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Numerical study for the fractional model of banks' competition using two efficient computational methods

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Abstract

In this paper, we examine and analyze how Egyptian banks compete for profits. Four fractional differential equations make up this model. As a first step toward mitigating its sudden negative impact, we present the appropriate optimal control technique for bank profits during crises (for instance, the Covid-19 crisis). The proposed system is then solved using the spectral collocation method (SCM). This approach efficiently solves the model (with Caputo-sense of fractional derivative) by approximating the solution using Gegenbauer wavelet polynomials (GWPs). In addition, we consider the same model but with another type of derivative known as Caputo-Fabrizio (CF), and use its properties to convert the proposed model into a system of fractional integral equations which are numerically estimated with the help of the Simpson's-1/3 rule as an efficient numerical technique for integration. By contrasting the findings of the given techniques with those of the RK4 method, we can confirm their efficiency and accuracy. The results show that the techniques are efficient tools for simulating the solution to such a problem. The sudden drop in bank profits that occurred in 2020 as a result of the Covid-19 crisis may have been offset by the control mechanism in place. Special attention is given to studying the effect of some parameters in the system to provide a complete numerical simulation of the system which can be used in making decisions related to banking operations.

Keywords: Banks' competition model, optimal control, fractional derivative, Gegenbauer wavelet polynomials, SCM, numerical integration, RK4 method.

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1. Introduction

Not only are mathematical models helpful for resolving problems in science, but they are also gaining popularity in fields related to economic growth, such as banking and finance. A mathematical competition model, consisting primarily of a set of ODEs, is typically used to simulate the competition between banks [19, 23]. The Lotka-Volterra system is used in the business world to compare the profits banks make [8, 41]. The authors in [10, 39] utilized the same model driven using many definitions of the fractional

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derivative to depict the competitive dynamics of Indonesian commercial and rural banks. According to the Central Bank of Egypt (CBE), the 33 banks that make up the majority of the banking system in Egypt fall into four categories: private, public, Arabic, and international investment partnership banks [7]. The directives made public by CBE apply to each of these banks. The CBE performs the dual roles of national monetary authority and central bank for the nation. As there is no clear distinction in their business sectors, this study focuses on the competition among these banks, there may be competition amongst the four bank groups previously described [26].

One area of mathematical analysis that has been researched in numerous real-world applications is fractional calculus (FC), as discussed in [15, 16]. Fractional analysis has drawn increasing attention in biology, where the scaling power law of fractional order generally serves as an empirical characterization of complicated events [2, 12, 18], and the bank competition model [9, 29].

Gegenbauer wavelets have been used to solve the fractional system of Burger's equations [28], fractional KdV-Burger-Kuramoto equation [35]. In this method, the domain discretization is not needed, and the approximation of the nonlinear term is an important advantage. This technique converts the system of fractional differential equations (FDEs) into a system of algebraic equations with unknown coefficients of the required approximate solution of the system under study.

Currently, there exists a substantial body of research pertaining to novel fractional derivatives, such as the distributed order definition, in conjunction with various numerical techniques, including the alternating direction implicit methods, localized kernel-based meshless methods, and local meshless methods. As shown in [11, 24, 25], these are used respectively to solve a wide range of problems, such as the multidimensional distributed-order fractional integro-differential problems, fractal transmission systems, and time fractional Tricomi-type models. The dual Bernstein operators, collocation-Galerkin method [22], and fractional B-spline functions, along with a robust numerical algorithm incorporating hypergeometric functions and the innovative coupling of the Daftardar-Jafari method, are utilized in various applied models, including solutions to the fractional convection-diffusion equation related to underground water pollution [33], analysis of delay fractional variational problems [34], nonlinear fractional KdV equation [32], and stochastic fractional integro-differential equations [40].

Numerous physical problems associated with fractional differential equations lack established accurate solutions, so numerical and approximate methods must be applied [14, 17]. Among these techniques, are the Adams-Bashforth with the CF-operator [27], and the trapezoidal scheme [13]. This study presents the development of the fractional Simpson's 1/3 approach for addressing the presented model [20, 30] with the CF-derivative with a fourth-order accuracy, as well as the SCM with Gegenbauer wavelets. The advantage of these methods is that they exhibit superior accuracy compared to current methodologies and are straightforward to deploy.

Definition 1.1. The Caputo fractional derivative D^{ν} of $\varphi(t)$ is formulated as follows:

$$D^{\nu}\varphi(t) = \frac{1}{\Gamma(1-\nu)}\int_0^t \frac{\varphi'(\tau)}{(t-\tau)^{\nu}}d\tau, \quad t>0, \quad 0<\nu<1.$$

Definition 1.2. For $\psi(t) \in \mathbb{H}^1(0,a)$, $0<\gamma<1$. Then the CF fractional derivative ${}^{CF}D^{\gamma}\psi(t)$ and its corresponding integral ${}^{CF}I^{\gamma}\psi(t)$, respectively, are defined by

$$\begin{aligned} {}^{CF}D^{\gamma}\psi(t) &= \frac{1}{1-\gamma}\int_0^t \text{Exp}\Big[-\frac{\gamma}{1-\gamma}(t-\tau)\Big]\dot{\psi}(\tau)d\tau, \\ {}^{CF}I^{\gamma}\psi(t) &= (1-\gamma)\psi(t)+\gamma\int_0^t \psi(\tau)d\tau. \end{aligned} \tag{1.1}$$

To read more and more [4, 31].

Here in this article, a study and numerical simulation of the banking system which is represented by differential equations in their fractional form was presented (where we dealt with the system with two different types of fractional derivatives). This study was done by using two methods. The first is analytical, which is the SCM with the help of Gegenbauer wavelet polynomials, and it has many applications as it is characterized by good properties in approximation, including ease of application and accuracy of the resulting in solutions, and we do not need complex calculations, and others. The second method is numerical, which is the Simpson's-1/3 method, which is used to solve the integral equations resulting from the proposed system, and it also has an important advantage in approximation, as the error in it, is of the order four, in addition to its ease of application and ease of programming with it. Finally, a comparison was made with the fourth-order Runge-Kutta method to know the accuracy and efficiency of the methods used, in addition to making a simulation by studying the effect of some parameters present in the system itself or the method such as the fractional order, and the rate of increase and others.

The primary framework of this study is delineated as follows. Section 2 outlines the solution technique utilizing the SCM, wherein we derive and articulate certain features of Gegenbauer polynomials, formulate the Gegenbauer wavelets, and approximate functions. In Section 3, we outline the solution technique for the model under investigation by employing the Simpson's 1/3 rule, including the derivation of the Simpson's 1/3 scheme for the CF-fractional integral. Section 4 presents the numerical simulation of the proposed model utilizing the SCM and Simpson's 1/3 rule. Ultimately, Section 5 contains the conclusion.

2. Procedure of solution using the SCM

2.1. Some concepts of the GWPs and approximate the functions

The continuous wavelet is created by making continuous changes in the transmission parameter $\tau \in \mathbb{R}$ as well as in the dilation parameter $\gamma \in \mathbb{R}^*$ of the mother wavelet $\phi(t)$, which takes the following form [5]:

$$\phi_{\gamma,\tau}(t) = |\gamma|^{-1/2}\, \phi\left(\frac{t-\tau}{\gamma}\right).$$

We can get the discrete wavelets (which provide a basis in $L^2(\mathbb{R})$) of the following form by taking the limit of two parameters $\gamma \to \gamma_0^{-\ell}$ and $\tau \to \epsilon\, \tau_0\, \gamma_0^{-\ell}$, where $\gamma_0 > 1$, τ_0, ℓ, $\epsilon > 0$ [37]:

$$\phi_{\ell,\epsilon}(t) = \gamma_0^{\ell/2}\, \phi(\gamma_0^{\ell}\, t - \epsilon\, \tau_0).$$

The orthonormal basis can be obtained in the case $\gamma_0 = 2$ and $\tau_0 = 1$, as follows [6]:

$$\phi_{\ell,\epsilon}(t) = 2^{\ell/2}\phi(2^{\ell}\, t - \epsilon).$$

The Gegenbauer polynomials $G_n^{\kappa}(t)$ of degree $n \in \mathbb{N}$ can be generated by using the following recurrence formula [38]:

$$G_{n+1}^{\kappa}(t) = \frac{1}{n+1}\big(2(n+\kappa)\, t\, G_n^{\kappa}(t) - (n+2\kappa-1)\, G_{n-1}^{\kappa}(t)\big), \quad G_0^{\kappa}(t) = 1, \quad G_1^{\kappa}(t) = 2\kappa\, t,$$

the ultraspherical parameter is known to be $\kappa > -0.5$. For $\kappa = 1/2$, $\kappa = 0$, and $\kappa = 1$, we obtain Legendre wavelets and the first and second forms of Chebyshev wavelets, respectively. The following polynomials exhibit orthogonality on the interval $[-1, 1]$:

$$\int_{-1}^{1} \frac{1}{(1-s^2)^{0.5-\kappa}}\, G_i^{\kappa}(s)\, G_j^{\kappa}(s)\, ds = L_i^{\kappa}\, \delta_{ij}.$$

We use the following L_i^{κ} for normalization:

$$L_i^{\kappa} = \frac{\pi\, 2^{1-2\kappa}\, \Gamma(n+2\kappa)}{n!\, (n+\kappa)(\Gamma(\kappa))^2}.$$

The Gegenbauer wavelets can be generated on $[0, 1]$ in the following form [35]:

$$\phi_{\epsilon,n}^{\kappa}(t) = \frac{1}{\sqrt{L_n^{\kappa}}}\, 2^{\ell/2}\, G_n^{\kappa}(2^{\ell}t - 2\,\epsilon + 1), \quad \frac{2\,\epsilon-2}{2^{\ell}} \leqq t \leqq \frac{2\,\epsilon}{2^{\ell}},$$

where $\epsilon = 1, 2, \ldots, 2^{\ell-1}$, $n = 0, 1, \ldots, N-1$ for $N \geqq 1$, and $\ell \in \mathbb{N}$ is the level of resolution. The first few

GWPs can be given as follows by taking $\ell = 1$, $N = 4$, and $\kappa = 30$:

$$\phi^{30}_{1,0}(t) = 2.49122, \quad \phi^{30}_{1,1}(t) = 39.2318\,t - 19.6159, \quad \phi^{30}_{1,2}(t) = 447.481t^2 - 447.481t + 110.066.$$

A function $\theta(t)$ can be approximated by the following finite series of the GWPs:

$$\theta_m(t) \cong \sum_{i=1}^{2^{\ell-1}} \sum_{j=0}^{m-1} c_{ij}\, \phi_{ij}(t). \tag{2.1}$$

Now, we can give an approximation of the fractional derivative $D^\nu \theta_m(t)$, with the help of the property that $D^\nu = D^s I^{s-\nu}$, for $s - 1 < \nu < s$, which enables us to differentiate and integrate the double sum in (2.1) term per term within the interval of convergence for specified ℓ and m. So

$$D^\nu \theta_m(t) = \sum_{i=1}^{2^{\ell-1}} \sum_{j=0}^{m-1} c_{ij} D^s I^{s-\nu}[\phi_{ij}(t)]. \tag{2.2}$$

2.2. Numerical implementation for the SCM

In this subsection, we present an implementation of the proposed technique to solve numerically the studied model.

2.2.1. The standard form of the banks' competition model

Here in this paper, we use the symbols $[P(t), A(t), F(t), I(t)] = [\psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t)]$ and define the fractional banks' competition model with derivative $0 < \nu \leqslant 1$ as follows:

$$D^\nu \psi_i(t) = \alpha_i \psi_i(t) \left(1 - \frac{\psi_i(t)}{\beta_i}\right) - \delta_i G(\psi_1, \psi_2, \psi_3, \psi_4), \quad i = 1, 2, 3, 4, \tag{2.3}$$

where the nonlinear term $G(\psi_1, \psi_2, \psi_3, \psi_4) = \psi_1(t)\psi_2(t)\psi_3(t)\psi_4(t)$ is the market region where the four banks engage. With the initial conditions are defined by:

$$\psi_i(0) = \hat{\hat{\psi}}_i, \quad i = 1, 2, 3, 4. \tag{2.4}$$

Also, the parameters included in the model are defined as follows (for $k = 1, 2, 3, 4$).

1. α_k are the growth rates of public/private ($P(t)$) and Arabic ($A(t)$), foreign ($F(t)$), and investment ($I(t)$) collaboration banks, respectively.
2. β_k are the maximum profits archived by the above four categories.
3. δ_k are the competition parameters.

Let us approximate the unknown functions $\psi_k(t)$, $k = 1, 2, 3, 4$, in terms of GWPs, by $\psi_{k,m}(t)$, as follows:

$$\psi_{k,m}(t) \cong \sum_{i=1}^{2^{\ell-1}} \sum_{j=0}^{m-1} a^k_{ij} \phi_{ij}(t), \quad k = 1, 2, 3, 4. \tag{2.5}$$

By substitution from (2.2) and (2.5) in the system (2.3), we get

$$\sum_{i=1}^{2^{\ell-1}} \sum_{j=0}^{m-1} a^k_{ij} D^s I^{s-\nu}[\phi_{ij}(t)] = \alpha_k \psi_{k,m}(t) \left(1 - \beta_k^{-1} \psi_{k,m}(t)\right) \delta_k G(a^1_{ij}, a^2_{ij}, a^3_{ij}, a^4_{ij}; t), \quad k = 1(1)4, \tag{2.6}$$

where

$$G(a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4; t) = \prod_{k=1}^{4} \left(\sum_{i=1}^{2^{\ell-1}} \sum_{j=0}^{m-1} a_{ij}^k \phi_{ij}(t) \right).$$

By the collocation of these last equations (2.6) at $t_r = \frac{(2r-1)h}{2^\ell N}$, where $r = 1, 2, \ldots, 2^{\ell-1}m$ and $0 \leqq t_i \leqq h$, we obtain the following system of algebraic equations:

$$\sum_{i=1}^{2^{\ell-1}} \sum_{j=0}^{m-1} a_{ij}^k D^s I^{s-\nu}[\phi_{ij}(t_r)] = \alpha_k \psi_{k,m}(t_r) \left(1 - \beta_k^{-1} \psi_{k,m}(t_r)\right) \delta_k G(a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4; t_r), \quad k = 1(1)4. \tag{2.7}$$

Also, upon substituting Eq. (2.5) into (2.4), the I.Cs (2.4) will be converted to:

$$\sum_{i=1}^{2^{\ell-1}} \sum_{j=0}^{m-1} a_{ij}^k \phi_{ij}(0) = \hat{\psi}_k, \quad k = 1, 2, 3, 4. \tag{2.8}$$

We implement the Newton iteration method for solving the nonlinear system (2.7)-(2.8) for the GW coefficients a_{ij}^k, $k = 1, 2, 3, 4$, $i = 0, 2, \ldots, 2^{\ell-1}m$, $j = 0, 1, \ldots, m-1$. Then the approximate solution can be obtained by substitution in (2.5).

2.2.2. Controlling of banks' profits

Currently, a range of problems in real-world applications have been successfully addressed by the optimal control theory [36]. An intriguing the question in our proposed banking system is how to influence the profits in the given competition model (2.3)-(2.4) by choosing an effective method to transition the system from an initial state $(P(0), A(0), F(0), I(0))$ to a specified objective, namely profit maximization, or a targeted final state $(P(T_f), A(T_f), F(T_f), I(T_f))$ within the time frame T_f. Let's think about the following system to do this:

$$D^\nu \psi_k(t) = \alpha_k \psi_k(t) \left(1 - \frac{\psi_k(t)}{\beta_k}\right) - \delta_k \psi_1 \psi_2 \psi_3 \psi_4 - \varepsilon_k O_c(t) \psi_k(t), \quad k = 1, 2, 3, 4, \tag{2.9}$$

with the same previous initial conditions. The constants ε_k, $k = 1, 2, 3, 4$ stand for the upper limit of investment for each of the four categories of banks, therefore $\varepsilon_k = 1$, $k = 1, 2, 3, 4$ indicates that there are no investment restrictions. Additionally, the control parameter $O_c(t) \in L^2[0, T_f]$ indicates how much of each population's profit is retained by the bank for reinvestment in the market or taken from the market as a result of outside causes. Theorem 4 in [26] may be used to derive and acquire the optimum choice for this function.

In the same way, as in the previous subsection, we can construct a numerical scheme for solving the modified system (2.9).

3. The procedure of solution using the Simpson's-1/3 rule

3.1. Derivation of the Simpson's-1/3 scheme for the CF-fractional integral

In this subsection, we formulate the fractional Simpson's-1/3 scheme to find the solution for CF-FDEs [3]. In the Simpson rule-1/3, we use the quadratic polynomials to approximate the integration's function, so it is a closed scheme of numerical integration. Taking into account, the following γ order FDE:

$$^{CF}D^\gamma u(t) = f(u(t)), \quad u(0) = u_0, \tag{3.1}$$

where f denotes a continuous function that fulfills the subsequent Lipschitz condition:

$$|f(u(t_1)) - f(u(t_2))| \leqslant \kappa |u(t_1) - u(t_2)\|, \quad \kappa > 0. \tag{3.2}$$

Utilizing the CF-fractional integral operator on Eq. (3.1) and the formula (1.1), we get ([1]):

$$u(t) = u_0 + {}^{CF}I^{\gamma} f(u(t)) = u_0 + (1-\gamma) f(u(t)) + \gamma \int_0^t f(u(s))ds. \tag{3.3}$$

Theorem 3.1 ([21])**.** *Consider the continuous function* $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ *satisfying the Lipschitz condition* (3.2)*, where* $0 < \gamma < 1$ *and* $T > 0$*. The IVP* (3.1) *possesses a unique solution on the interval* $C[0, T]$ *given the following condition:*

$$\frac{(2(1-\gamma) + 2\gamma T)\kappa}{(2-\gamma)} < 1.$$

Now we will estimate the value of the integral in equation (3.3) by approximating the integral function f by using the second-degree polynomial P_2 and that function will be evaluated at t_0, t_1, and t_2, with $t_0 < t_1 < t_2$, where that time interval with width $2h$ is divided into two sub-intervals like $t_1 - t_0 = t_2 - t_1 = h$. These concepts are explained below:

$$I_2 f(u(t)) = \int_a^b f(u(t))dt \approx \int_{t_0}^{t_2} P_2(t)dt = \int_{t_0}^{t_2} \left[L_0(t) f(u(t_0)) + L_1(t) f(u(t_1)) + L_2(t) f(u(t_2))\right] dt,$$

where $L_0(t)$, $L_1(t)$, and $L_2(t)$ are second-order Lagrange polynomials. Integrating $L_0(t)$, with $h = \frac{t_2 - t_0}{2}$ and substituting "$t = s + t_0$", gives us:

$$\begin{aligned}\int_{t_0}^{t_2} L_0(t)dt &= \frac{1}{2h^2}\int_{t_0}^{t_0+2h} (t-t_1)(t-t_2)\,dt \\ &= \frac{1}{2h^2}\int_0^{2h} (s+t_0-t_2)(s+t_0-t_1)\,ds = \frac{1}{2h^2}\int_0^{2h}(s-2h)(s-h)ds = \frac{h}{3}.\end{aligned}$$

Upon simplifying the remaining terms, we can get:

$$I_2(f) = \frac{h}{3}\left[f(u(t_0)) + 4f(u(t_1)) + f(u(t_2))\right].$$

Using Eq. (3.3), we get:

$$u(t_n) = u_0 + (1-\gamma) f(u(t_n)) + \frac{h}{3}\gamma\left[f(u(t_0)) + 4f(u(t_1)) + f(u(t_2))\right], \quad n = 0, 1, 2.$$

Now, to significantly improve the precision of the numerical integration, we partition the integration interval $[a, b]$ into $n \geqslant 2$ subintervals and apply the quadrature rule to each pair of adjacent subintervals at the same time. We define

$$h = \frac{b-a}{n}, \quad t_k = a + kh, \quad k = 0, 1, 2, \ldots, n.$$

By implementing the Simpson's-1/3 scheme to the sub-intervals $\left[t_{2k}, t_{2(k+1)}\right]$, $k = 0, 1, 2, \ldots, \frac{n-2}{2}$, we get:

$$\begin{aligned}I_n(f) &= \int_{t_0}^{t_2} f(u(t))dt + \int_{t_2}^{t_4} f(u(t))dt + \cdots + \int_{t_{n-2}}^{t_n} f(u(t))dt \\ &= \sum_{k=0}^{\frac{n-2}{2}} \int_{t_{2k}}^{t_{2k+2}} f(u(t))dt = \sum_{k=0}^{\frac{n-2}{2}} \left(\frac{h}{3}\left[f(u(t_{2k})) + 4f(u(t_{2k+1})) + f(u(t_{2k+2}))\right]\right).\end{aligned}$$

Using Eq. (3.3), we can define the Simpson's-1/3 scheme for the CF-FDE (3.1), for $n = 0, 1, 2, \ldots, m-1$ as follows:

$$u_{n+1} = u_0 + (1-\gamma) f(u_{n+1}) + \gamma\frac{h}{3}\left[f(u(t_0)) + 4\sum_{i=2,4,6}^{n} f(u(t_i)) + 2\sum_{j=1,3,5}^{n-1} f\left(u\left(t_j\right)\right) + f(u(t_{n+1}))\right]. \tag{3.4}$$

Denote $u(t_n) \simeq u_n$. Then (3.4) can be formatted as

$$u_{n+1} = u_0 + (1-\gamma) f(u_{n+1}) + \gamma h \sum_{r=0}^{n+1} \xi_r f(t_r), \quad n = 0,1,2,\dots,m-1, \tag{3.5}$$

where

$$\xi_r = \begin{cases} 1/3, & r = 0, n+1, \\ 2/3, & r = 1,3,5,\dots, \\ 4/3, & r = 2,4,6,\dots. \end{cases}$$

Lemma 3.2. *Assuming that* $f(u(t)) \in C^4([a,b])$*, the error of the Simpson's rule is approximated by:*

$$\left| \int_{t_0}^{t_{n+1}} f(u(s))ds - \gamma h \sum_{i=0}^{n+1} \xi_i f(u(t_i)) \right| \leqslant \hat{c} h^4,$$

where $\hat{c} = \frac{(b-a)f^{(4)}(\zeta)}{180}$*, for some constant* $a < \zeta < b$*,* $h = \frac{b-a}{n}$*, and* $t_k = a + hk$*,* $k = 0,1,\dots,n+1$.

Theorem 3.3. *The newly constructed fractional numerical approach* (3.5) *is conditionally convergent of order 4 with the following estimation:*

$$\| u(t_{n+1}) - u_{n+1} \| \leqslant \gamma \hat{c} c_h h^4.$$

3.2. *Numerical implementation using the Simpson's-1/3 rule*

In this subsection, we implement the proposed methodology to numerically resolve the model under examination in its fractional representation.

3.2.1. *The standard form of the banks' competition model*

The proposed fractional competition model between them with CF-fractional derivative $0 < \nu \leqslant 1$ is given as follows:

$$^{CF}D^{\nu} \psi_k(t) = \alpha_k \psi_k(t) \left(1 - \frac{\psi_k(t)}{\beta_k}\right) - \delta_k G(\psi_1, \psi_2, \psi_3, \psi_4), \qquad k = 1,2,3,4, \tag{3.6}$$

where the nonlinear term $G(\psi_1, \psi_2, \psi_3, \psi_4)$ is defined in (2.3).

The majority of current numerical approaches exhibit delayed convergence for this type of issue, leading to imprecise approximations [2]. This study numerically integrates the discretized system of the CF-FDEs using Simpson's 1/3 rule for integration. To establish a numerical scheme for the proposed model (3.6), it can be rewritten as:

$$^{CF}D^{\nu} \bar{\Psi}(t) = \mathbb{F}(\bar{\Psi}(t), t), \tag{3.7}$$

where $\bar{\Psi}(t) = [\psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t)]^T$, $\bar{\Psi}(0) = [\psi_1(0), \psi_2(0), \psi_3(0), \psi_4(0)]^T$,

$$\mathbb{F}(\bar{\Psi}(t), t) = [\mathbf{f}_1(\psi_1, \psi_2, \psi_3, \psi_4, t),\ \mathbf{f}_2(\psi_1, \psi_2, \psi_3, \psi_4, t), \mathbf{f}_3(\psi_1, \psi_2, \psi_3, \psi_4, t),\ \mathbf{f}_4(\psi_1, \psi_2, \psi_3, \psi_4, t)]^T, \tag{3.8}$$

where

$$\mathbf{f}_k(\psi_1, \psi_2, \psi_3, \psi_4, t) = \alpha_k \psi_k(t) \left(1 - \beta_k^{-1} \psi_k(t)\right) - \delta_k G(\psi_1, \psi_2, \psi_3, \psi_4), \quad k = 1,2,3,4.$$

By applying the CF-fractional integral operator to Eq. (3.7) and utilizing Proposition 3 in [1] along with formula (1.1), we obtain:

$$\bar{\Psi}(t) = \bar{\Psi}(0) + {}^{CF}I^{\nu} \mathbb{F}(\bar{\Psi}(t), t) = \bar{\Psi}(0) + (1-\nu) \mathbb{F}(\bar{\Psi}(t), t) + \nu \int_0^t \mathbb{F}(\bar{\Psi}(s), s) ds. \tag{3.9}$$

Applying the derived Simpson's-1/3 method for the integration on the RHS of (3.9), we get the following numerical scheme as constructed in the formula (3.5):

$$\bar{\Psi}_{n+1} = \bar{\Psi}(0) + (1-\nu)\mathbb{F}(\bar{\Psi}_{n+1}, t_{n+1}) + \nu h \sum_{r=0}^{n+1} \xi_r \mathbb{F}(\bar{\Psi}_r, t_r), \quad n = 0, 1, 2, \ldots, m-1,$$

where ξ_r, $r = 0, 1, \ldots, n+1$ are defined in (3.5). So, (3.6) transforms to:

$$\begin{aligned} \psi_{k,n+1} &= \psi_{k,0} + (1-\nu)\mathbf{f}_k(\psi_{1,n+1}, \psi_{2,n+1}, \psi_{3,n+1}, \psi_{4,n+1}, t_{n+1}) \\ &\quad + \nu h \sum_{r=0}^{n+1} \xi_r \mathbf{f}_k(\psi_{1,r}, \psi_{2,r}, \psi_{3,r}, \psi_{4,r}, t_r), \end{aligned} \tag{3.10}$$

where the functions $\mathbf{f}_k$, $k = 1, 2, 3, 4$ are defined in (3.8).

3.2.2. Controlling of banks' profits

In the same way, as in the previous subsection, we can construct the numerical scheme for solving the modified system (2.9).

4. Numerical simulation for the proposed model

4.1. Case 1: Using the SCM

To confirm the validity and excellence of the provided scheme, we present some simulations of the proposed system (2.3) and its modification (2.9) with optimal control in the interval $[0, 9]$, for different values of ν, m. The values for the parameters of [26] are taken in all Figures:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.7, 0.5, 0.45, 0.3), \qquad (\beta_1, \beta_2, \beta_3, \beta_4) = (33696, 31162, 11679, 3710),$$
$$(\delta_1, \delta_2, \delta_3, \delta_4) = (1.9, 2.3, 1.02, 5.0) \times 10^{-18}, \qquad (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (0.392, 0.327, 0.245, 0.295).$$

We consider the following initial conditions [26]:

$$\hat{\hat{p}} = 2.25, \quad \hat{\hat{a}} = 2.3, \quad \hat{\hat{f}} = 0.74, \quad \hat{\hat{\imath}} = 0.6250.$$

Furthermore, to evaluate the efficacy of the proposed SCM scheme, we juxtapose the findings derived from the recommended technique with those acquired by the RK4 method. Figures 1-4 present the numerical outcomes for the analyzed model by using the SCM scheme (2.7)-(2.8).

1. Figure 1 shows the solution with $\nu = 1.0, 0.95, 0.9, 0.85$, and $m = 7$.
2. In Figure 2, we compare the outcomes of the suggested method with those of the RK4 at ($\nu = 1$) with $m = 8$.
3. In Figure 3, the approximate solution of the modified system with the optimal control $O_c(t) = -2e^{-2t}$ by using the proposed method at ($\nu = 0.95$) and RK4 ($\nu = 1$) methods, at $m = 7$.
4. In Figure 4, the effect of the approximation order; $m = 6, 9$ to the approximate solution of the proposed model with and without optimal control is given.

The results indicate that the proposed technique is appropriate for addressing the problem in its Caputo fractional form, since they reveal that the characteristics of the numerical solution are contingent upon the values of ν and m. The great importance here is that we were able to overcome this rapid decline in bank profits during crises, by building an optimization function for monitoring and reinvestment as a proposal to build a pre-reinvestment monitoring system that can absorb this rapid loss in profits, and this simulation clearly shows how to manage Banks to make good profits. Finally, the proposed method significantly enhances efficiency and output of the given scheme.

4.2. Case 2: Using Simpson's-1/3 rule

To confirm the precision of the Simpson's-1/3 rule method (SRM), we also provide a comparison of the outcomes of the given scheme (3.10) with those acquired by employing the RK4 method.

Figures 5-11 present the numerical findings that were achieved for the model under study using the suggested technique (Simpson's-1/3 rule), with the same initial conditions considered above, and the same values of the remainder parameters.

1. The behavior of the numerical solution is shown in Figure 5 for $n = 50$, for $\nu = 1.0, 0.95, 0.9, 0.85$.
2. In Figure 6, we show a comparison between the results using the RK4 method at ($\nu = 1$) with $n = 90$ and the results using the suggested technique.
3. Figure 7 shows the approximate solution of the modified system with the optimal control $O_c(t) = -2e^{-2t}$ by using the proposed method at ($\nu = 0.95$) and RK4. In Figure 6 we show ($\nu = 1$) methods at $n = 90$.
4. In Figure 8, the effect of the approximation order n to the approximate solution of the proposed model with and without optimal control is given.
5. In Figure 9, the effect of the growth rates α_k, $k = 1, 2, 3, 4$ on the numerical solution $P(t)$, $A(t)$, $F(t)$, $I(t)$, respectively, of the model is given at $\nu = 0.95$, $n = 80$.
6. In Figure 10, the effect of the maximum profits β_k, $k = 1, 2, 3, 4$ on the numerical solution $P(t)$, $A(t)$, $F(t)$, $I(t)$, respectively, of the proposed model is given at $\nu = 0.95$, $n = 80$.
7. In Figure 11, the effect of the maximum level of investment ε_k, $k = 1, 2, 3, 4$ on the numerical solution $P(t)$, $A(t)$, $F(t)$, $I(t)$, respectively, of the proposed model is given at $\nu = 0.95$, $n = 80$.

Based on Figures 5-8, we can conclude that the suggested technique is appropriate for solving the system in its CF form because the behavior of the numerical solution that is obtained after applying the given technique depends on ν, n. From Figures 9-11, we can see that the effect of the growth rates α_k, the maximum profits β_k, and the maximum level of investment ε_k, $k = 1, 2, 3, 4$ on all components of the numerical solution consist of the natural effect as it is known in the banking system. This indicates that the proposed numerical technique has been well implemented to obtain the required solution for the system under study.

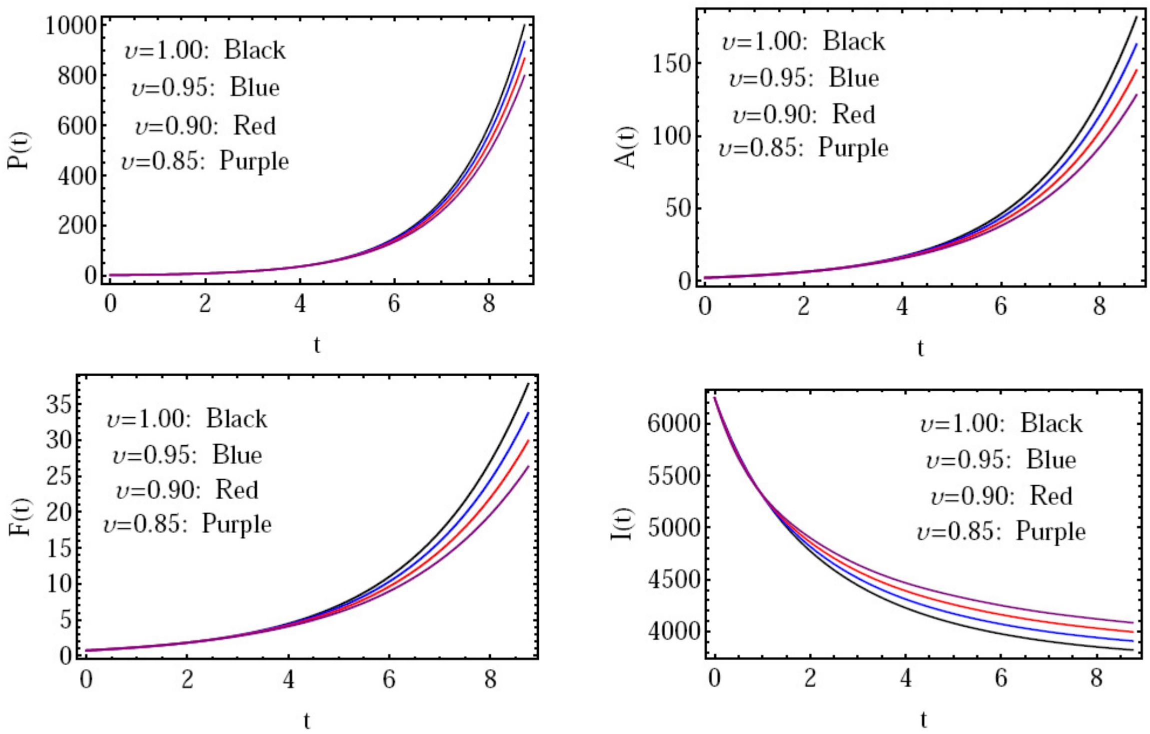

Figure 1: The approximate solution (SCM) against distinct values of ν.

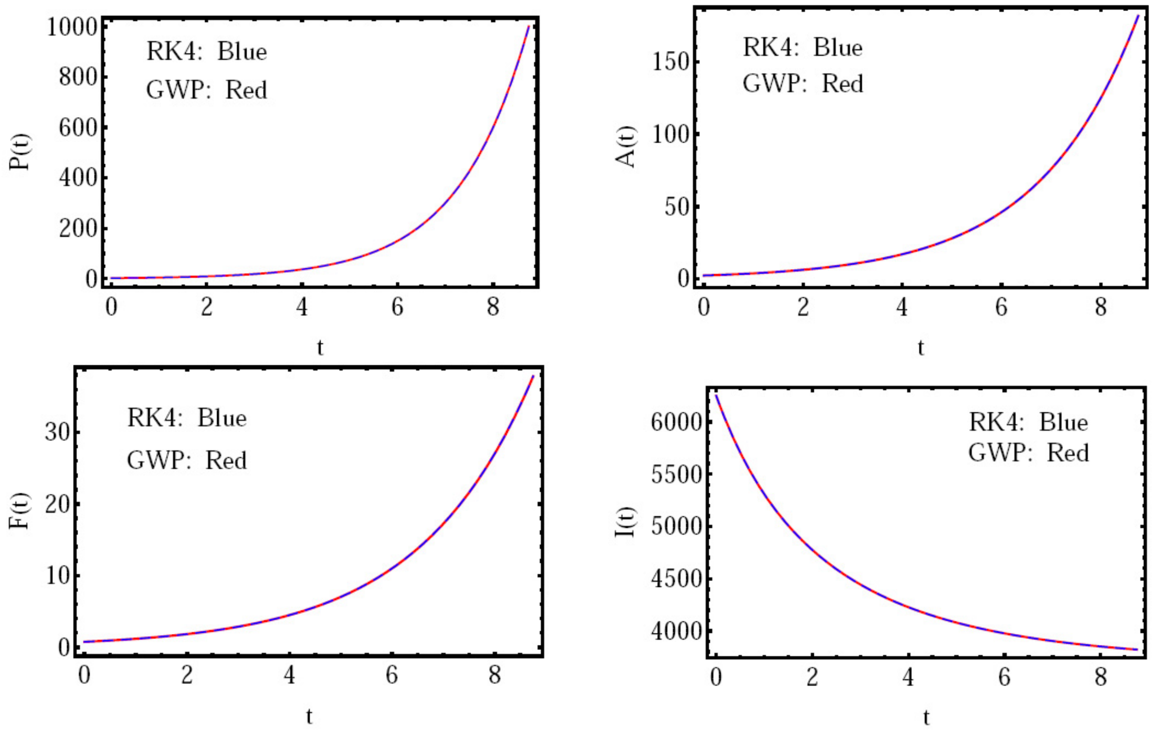

Figure 2: The approximate solution by the SCM and RK4 method $\nu = 1$.

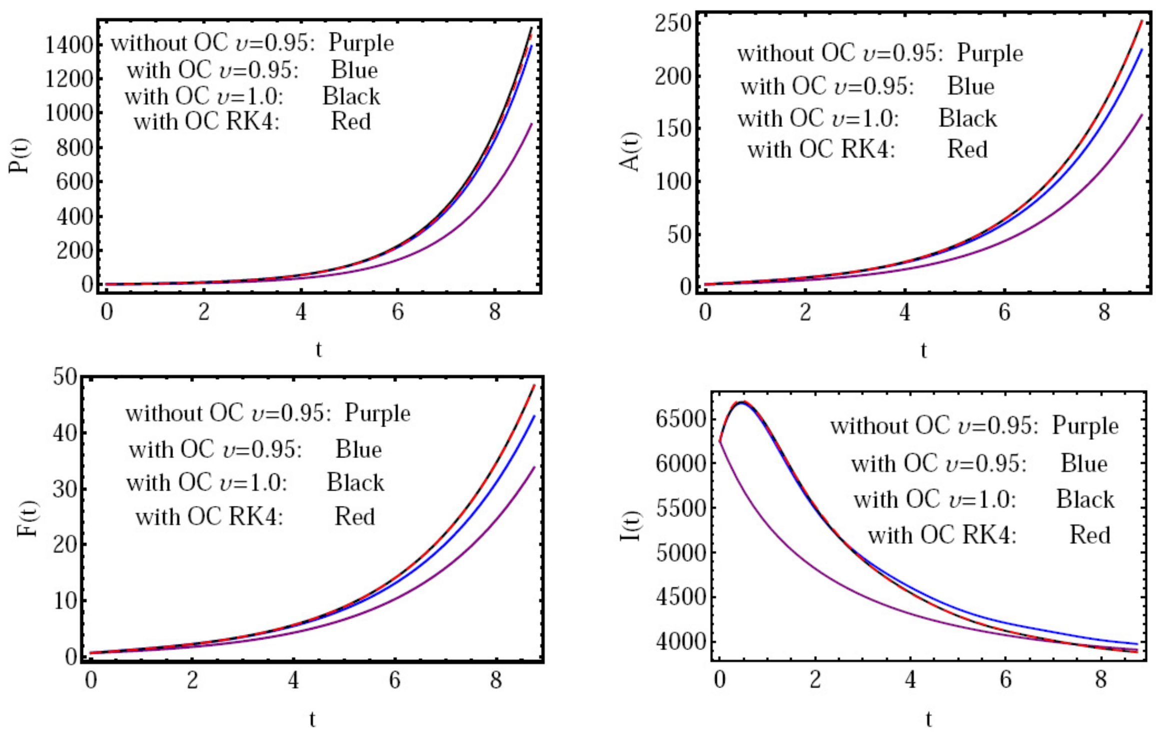

Figure 3: The approximate solution with optimal control by GWPs ($\nu = 0.95$) and RK4 ($\nu = 1$) methods.

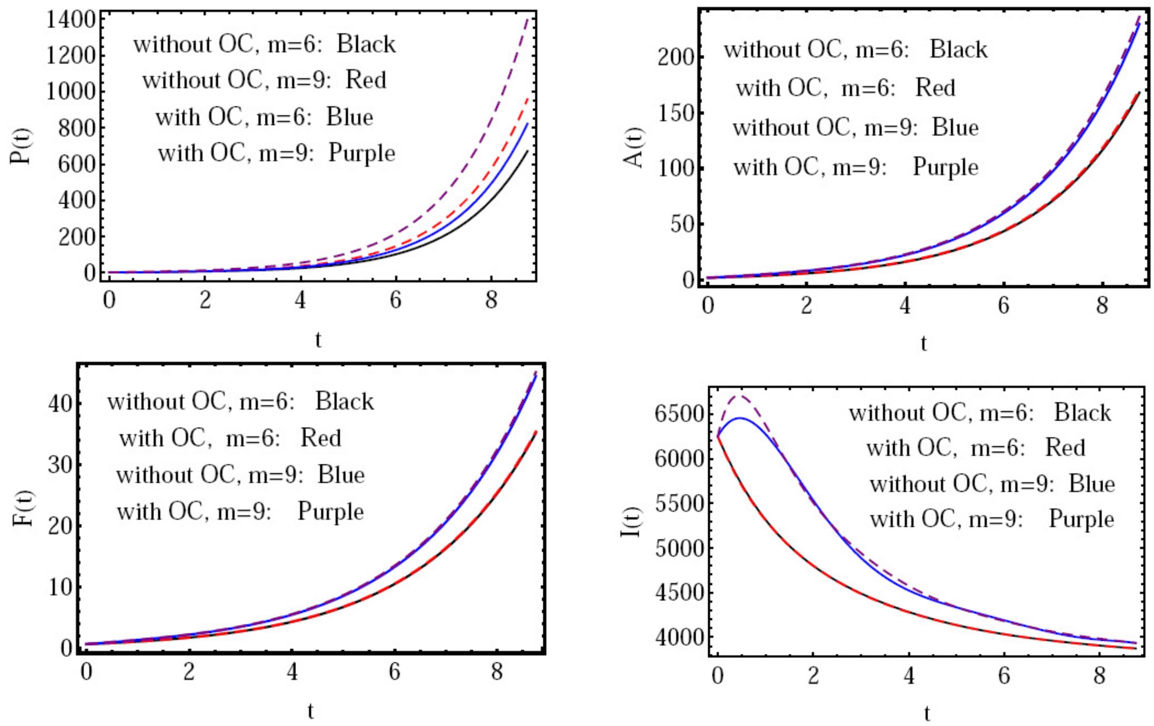

Figure 4: The approximate solution with and without optimal control against distinct values of m.

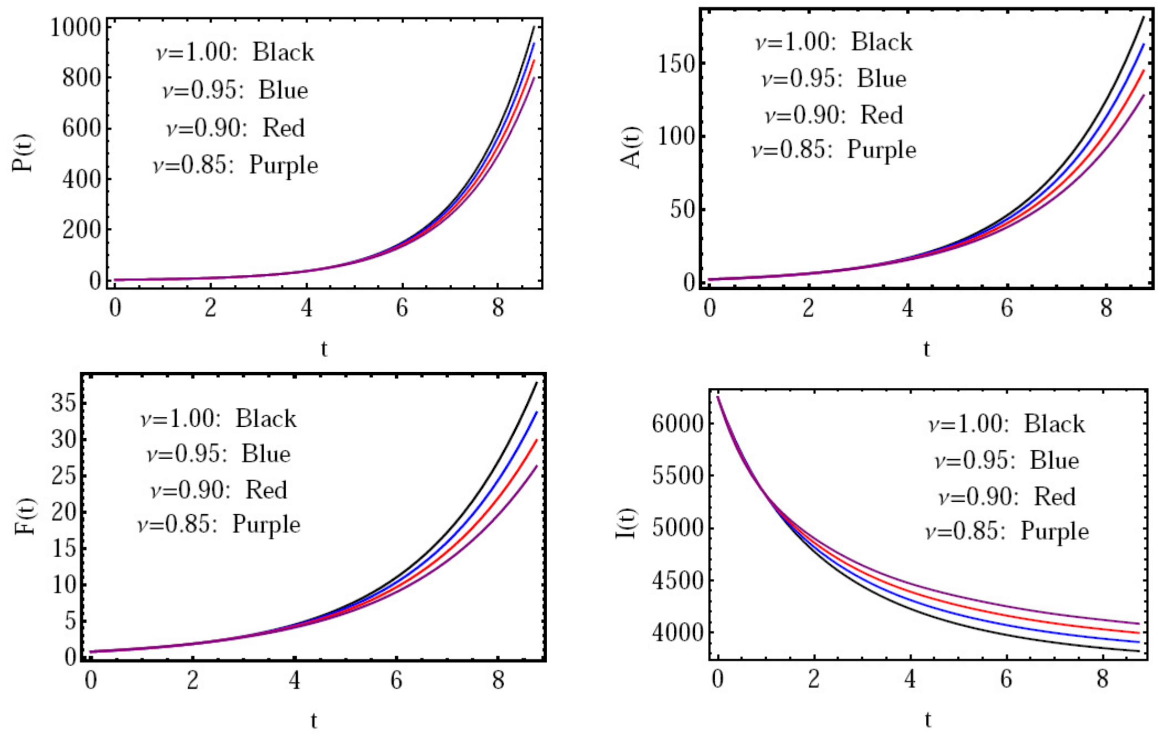

Figure 5: The numerical solution (SRM) against distinct values of ν.

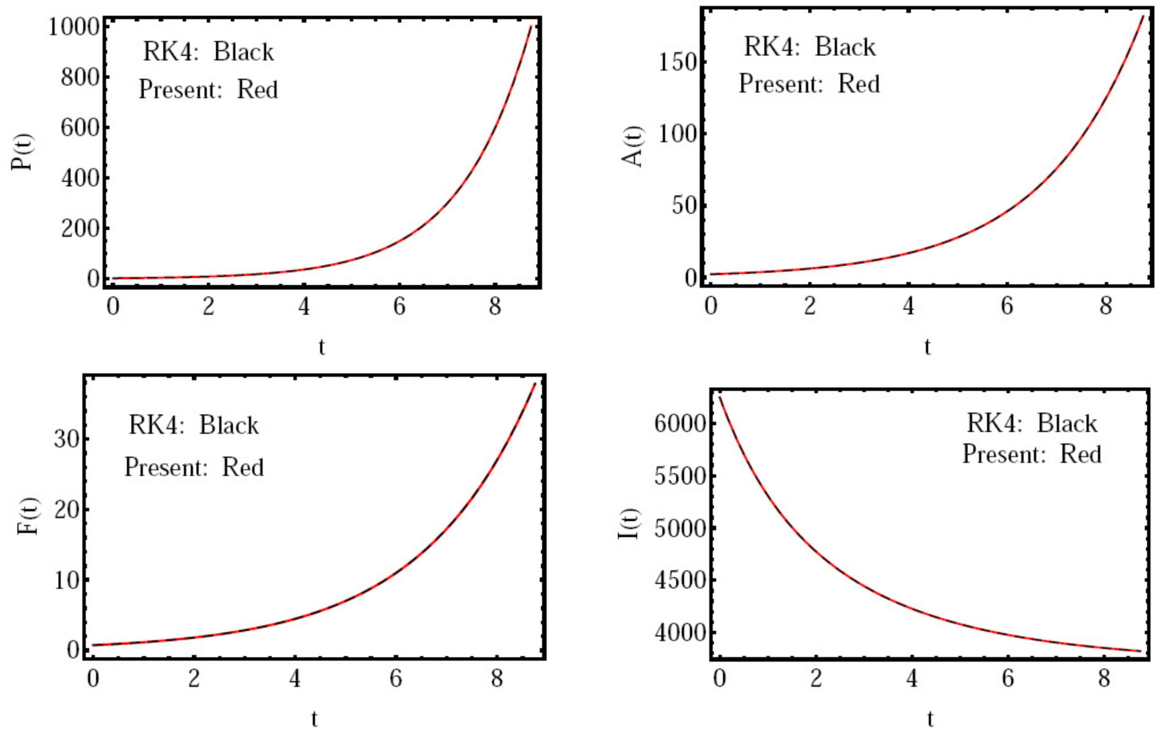

Figure 6: The numerical solution by the SRM and RK4 methods at $\nu = 1$.

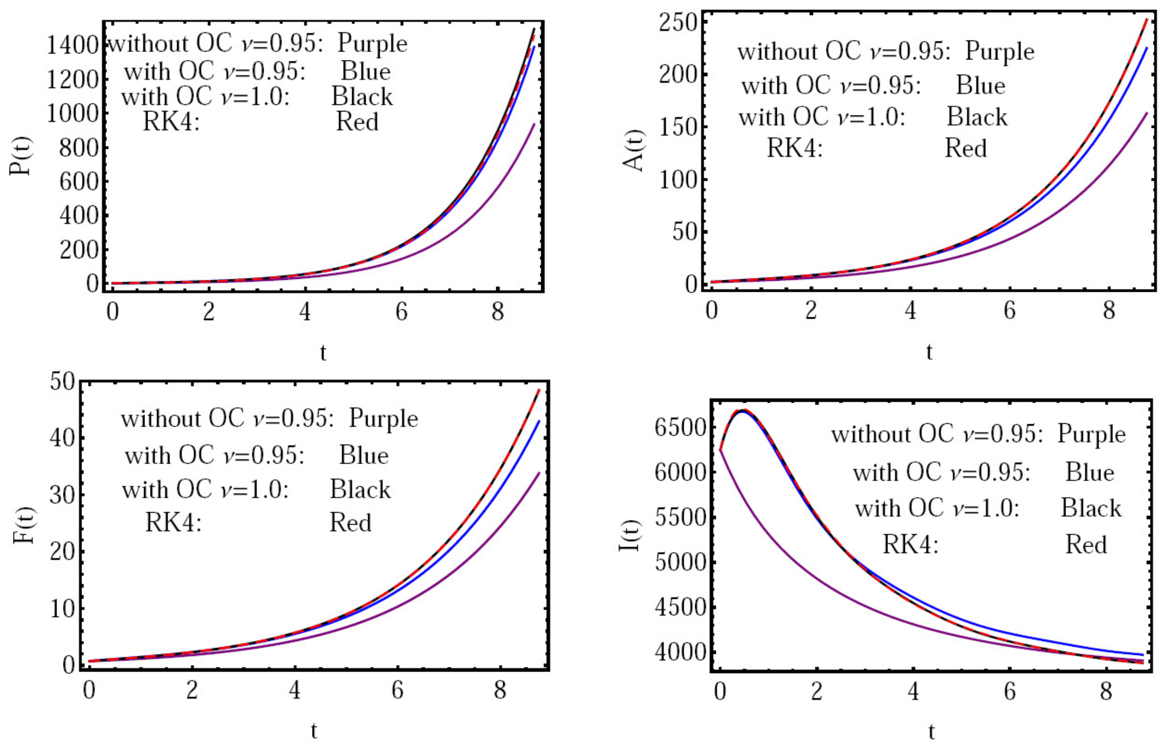

Figure 7: The numerical solution with optimal control by the SRM ($\nu = 0.95$) and RK4 method ($\nu = 1$).

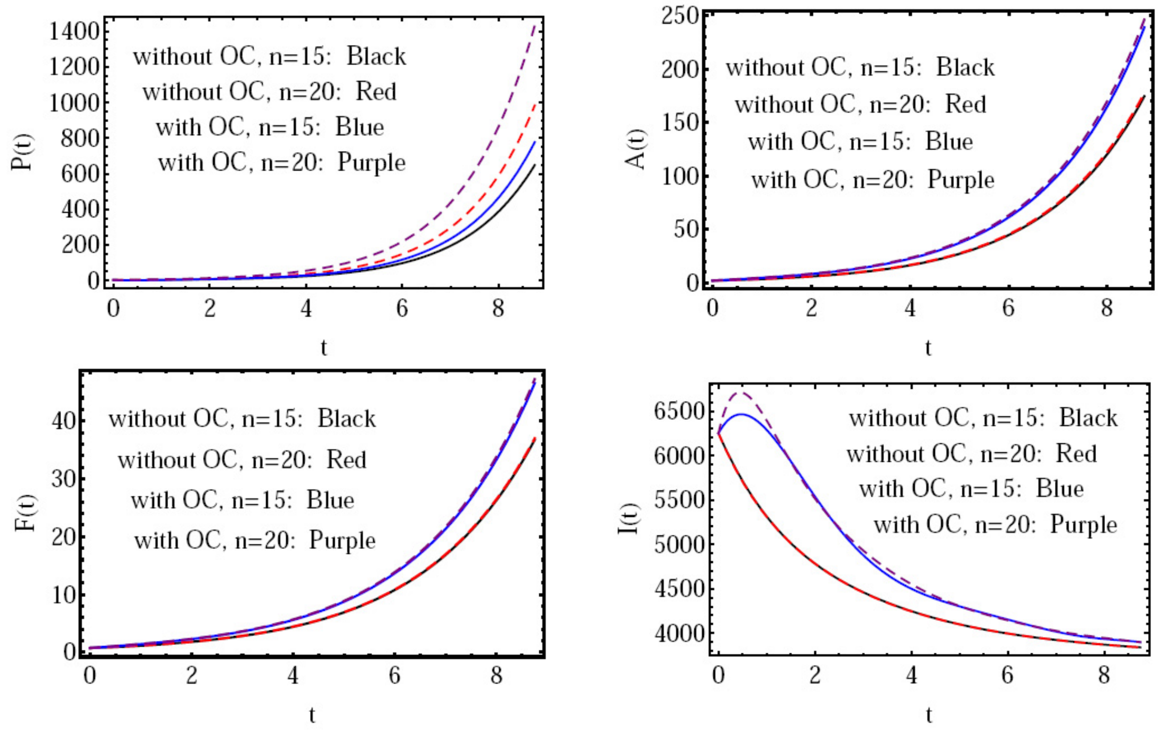

Figure 8: The numerical solution with and without optimal control against distinct values of n.

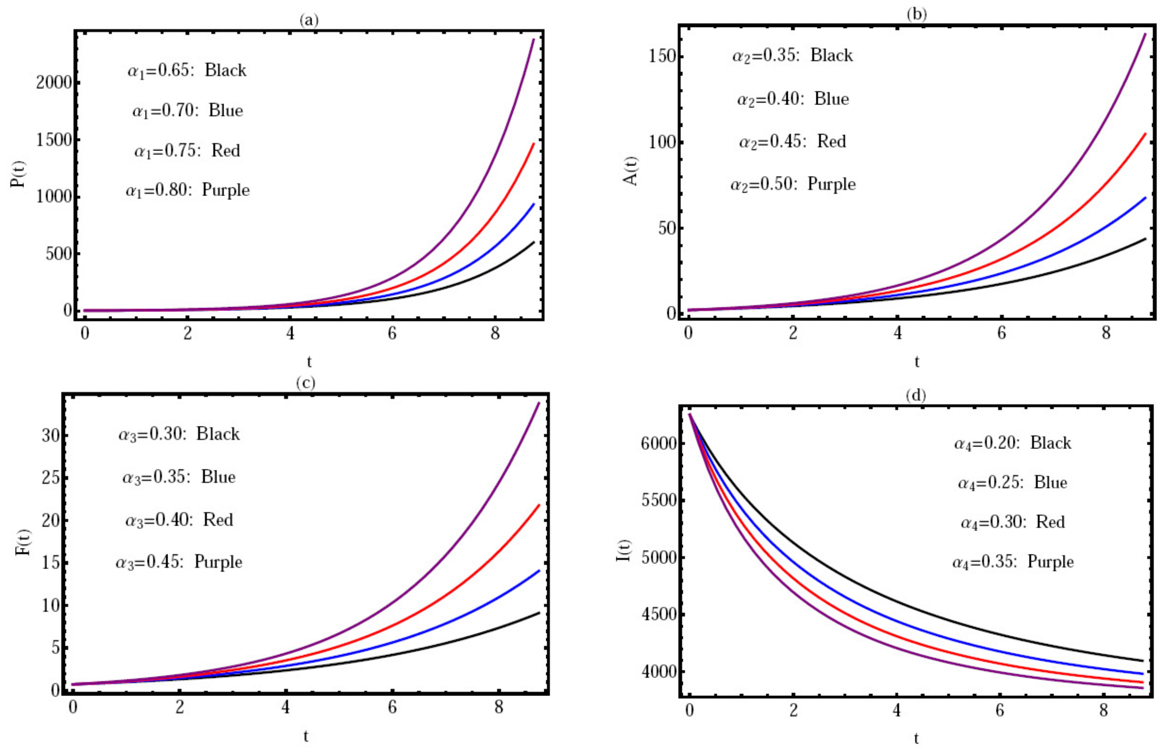

Figure 9: The effect of the growth rates α_k, $k = 1, 2, 3, 4$ (a, b, c, d, respectively) on the numerical solution.

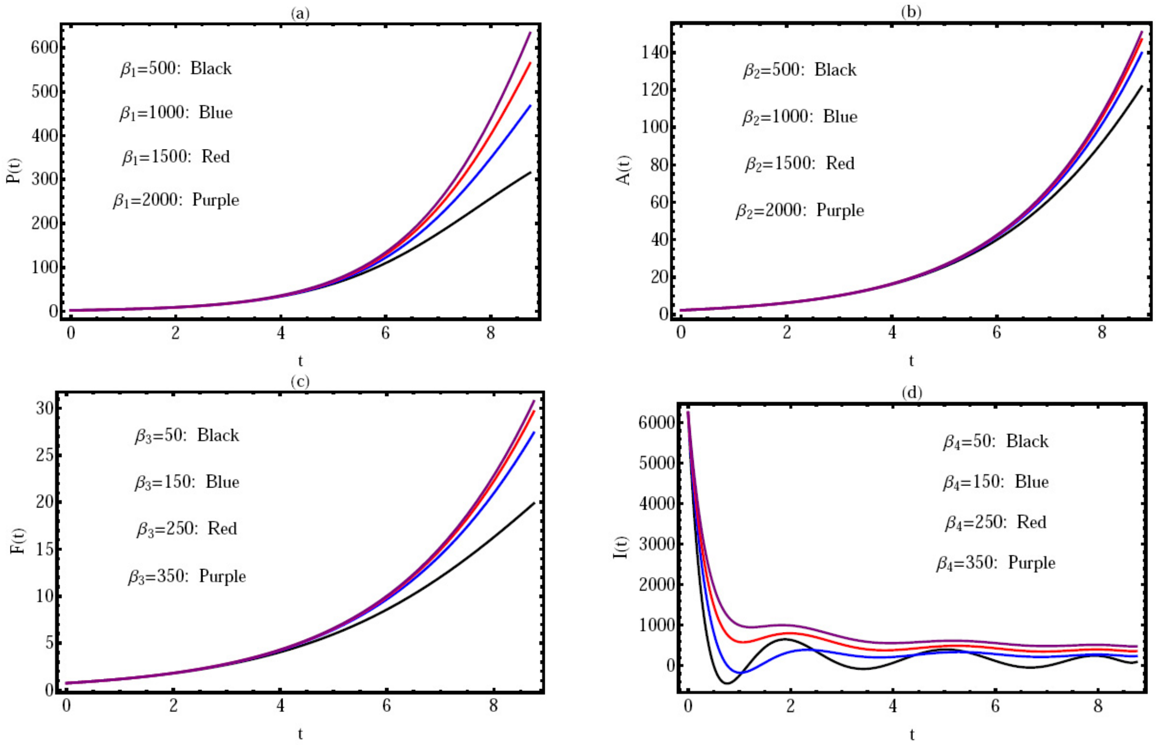

Figure 10: The effect of the maximum profits $\beta_k, k = 1,2,3,4$ (a, b, c, d, respectively) on the solution.

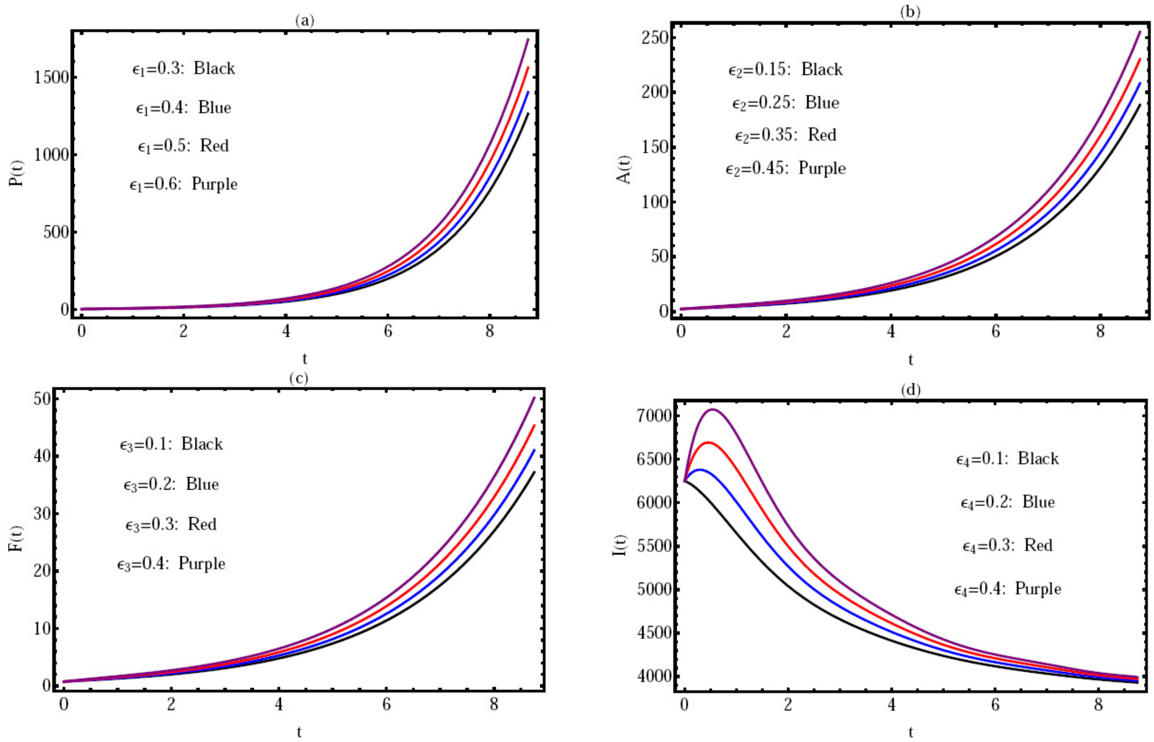

Figure 11: The effect of the maximum level of investment ε_k (a, b, c, d, respectively) on the solution.

5. Conclusions

This paper used the benefits of FC to simulate the behavior of the bank competition model. When applying the GWPs including with the SCM, the proposed model of fractional equations is numerically calculated. Through this work, several values of ν and m were used to calculate the numerical solutions of the model. By incorporating more terms from the approximation solution series or by augmenting m, we may also control the precision of the error. The numerical estimation of the conversion system for fractional integral equations is conducted using Simpson's 1/3 rule with fourth-order accuracy. The proposed methods' suitability for successfully studying this model was confirmed. In conclusion, we assert that for the model examined in this research, numerical simulations using the Caputo and CF operators are more appropriate. The results obtained graphically and derived using the RK4 approach, were comparable. The outcomes also demonstrate the efficiency of the suggested techniques. The key finding is the ability to overcome the rapid decline in bank profits during crises, by building an optimization function to monitor and reinvest, proposing a pre-reinvestment monitoring system capable of accommodating this swift decline in profits, and this simulation clearly shows how banks can be managed to make good profits. The numerical simulation work is performed with the aid of the Mathematica software package. As an extension of this work, we will address the problem in the Future with a more in-depth study, either by presenting a theoretical study to address the stability and convergence of the methods used, or by trying to provide an improvement in these methods, and finally by trying to use another type of polynomials that have properties and advantages in terms of ease of application or accuracy of the solutions resulting from them, compared to those we have discussed.

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