extended Jensen's double integral inequality, the extended Wirtinger's double integral inequality, new Lyapunov-Krasovskii functional, and application of zero equations, sufficient conditions for asymptotic stability are in the form of linear matrix inequality for this system.
- We provide numerical examples to illustrate the theorem's feasibility and effectiveness.

This research contributes to understanding how two additive delays interact with asymptotic stability in uncertain linear systems. Exploring these elements lays a foundation for future research and applications in various scientific, engineering, economics, and medical fields.

2. Preliminaries

Notation: The notations employed throughout this paper are standard. $\mathbb{R}$ denotes the set of real constants; the set of positive real constants is denoted by $\mathbb{R}^+$; $\mathbb{R}^n$ denotes n-dimensional spaces with the vector norm $\|\cdot\|$; $\|\zeta\|$ denotes the Euclidean vector norm of $\zeta \in \mathbb{R}^n$; $\mathbb{R}^{n\times r}$ denotes the set $n \times r$ real matrices; the transpose of the matrix P is denoted by P^T; matrix P is symmetric if $P = P^T$; matrix P is called semi-positive definite ($P \geqslant 0$) if $\zeta^T P\zeta \geqslant 0$, for all $\zeta \in \mathbb{R}^n$; P is called positive definite ($P > 0$) if $\zeta^T P\zeta > 0$, for all $\zeta \in \mathbb{R}^n/\{0\}$; matrix Q is called semi-negative definite ($Q \leqslant 0$) if $\zeta^T Q\zeta \leqslant 0$, for all $\zeta \in \mathbb{R}^n$; Q is called negative definite ($Q < 0$) if $\zeta^T Q\zeta < 0$, for all $\zeta \in \mathbb{R}^n/\{0\}$; $\zeta_s = \zeta(s + t), t \in [-h, 0]$; $*$ denotes the elements situated below the main diagonal of a symmetric matrix; and I denotes the identity matrix.

We consider the linear model with two additive time-varying delays and nonlinear perturbations of the form

$$\begin{aligned} \dot{\zeta}(s) &= A\zeta(s) + B\zeta(s - h_1(s) - h_2(s)) + Cf(s, \zeta(s)) + Dg(s, \zeta(s - h_1(s) - h_2(s))), \\ \zeta(s) &= \phi(s), \ \forall s \in [-(h_1 + h_2), 0], \end{aligned} \tag{2.1}$$

where $\zeta(s) \in \mathbb{R}^n$, $A, B, C,$ and D are real matrices $n \times n$, $\phi(s)$ is a given continuously differentiable function on $s \in [-(h_1 + h_2), 0]$, and $h_1(s)$ and $h_2(s)$ are discrete time-varying delays

$$0 \leqslant h_1(s) \leqslant h_1, \quad \dot{h}_1(s) \leqslant \tau_1, \quad 0 \leqslant h_2(s) \leqslant h_2, \quad \dot{h}_2(s) \leqslant \tau_2,$$

where $h_1, h_2, \tau_1,$ and τ_2 are given positive real constants. Let $h_3 = h_1 + h_2$, $\dot{h}_3(s) = \dot{h}_1(s) + \dot{h}_2(s)$, $h_3(s) = h_1(s) + h_2(s), \tau_3 = \tau_1 + \tau_2$. The uncertainties $f(s, \zeta(s))$ and $g(s, \zeta(s - h_3(s)))$ represent nonlinear parameter perturbations concerning the current state $\zeta(s)$ and the delayed state $\zeta(s-h_3(s))$ and are bounded in magnitude in the form

$$f^T(s,\zeta(s))f(s,\zeta(s)) \leqslant \eta^2\zeta^T(s)\zeta(s), \tag{2.2}$$

$$g^T(s,\zeta(s-h_3(s)))g(s,\zeta(s-h_3(s))) \leqslant \beta^2\zeta^T(s-h_3(s))\zeta(s-h_3(s)), \tag{2.3}$$

where η and β are known real positive scalars.

Definition 2.1 ([6]). A functional $\vee : \mathbb{R}^+ \times C([-h,0],\mathbb{R}^n) \to \mathbb{R}^+$ is defined as a Lyapunov-Krasovskill functional for the system (2.1) if it satisfies the following properties: there exist constants $\lambda_1,\lambda_2,\lambda_3 > 0$ such that

(i) $\lambda_1\|\zeta(s)\|^2 \leqslant \vee(s,\zeta_s) \leqslant \lambda_2\|\zeta_s\|^2$;
(ii) $\dot{\vee}(s,\zeta_s) \leqslant -\lambda_3\|\zeta(s)\|^2$.

Lemma 2.2 ([6]). *Consider the time-delay system (2.1). If there exists a Lyapunov-Krasovskill function $\vee(s,\zeta_s)$ and $\lambda_1,\lambda_2,\lambda_3 > 0$ such that for every solution $\zeta(s)$ of the system, the following conditions hold:*

(i) $\lambda_1\|\zeta(s)\|^2 \leqslant \vee(s,\zeta_s) \leqslant \lambda_2\|\zeta_s\|^2$;
(ii) $\dot{\vee}(s,\zeta_s) \leqslant -\lambda_3\|\zeta(s)\|^2$,

then the solution of the system (2.1) is asymptotically stable.

Lemma 2.3 ([9]). *If any matrix $P = P^T > 0$, $j(s)$ is time-varying delays with $0 \leqslant j_1 \leqslant j(s) \leqslant j_2$, $j_1, j_2 \in \mathbb{R}$ and a vector-valued function $\zeta : [-j_2,-j_1] \to \mathbb{R}^n$ such that the integrals are well-defined, then*

$$-\frac{(j_2-j_1)^2}{2}\int_{j_1}^{j_2}\int_r^{j_2}\zeta^T(u)P\zeta(u)dudr \leqslant -\left(\int_{j_1}^{j_2}\int_r^{j_2}\zeta(u)dudr\right)^T P\left(\int_{j_1}^{j_2}\int_r^{j_2}\zeta(u)dudr\right).$$

Lemma 2.4 ([9]). *If a given matrix $P = P^T > 0$, $k(s)$ is time-varying delays with $0 \leqslant k_1 \leqslant k(s) \leqslant k_2$, $k_1, k_2 \in \mathbb{R}$ and a vector-valued functions $\zeta : [-k_2,-k_1] \to \mathbb{R}^n$ such that the integrals are well-defined, then*

$$(k_2-k_1)\int_{k_1}^{k_2}\zeta^T(u)P\zeta(u)du \geqslant \Omega_0^T P\Omega_0 + 3\Omega_1^T P\Omega_1 + 5\Omega_2^T P\Omega_2,$$

where

$$\begin{aligned}
\Omega_0 &= \int_{k_1}^{k_2}\zeta(u)du,\\
\Omega_1 &= \int_{k_1}^{k_2}\zeta(u)du - \frac{2}{k_2-k_1}\int_{k_1}^{k_2}\int_{k_1}^{u}\zeta(r)drdu,\\
\Omega_2 &= \int_{k_1}^{k_2}\zeta(u)du + \frac{6}{k_2-k_1}\int_{k_1}^{k_2}\int_{k_1}^{u}\zeta(r)drdu - \frac{12}{(k_2-k_1)^2}\int_{k_1}^{k_2}\int_{k_1}^{u}\int_{k_1}^{\theta}\zeta(r)drd\theta du.
\end{aligned}$$

Lemma 2.5 ([14]). *For given $n \times n$ real matrices $\wp_1, \wp_2, \wp_3$ such that $\wp_1 \geqslant 0, \wp_3 > 0$, $\begin{bmatrix} \wp_1 & \wp_2 \\ * & \wp_3 \end{bmatrix} \geqslant 0$, $\kappa(s)$ is time-varying delays with $0 \leqslant \kappa_1 \leqslant \kappa(s) \leqslant \kappa_2$, $\kappa_1, \kappa_2 \in \mathbb{R}$, vector-valued functions ζ and $\dot{\zeta} : [-\kappa_2,-\kappa_1] \to \mathbb{R}^n$ such that the following integrals are well-defined, then*

$$\begin{aligned}
&-(\kappa_2-\kappa_1)\int_{s-\kappa_2}^{s-\kappa_1}\begin{bmatrix}\zeta(u)\\ \dot{\zeta}(u)\end{bmatrix}^T\begin{bmatrix}\wp_1 & \wp_2\\ * & \wp_3\end{bmatrix}\begin{bmatrix}\zeta(u)\\ \dot{\zeta}(u)\end{bmatrix}ds\\
&\leqslant \begin{bmatrix}\zeta(s-\kappa_1)\\ \zeta(s-\kappa(s))\\ \zeta(s-\kappa_2)\\ \int_{s-\kappa(s)}^{s-\kappa_1}\zeta(u)du\\ \int_{s-\kappa_2}^{s-\kappa(s)}\zeta(u)du\end{bmatrix}^T
\begin{bmatrix}-\wp_3 & \wp_3 & 0 & -\wp_2^T & 0\\ * & -\wp_3-\wp_3^T & \wp_3 & \wp_2^T & -\wp_2^T\\ * & * & -\wp_3 & 0 & \wp_2^T\\ * & * & * & -\wp_1 & 0\\ * & * & * & * & -\wp_1\end{bmatrix}
\begin{bmatrix}\zeta(s-\kappa_1)\\ \zeta(s-\kappa(s))\\ \zeta(s-\kappa_2)\\ \int_{s-\kappa(s)}^{s-\kappa_1}\zeta(u)du\\ \int_{s-\kappa_2}^{s-\kappa(s)}\zeta(u)du\end{bmatrix}.
\end{aligned}$$

3. Main result

In this section, we let some notations for later use:

$$\begin{aligned}
\rho_1 &= P_1A + A^TP_1 + h_1P_2 + h_2P_3 + h_3P_4 + P_7 + P_8 + P_9 + \epsilon_1\eta^2 I + \epsilon_2\rho^2 I + h_1^2P_{10} + h_2^2P_{11} + h_3^2P_{12}\\
&\quad - h_1^2R_1 - h_2^2R_2 - h_3^2R_3 + h_1^2G_1 + h_2^2G_4 + h_3^2G_7 - G_3 - G_6 - G_9,\\
\rho_2 &= A^TQ^T + h_1^2G_2 + h_2^2G_5 + h_3^2G_8,\\
\rho_3 &= -h_1P_2 + h_1\tau_1P_2 - G_3 - G_3^T,\\
\rho_4 &= -h_2P_3 + h_2\tau_2P_3 - G_6 - G_6^T,\\
\rho_5 &= -h_3P_4 + h_3\tau_3P_4 - G_9 - G_9^T,\\
\rho_6 &= \frac{h_1^4}{4}R_1 + \frac{h_2^4}{4}R_2 + \frac{h_3^4}{4}R_3 - Q_1 - Q_1^T + h_1^2G_3 + h_2^2G_6 + h_3^2G_9,\\
\Sigma &= \left[\Delta_{(i,j)}\right]_{27\times 27},
\end{aligned}$$

where $\Delta_{(i,j)} = \Delta^T_{(j,i)}, i, j = 1, 2, 3, \dots, 27$, except

$\Delta_{(1,1)} = \rho_1$, $\Delta_{(1,2)} = G_3$, $\Delta_{(1,3)} = G_6$, $\Delta_{(1,4)} = P_1B + G_6$,
$\Delta_{(1,10)} = h_1^2R_1$, $\Delta_{(1,13)} = h_2^2R_2$, $\Delta_{(1,16)} = h_3^2R_3$, $\Delta_{(1,19)} = \rho_2$,
$\Delta_{(1,20)} = -G_2^T$, $\Delta_{(1,22)} = -G_5^T$, $\Delta_{(1,24)} = -G_8^T$, $\Delta_{(1,26)} = P_1C$,
$\Delta_{(1,27)} = P_1D$, $\Delta_{(1,23)} = 360N_2$, $\Delta_{(1,24)} = 12N_3$, $\Delta_{(2,2)} = \rho_3$,
$\Delta_{(2,5)} = G_3$, $\Delta_{(2,20)} = G_2^T$, $\Delta_{(2,21)} = -G_2^T$, $\Delta_{(3,3)} = \rho_4$,
$\Delta_{(3,7)} = G_6$, $\Delta_{(3,22)} = G_5^T$, $\Delta_{(3,23)} = -G_5^T$, $\Delta_{(4,4)} = \rho_5$,
$\Delta_{(4,9)} = G_9$, $\Delta_{(4,19)} = B^TQ^T$, $\Delta_{(4,24)} = G_8^T$, $\Delta_{(4,25)} = -G_8^T$,
$\Delta_{(5,5)} = P_5 - P_7 - G_3$, $\Delta_{(5,21)} = G_2^T$, $\Delta_{(6,6)} = -P_5 + \tau_2P_5$, $\Delta_{(7,7)} = P_6 - P_8 - G_6$,
$\Delta_{(7,23)} = G_5^T$, $\Delta_{(8,8)} = -P_6 + \tau_1P_6$, $\Delta_{(9,9)} = -P_9 - G_9$, $\Delta_{(9,25)} = G_8^T$,
$\Delta_{(10,10)} = -9h_1^2P_{10} - h_1^2R_1$, $\Delta_{(10,11)} = 3h_1^2P_{10}$, $\Delta_{(10,12)} = -5h_1^2P_{10}$, $\Delta_{(11,11)} = -\frac{4}{3}h_1^2P_{10}$,
$\Delta_{(11,12)} = \frac{5}{2}h_1^2P_{10}$, $\Delta_{(12,12)} = -5h_1^2P_{10}$, $\Delta_{(13,13)} = -9h_2^2P_{11} - h_2^2R_2$, $\Delta_{(13,14)} = 3h_2^2P_{11}$,
$\Delta_{(13,15)} = -5h_2^2P_{11}$, $\Delta_{(14,14)} = -\frac{4}{3}h_2^2P_{11}$, $\Delta_{(14,15)} = \frac{5}{2}h_2^2P_{11}$, $\Delta_{(15,15)} = -5h_2^2P_{11}$,
$\Delta_{(16,16)} = -9h_3^2P_{12} - h_3^2R_3$, $\Delta_{(16,17)} = 3h_3^2P_{12}$, $\Delta_{(16,18)} = -5h_3^2P_{12}$, $\Delta_{(17,17)} = -\frac{4}{3}h_3^2P_{12}$,
$\Delta_{(17,18)} = \frac{5}{2}h_3^2P_{12}$, $\Delta_{(18,18)} = -5h_3^2P_{12}$, $\Delta_{(19,19)} = \rho_6$, $\Delta_{(19,26)} = QC$,
$\Delta_{(19,27)} = QD$, $\Delta_{(20,20)} = -G_1$, $\Delta_{(21,21)} = -G_1$, $\Delta_{(22,22)} = -G_4$,
$\Delta_{(23,23)} = -G_4$, $\Delta_{(24,24)} = -G_7$, $\Delta_{(25,25)} = -G_7$, $\Delta_{(26,26)} = -\epsilon_1 I$,
$\Delta_{(27,27)} = -\epsilon_2 I$, and otherterms are 0.

Theorem 3.1. *The system (2.1) is asymptotically stable, if for given scalars $h_m, \tau_m, m = 1, 2, 3$, there exist positive definite matrices G_1, G_3, G_4, G_6, G_7, G_9, $P_i, R_j, i = 1, 2, \dots, 12$ and $j = 1, 2, 3$, any appropriate dimensional matrices Q, G_2, G_5, G_8, and any scalars $\eta \geqslant 0, \beta \geqslant 0$ such that the symmetric linear matrix inequalities hold:*

$$\begin{bmatrix} G_1 & G_2 \\ G_2^T & G_3 \end{bmatrix} > 0, \quad \begin{bmatrix} G_4 & G_5 \\ G_5^T & G_6 \end{bmatrix} > 0, \quad \begin{bmatrix} G_7 & G_8 \\ G_8^T & G_9 \end{bmatrix} > 0, \quad \Sigma < 0. \tag{3.1}$$

Proof. Let a Lyapunov-Krasovskii functional for the system (2.1) be

$$V(s)=\sum_{i=1}^{5}V_i(s), \tag{3.2}$$

where

$$\begin{aligned}
V_1(s)&=\zeta^T(s)P_1\zeta(s),\\
V_2(s)&=h_1\int_{s-h_1(s)}^{s}\zeta^T(u)P_2\zeta(u)du+h_2\int_{s-h_2(s)}^{s}\zeta^T(u)P_3\zeta(u)du+h_3\int_{s-h_3(s)}^{s}\zeta^T(u)P_4\zeta(u)du\\
&\quad+\int_{s-h_1-h_2(s)}^{s-h_1}\zeta^T(u)P_5\zeta(u)du+\int_{s-h_2-h_1(s)}^{s-h_2}\zeta^T(u)P_6\zeta(u)du+\int_{s-h_1}^{s}\zeta^T(u)P_7\zeta(u)du\\
&\quad+\int_{s-h_2}^{s}\zeta^T(u)P_8\zeta(u)du+\int_{s-h_3}^{s}\zeta^T(u)P_9\zeta(s)du,\\
V_3(s)&=h_1\int_{s-h_1}^{s}\int_{\theta}^{s}\zeta^T(u)P_{10}\zeta(u)dud\theta+h_2\int_{s-h_2}^{s}\int_{\theta}^{s}\zeta^T(u)P_{11}\zeta(u)dud\theta\\
&\quad+h_3\int_{s-h_3}^{s}\int_{\theta}^{s}\zeta^T(u)P_{12}\zeta(u)dud\theta,\\
V_4(s)&=\frac{h_1^2}{2}\int_{s-h_1}^{s}\int_{\theta}^{s}\int_{\alpha}^{s}\dot\zeta^T(u)R_1\dot\zeta(u)dud\alpha d\theta+\frac{h_2^2}{2}\int_{s-h_2}^{s}\int_{\theta}^{s}\int_{\alpha}^{s}\dot\zeta^T(u)R_2\dot\zeta(u)dud\alpha d\theta\\
&\quad+\frac{h_3^2}{2}\int_{s-h_3}^{s}\int_{\theta}^{s}\int_{\alpha}^{s}\dot\zeta^T(u)R_3\dot\zeta(u)dud\alpha d\theta,\\
V_5(s)&=h_1\int_{-h_1}^{0}\int_{s+u}^{s}\begin{bmatrix}\zeta(\theta)\\ \dot\zeta(\theta)\end{bmatrix}^T\begin{bmatrix}G_1 & G_2\\ * & G_3\end{bmatrix}\begin{bmatrix}\zeta(\theta)\\ \dot\zeta(\theta)\end{bmatrix}d\theta du+h_2\int_{-h_2}^{0}\int_{s+u}^{s}\begin{bmatrix}\zeta(\theta)\\ \dot\zeta(\theta)\end{bmatrix}^T\begin{bmatrix}G_4 & G_5\\ * & G_6\end{bmatrix}\begin{bmatrix}\zeta(\theta)\\ \dot\zeta(\theta)\end{bmatrix}d\theta du\\
&\quad+h_3\int_{-h_3}^{0}\int_{s+u}^{s}\begin{bmatrix}\zeta(\theta)\\ \dot\zeta(\theta)\end{bmatrix}^T\begin{bmatrix}G_7 & G_8\\ * & G_9\end{bmatrix}\begin{bmatrix}\zeta(\theta)\\ \dot\zeta(\theta)\end{bmatrix}d\theta du.
\end{aligned}$$

The time derivative of $V(s)$ along the trajectory of the system described by (2.1) is expressed as

$$\dot V(s)=\sum_{i=1}^{5}\dot V_i(s). \tag{3.3}$$

The time derivative of $V_1(s)$ is calculated as

$$\dot V_1(s)=2\zeta^T(s)P_1\left[A\zeta(s)+B\zeta(s-h_1(s)-h_2(s))+Cf(s,\zeta(s))+Dg(s,\zeta(s-h_1(s)-h_2(s)))\right]. \tag{3.4}$$

We compute $\dot V_2(s)$ as

$$\begin{aligned}
\dot V_2(s)\leqslant{}& h_1\zeta^T(s)P_2\zeta(s)-h_1(1-\tau_1)\zeta^T(s-h_1(s))P_2\zeta(s-h_1(s))+h_2\zeta^T(s)P_3\zeta(s)\\
&-h_2(1-\tau_2)\zeta^T(s-h_2(s))P_3\zeta(s-h_2(s))+h_3\zeta^T(s)P_4\zeta(s)-h_3(1-\tau_3)\zeta^T(s-h_3(s))P_4\zeta(s-h_3(s))\\
&+\zeta^T(s-h_1)P_5\zeta(s-h_1)-(1-\tau_2)\zeta^T(s-h_1-h_2(s))P_5\zeta(s-h_1-h_2(s))+\zeta^T(s-h_2)P_6\zeta(s-h_2)\\
&-(1-\tau_1)\zeta^T(s-h_2-h_1(s))P_6\zeta(s-h_2-h_1(s))+\zeta^T(s)P_7\zeta(s)-\zeta^T(s-h_1)P_7\zeta(s-h_1)\\
&+\zeta^T(s)P_8\zeta(s)-\zeta^T(s-h_2)P_8\zeta(s-h_2)+\zeta^T(s)P_9\zeta(s)-\zeta^T(s-h_3)P_9\zeta(s-h_3).
\end{aligned}$$

Utilizing Lemma 2.4, we can derive $V_3(s)$ as follows

$$\dot V_3(s)\leqslant h_1^2\zeta^T(s)P_{10}\zeta(s)+h_2^2\zeta^T(s)P_{11}\zeta(s)+h_3^2\zeta^T(s)P_{12}\zeta(s)+h_1^2\begin{bmatrix}\frac{1}{h_1}\int_{s-h_1}^{s}\zeta(u)du\\ \frac{12}{h_1^2}\int_{s-h_1}^{s}\int_{s-h_1}^{\theta}\zeta(u)dud\theta\\ \frac{12}{h_1^3}\int_{s-h_1}^{s}\int_{s-h_1}^{\theta}\int_{s-h_1}^{u}\zeta(r)drdud\theta\end{bmatrix}^T$$

$$
\begin{aligned}
&\times \begin{bmatrix} -9P_{10} & 3P_{10} & -5P_{10} \\ 3P_{10} & -\frac{4}{3}P_{10} & \frac{5}{2}P_{10} \\ -5P_{10} & \frac{5}{2}P_{10} & -5P_{10} \end{bmatrix} \begin{bmatrix} \frac{1}{h_1}\int_{s-h_1}^{s}\zeta(u)du \\ \frac{12}{h_1^2}\int_{s-h_1}^{s}\int_{s-h_1}^{\theta}\zeta(u)dud\theta \\ \frac{12}{h_1^3}\int_{s-h_1}^{s}\int_{s-h_1}^{\theta}\int_{s-h_1}^{u}\zeta(r)drdud\theta \end{bmatrix} \\
&+ h_2^2 \begin{bmatrix} \frac{1}{h_2}\int_{s-h_2}^{s}\zeta(u)du \\ \frac{12}{h_2^2}\int_{s-h_2}^{s}\int_{s-h_2}^{\theta}\zeta(u)dud\theta \\ \frac{12}{h_2^3}\int_{s-h_2}^{s}\int_{s-h_2}^{\theta}\int_{s-h_2}^{u}\zeta(r)drdud\theta \end{bmatrix}^T \begin{bmatrix} -9P_{11} & 3P_{11} & -5P_{11} \\ 3P_{11} & -\frac{4}{3}P_{11} & \frac{5}{2}P_{11} \\ -5P_{11} & \frac{5}{2}P_{11} & -5P_{11} \end{bmatrix} \\
&\times \begin{bmatrix} \frac{1}{h_2}\int_{s-h_2}^{s}\zeta(u)du \\ \frac{12}{h_2^2}\int_{s-h_2}^{s}\int_{s-h_2}^{\theta}\zeta(u)dud\theta \\ \frac{12}{h_2^3}\int_{s-h_2}^{s}\int_{s-h_2}^{\theta}\int_{s-h_2}^{u}\zeta(r)drdud\theta \end{bmatrix} + h_3^2 \begin{bmatrix} \frac{1}{h_3}\int_{s-h_3}^{s}\zeta(u)du \\ \frac{12}{h_3^2}\int_{s-h_3}^{s}\int_{s-h_3}^{\theta}\zeta(u)dud\theta \\ \frac{12}{h_3^3}\int_{s-h_3}^{s}\int_{s-h_3}^{\theta}\int_{s-h_3}^{u}\zeta(r)drdud\theta \end{bmatrix}^T \\
&\times \begin{bmatrix} -9P_{12} & 3P_{12} & -5P_{12} \\ 3P_{12} & -\frac{4}{3}P_{12} & \frac{5}{2}P_{12} \\ -5P_{12} & \frac{5}{2}P_{12} & -5P_{12} \end{bmatrix} \begin{bmatrix} \frac{1}{h_3}\int_{s-h_3}^{s}\zeta(u)du \\ \frac{12}{h_3^2}\int_{s-h_3}^{s}\int_{s-h_3}^{\theta}\zeta(u)dud\theta \\ \frac{12}{h_3^3}\int_{s-h_3}^{s}\int_{s-h_3}^{\theta}\int_{s-h_3}^{u}\zeta(r)drdud\theta \end{bmatrix}.
\end{aligned} \tag{3.5}
$$

Using Lemma 2.3, an upper bound of $V_4(s)$ can be obtained as

$$
\begin{aligned}
\dot{V}_4(s) \leqslant{}& \frac{h_1^4}{4}\dot{\zeta}^T(s)R_1\dot{\zeta}(s) - \left(\zeta(s) - \frac{1}{h_1}\int_{s-h_1}^{s}\zeta(u)du\right)^T h_1^2R_1\left(\zeta(s) - \frac{1}{h_1}\int_{s-h_1}^{s}\zeta(u)du\right) \\
&+ \frac{h_2^4}{4}\dot{\zeta}^T(s)R_2\dot{\zeta}(s) - \left(\zeta(s) - \frac{1}{h_2}\int_{s-h_2}^{s}\zeta(u)du\right)^T h_2^2R_2\left(\zeta(s) - \frac{1}{h_2}\int_{s-h_2}^{s}\zeta(u)du\right) \\
&+ \frac{h_3^4}{4}\dot{\zeta}^T(s)R_3\dot{\zeta}(s) - \left(\zeta(s) - \frac{1}{h_3}\int_{s-h_3}^{s}\zeta(u)du\right)^T h_3^2R_3\left(\zeta(s) - \frac{1}{h_3}\int_{s-h_3}^{s}\zeta(u)du\right)
\end{aligned} \tag{3.6}
$$

Based on Lemma 2.5, $\dot{V}_5(s)$ can be derived as follows

$$
\begin{aligned}
\dot{V}_5(s) \leqslant{}& h_1^2\begin{bmatrix}\zeta(s)\\ \dot{\zeta}(s)\end{bmatrix}^T\begin{bmatrix}G_1 & G_2\\ G_2^T & G_3\end{bmatrix}\begin{bmatrix}\zeta(s)\\ \dot{\zeta}(s)\end{bmatrix} + h_2^2\begin{bmatrix}\zeta(s)\\ \dot{\zeta}(s)\end{bmatrix}^T\begin{bmatrix}G_4 & G_5\\ G_4^T & G_6\end{bmatrix}\begin{bmatrix}\zeta(s)\\ \dot{\zeta}(s)\end{bmatrix} + h_3^2\begin{bmatrix}\zeta(s)\\ \dot{\zeta}(s)\end{bmatrix}^T\begin{bmatrix}G_7 & G_8\\ G_8^T & G_9\end{bmatrix}\begin{bmatrix}\zeta(s)\\ \dot{\zeta}(s)\end{bmatrix} \\
&+ \begin{bmatrix}\zeta(s)\\ \zeta(s-h_1(s))\\ \zeta(s-h_1)\\ \int_{s-h_1(s)}^{s}\zeta(u)du\\ \int_{s-h_1}^{s-h_1(s)}\zeta(u)du\end{bmatrix}^T \begin{bmatrix} -G_3 & G_3 & 0 & -G_2^T & 0\\ * & -G_3-G_3^T & G_3 & G_2^T & -G_2^T\\ * & * & -G_3 & 0 & G_2^T\\ * & * & * & -G_1 & 0\\ * & * & * & * & -G_1\end{bmatrix} \begin{bmatrix}\zeta(s)\\ \zeta(s-h_1(s))\\ \zeta(s-h_1)\\ \int_{s-h_1(s)}^{s}\zeta(u)du\\ \int_{s-h_1}^{s-h_1(s)}\zeta(u)du\end{bmatrix} \\
&+ \begin{bmatrix}\zeta(s)\\ \zeta(s-h_2(s))\\ \zeta(s-h_2)\\ \int_{s-h_2(s)}^{s}\zeta(u)du\\ \int_{s-h_2}^{s-h_2(s)}\zeta(u)du\end{bmatrix}^T \begin{bmatrix} -G_6 & G_6 & 0 & -G_5^T & 0\\ * & -G_6-G_6^T & G_6 & G_5^T & -G_5^T\\ * & * & -G_6 & 0 & G_5^T\\ * & * & * & -G_4 & 0\\ * & * & * & * & -G_4\end{bmatrix} \begin{bmatrix}\zeta(s)\\ \zeta(s-h_2(s))\\ \zeta(s-h_2)\\ \int_{s-h_2(s)}^{s}\zeta(u)du\\ \int_{s-h_2}^{s-h_2(s)}\zeta(u)du\end{bmatrix} \\
&+ \begin{bmatrix}\zeta(s)\\ \zeta(s-h_3(s))\\ \zeta(s-h_3)\\ \int_{s-h_3(s)}^{s}\zeta(u)du\\ \int_{s-h_3}^{s-h_3(s)}\zeta(u)du\end{bmatrix}^T \begin{bmatrix} -G_9 & G_9 & 0 & -G_8^T & 0\\ * & -G_9-G_9^T & G_9 & G_8^T & -G_8^T\\ * & * & -G_9 & 0 & G_8^T\\ * & * & * & -G_7 & 0\\ * & * & * & * & -G_7\end{bmatrix} \begin{bmatrix}\zeta(s)\\ \zeta(s-h_3(s))\\ \zeta(s-h_3)\\ \int_{s-h_3(s)}^{s}\zeta(u)du\\ \int_{s-h_3}^{s-h_3(s)}\zeta(u)du\end{bmatrix}.
\end{aligned}
$$

From the Leibniz-Newton formula, the given equation is true for any real scalar matrix Q with appropriate dimensions

$$2\dot{\zeta}^T(s)Q\left[-\dot{\zeta}(s)+A\zeta(s)+B\zeta(s-h_1(s)-h_2(s))+Cf(s,\zeta(s))+Dg(s,\zeta(s-h_1(s)-h_2(s)))\right]=0. \quad (3.7)$$

From (2.2) and (2.3), we get, for any positive real scalars ϵ_1 and ϵ_2,

$$0 \leqslant \epsilon_1\eta^2\zeta^T(s)\zeta(s)-\epsilon_1 f^T(s,\zeta(s))f(s,\zeta(s)), \quad (3.8)$$

$$0 \leqslant \epsilon_2\beta^2\zeta^T(s-h_3(s))\zeta(s-h_3(s))-\epsilon_2 g^T(s,\zeta(s-h_3(s)))g(s,\zeta(s-h_3(s))). \quad (3.9)$$

According to (3.1)-(3.9), it is straightforward to see that

$$\dot{V}(s) \leqslant \eta^T(s)\Sigma\eta(s), \quad (3.10)$$

where

$$\begin{aligned}\eta(s)=\mathrm{col}\{&\zeta(s),\ \zeta(s-h_1(s)),\ \zeta(s-h_2(s)),\ \zeta(s-h_3(s)),\ \zeta(s-h_1),\ \zeta(s-h_1-h_2(s)),\\ &\zeta(s-h_2),\ \zeta(s-h_2-h_1(s)),\ \zeta(s-h_3),\ \frac{1}{h_1}\int_{s-h_1}^{s}\zeta(u)du,\ \frac{12}{h_1^2}\int_{s-h_1}^{s}\int_{s-h_1}^{\theta}\zeta(u)dud\theta,\\ &\frac{12}{h_1^3}\int_{s-h_1}^{s}\int_{s-h_1}^{\theta}\int_{s-h_1}^{u}\zeta(r)drdud\theta,\ \frac{12}{h_2}\int_{s-h_2}^{s}\zeta(u)du,\ \frac{12}{h_2^2}\int_{s-h_2}^{s}\int_{s-h_2}^{\theta}\zeta(u)dud\theta,\\ &\frac{12}{h_2^3}\int_{s-h_2}^{s}\int_{s-h_2}^{\theta}\int_{s-h_2}^{u}\zeta(r)drdud\theta,\ \frac{1}{h_3}\int_{s-h_3}^{s}\zeta(u)du,\ \frac{12}{h_3^2}\int_{s-h_3}^{s}\int_{s-h_3}^{\theta}\zeta(u)dud\theta,\\ &\frac{12}{h_3^3}\int_{s-h_3}^{s}\int_{s-h_3}^{\theta}\int_{s-h_3}^{u}\zeta(r)drdud\theta,\ \dot{\zeta}(s),\ \int_{s-h_1(s)}^{s}\zeta(u)du,\ \int_{s-h_1}^{s-h_1(s)}\zeta(u)du,\\ &\int_{s-h_2(s)}^{s}\zeta(u)du,\ \int_{s-h_2}^{s-h_2(s)}\zeta(u)du,\ \int_{s-h_3(s)}^{s}\zeta(u)du,\ \int_{s-h_3}^{s-h_3(s)}\zeta(u)du,\\ &f(s,\zeta(s)),\ g(s,\zeta(s-h_3(s)))\}.\end{aligned}$$

For (3.10), if the conditions (3.1) hold, then $\dot{V}(s) \leqslant -\delta\|\zeta(s)\|^2$ for some $\delta > 0$. Therefore, system (2.1) is asymptotically stable. The proof of this statement is complete. □

Remark 3.2. When $C = 0, D = 0$, the system (2.1) without nonlinear term is reduced to the following linear system

$$\dot{\zeta}(s)=A\zeta(s)+B\zeta(s-h_1(s)-h_2(s)), \quad \zeta(s)=\phi(s), \quad \forall s\in[-(h_1+h_2),0],$$

and analogous to the logic of Theorem 3.1, Corollary 3.3 is directly acquired as follows. By employing the Lyapunov-Krasovskii functional, we get that it is the same as Theorem 3.1. We introduce the following notations for later use:

$$\hat{\sum}=[\hat{\Delta}_{(i,j)}]_{25\times 25},$$

where $\hat{\Delta}_{(i,j)}=\hat{\Delta}^T_{(j,i)}=\Delta_{(i,j)}, i,j=1,2,3,\dots,25$, except $\epsilon_1=0$ and $\epsilon_2=0$.

Corollary 3.3. *The system (2.1) is asymptotically stable, if for given scalars $h_m, \tau_m, m=1,2,3$, there exist positive definite matrices G_1, G_3, G_4, G_6, G_7, G_9, $P_i, R_j, i=1,2,\dots,12$ and $j=1,2,3$ and any appropriate dimensional matrices Q, G_2, G_5, G_8 such that the symmetric linear matrix inequalities hold:*

$$\begin{bmatrix}G_1 & G_2\\ * & G_3\end{bmatrix}>0, \quad \begin{bmatrix}G_4 & G_5\\ * & G_6\end{bmatrix}>0, \quad \begin{bmatrix}G_7 & G_8\\ * & G_9\end{bmatrix}>0, \quad \hat{\sum}<0.$$

Proof. The proof is similar to that in Theorem 3.1 and so it is omitted. □

4. Numerical examples

Example 4.1. Consider the system (2.1) with the following parameters:

$$A = \begin{bmatrix} -3.4 & -3.2 \\ 1 & -5.8 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1 & 0.1 \\ -0.8 & 0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad D = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.4 \end{bmatrix}. \tag{4.1}$$

By using the LMI Toolbox in MATLAB to apply Theorem 3.1 to system (2.1) with (4.1), the maximum upper bound of τ_2 is found to be 0.9 for the parameters $h_1 = 0.1$, $h_2 = 0.3$, $\tau_1 = 0.2$, $\eta = 0.3$, and $\beta = 0.04$. Furthermore, following the criteria of [1], we obtain that the system is asymptotically stable for $\tau_2 = 0.6$.

The response solution $\zeta(s)$ in Example 4.1 is shown in Figure 1. Here, the parameters are fixed by $f(s, \zeta(s)) = \begin{bmatrix} \sin(s)\zeta_1(s) \\ \sin(s)\zeta_2(s) \end{bmatrix}$, $g(s, \zeta(s - h_3(s))) = \begin{bmatrix} \sin(s)\zeta_1(s - h_1(s) - h_2(s)) \\ \sin(s)\zeta_2(s - h_1(s) - h_2(s)) \end{bmatrix}$, $h_1(s) = 0.1\sin^2(s)$, $h_2(s) = 0.3\sin^2(s)$ and the initial condition $\phi(s) = \begin{bmatrix} 4 & -4 \end{bmatrix}^T$, $t \in [-0.4, 0]$.

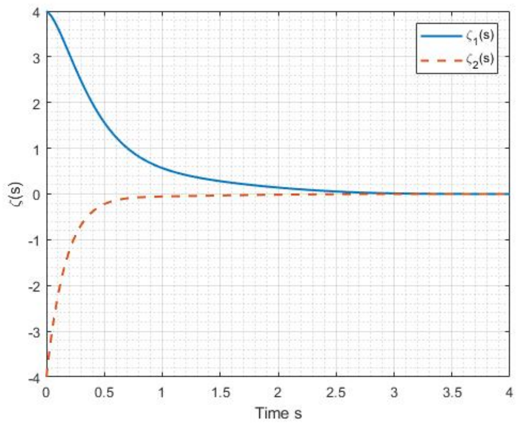

Figure 1: The trajectories of solutions $\zeta(s)$ of system (2.1).

5. Conclusion

This research introduces a new criterion for determining the asymptotic stability of linear system that exhibit two additive time-varying delays and nonlinear perturbations. The approach relies on various mathematical techniques, including the decomposition of coefficient constants, the Newton-Leibniz formula, the extended Jensen's double integral inequality, the extended Wirtinger's integral inequality, the creation of a new Lyapunov-Krasovskii functional, and the application of the zero equation principle. These combined techniques lead to the establishment of sufficient conditions for asymptotic stability, which are expressed in the form of Linear Matrix Inequalities (LMIs). In addition, we introduce new stability conditions that depend on delay for linear systems with two additive time-varying delays. The research also includes numerical examples to illustrate the practical applicability and effectiveness of the proposed theorem. Additionally, considering the asymptotic stability of a nonlinear system with two additive time-varying delays and distributed delays is suggested as a future research topic.

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