On fuzzy homomorphisms between Hypernear-rings

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Abstract

We focus on the study of the structure of hypernear-rings,in this paper,we recall the basics of crisp homomorphism between hyperstructures,particularly,between hypernear-rings and, then,the notion of fuzzy homomorphism between hypernear-rings is established and its main properties are analysed.

Keywords: Hypernear-rings,fuzzy homomorphisms,fuzzy ideals,fuzzy congruences.

1   Introduction

The study of hyperstructures started seventy-five years ago with Marty's paper [18] which firstly used a multiple-valued operator.Nowadays,the theory of hyperstructures is being thoroughly studied,focusing in particular classes such as hypergroups, hypernear-rings,semihyperrings,join spaces,etc. Mean while,fuzzy sets were introduced by Zadeh [22] and since then there has been a number of authors who started the development of fuzzy algebra. The two types of extensions presented in the previous paragraphs have started to be studied jointly,giving rise to the so-called fuzzy hyperalgebra.Several areas have benefitted from the developments in the area of hyperstructures and fuzzy set theory,in particular, artificial intelligence and soft computing:

• hyperstructures can be used as a useful tool to modeling computing with uncertainty,in that the result of an operation is not a single value and can be considered a set of possible values;

• on the other hand,some ideas underlying fuzzy set theory form the crux on which the development of the different approaches to multiple-valued and fuzzy logics has been based.

The use of hyperstructures in conjunction with fuzzy Logic has been shown to be fruitful in some areas certainly related to artificial intelligence and soft computing, such as fuzzy logic programming with multilattices [19]. The study of hyperstructures and its (fuzzy) congruences is important both from a theoretical standpoint and for its applications in the field of logic-based approaches to uncertainty. Regarding applications, the notion of congruence is intimately related to the foundations of fuzzy reasoning and its relationships with other logics of uncertainty.Morefocused on the theoretical aspects of Computer Science, some authors [2, 20] have pointed out the relation be tween congruences,fuzzy automata and determinism.

The structure of the paper is the following: after stating the preliminary definitions,we recall the basics of the theory of crisp homomorphisms between hyperstructures,recalling specially the results and isomorphism theorems which relate homomorphisms ,congruences and ideals.Then,the main contribution of the paper is presented:the extension of the previous relations to the fuzzy case.

2   Preliminaries

Firstly,let us introduce some preliminary concepts:

Definition 2.1. *A hypergroupoid is a pair $(H, +)$ consisting of a nonempty set H together with a hyperoperation $+ : H \times H \to 2^H \smallsetminus \emptyset$.*

If A and B are nonempty subsets of H, then we denote:
$$A + B = \bigcup_{a \in A, b \in B} a + b, \quad A + x = A + \{x\}, \quad x + B = \{x\} + B.$$
A hypergroupoid $(H, +)$ is called a *semihypergroup* if for all x, y, z of H we have $(x + y) + z = x + (y + z)$, which means that

$$\bigcup_{u \in x+y} u + z = \bigcup_{v \in y+z} x + v.$$

An element e of H is called an *identity* (scalar identity) of $(H, +)$ if for all $a \in H$, we have $a \in (e + a) \cap (e + a)$, $(\{a\} = (e + a) \cap (e + a))$.

A *hypergroup* is a semihypergroup such that for all $x \in H$, we have $x + H = H = H + x$.

A *subhypergroup* $(K, +)$ of $(H, +)$ is a nonempty set K, such that for all $k \in K$, we have $k + K = K = k + K$.

Definition 2.2. [17] *A hypergroupoid $(H, +)$ is a canonical hypergroup if the following conditions hold:*

(i) $(x + y) + z = x + (y + z)$, $\quad \forall\, x, y, z \in H$;

(ii) $x + y = y + x$, *for all* $x, y \in H$;

(iii) $\exists 0 \in H$ *such that* $x + 0 = x$, $\quad \forall x \in H$;

(iv) $\forall x \in H$, $\exists! x' \in H$ *such that* $0 \in x + x'$;

(v) $\forall x, y, z \in H$ *and* $z \in x + y \implies y \in x - z$.

Definition 2.3. [9] *A hypergroupoid $(H, +)$ is a quasicanonical hypergroup if the following conditions hold:*

(i) $(x + y) + z = x + (y + z)$, $\quad \forall x, y, z \in H$;

(ii) $\exists 0 \in H$ *such that* $x + 0 = x = 0 + x$, $\quad \forall x \in R$;

(iii) $\forall x \in H$, $\exists! x' \in H$ *such that* $0 \in (x + x') \cap (x' + x)$;

(iv) $\forall x, y, z \in H$ *and* $z \in x + y \implies x \in z + (-y)$, $\quad y \in (-x) + z$.

the following equalities follow easily from the axioms:

$-(-x) = x$ *and* $-(x + y) = -x - y$.

Note that in the rest of the paper we will frequently write singletons without braces.

Definition 2.4. [9] *The triple $(H, +, \cdot)$ is a hypernear $-$ ring if:*

(1) $(H, +)$ *is a quasicanonical hypergroup, i.e. the following axioms hold for $(R, +)$:*

(i) $(x + y) + z = x + (y + z)$, $\quad \forall x, y, z \in H$;

(ii) $\exists 0 \in R$ *such that* $x + 0 = x = 0 + x$, $\quad \forall x \in H$;

(iii) $\forall x \in H,$ $\exists! x' \in H$ $such$ $that$ $0 \in (x + x') \cap (x' + x);$

(iv) $\forall x, y, z \in H$ and $z \in x + y \Longrightarrow x \in z + (-y),$ $y \in (-x) + z.$

(2) $(H, \cdot)$ is a $semihypergroup$ $having$ 0 as a $right$ $absorbing$ $element,$ $i.e.$ $0 \cdot x = 0,$ $\forall x \in H;$

(3) $(x + y) \cdot z = x \cdot z + y \cdot z,$ $\forall x, y, z \in H.$

A subhypergroup $A \subseteq H$ is *normal* if we have $x + A - x \subseteq A.$

3 On crisp homomorphisms

We begin by discussing the different versions of the concept of homomorphism on hypegroupoids (also called multigroupoids) appearing in the literature.Some authors that deal with these and other hyperstructures use the following definitions of homomorphism[8].

Definition 3.1. [6] *Let* $(H_1,)$ *and* $(H_2,)$ *be two hypergroupoids.A map* $h : H_1 \to H_2$ *is said to be:*

- $Benado\text{-}homomorphism$ if $h(ab) \subseteq h(a)h(b),$ for all $a, b \in H_1.$
- $Algebraic\text{-}homomorphism$ if $h(ab)=h(a)h(b), for$ all $a, b \in H_1.$

Regarding the terminology,we depart here a bit from the usual one.The first one was the original definition by Benado [3] ,which has been used in several recent papers [8, 10]. However,it is noticeable that,finally, the authors concentrate mostly on the equality-based definition.

The terminology used in those papers is to call homo Morphism to Benado՚s ones and call good (or strong) Homomorphism to algebraic ones.We have adopted The term algebraic instead of good or strong because This type of homomorphism immediate allows the lifting of classical homomorphisms to the so-called powerset extension.Obviously,the advantage of using algebraic homomorphisms is that one can transfer properties from the powerset to the hypergroupoid very easily ,so that the presentation of multivalued concepts is greatly simplified.

The term homomorphism should induce the properties of the initial hyperalge-bra on the image set.It can be easily checked that this is the case for algebraic-homomorphisms but,in general,it is not true for Benado-homomorphisms.The notion of homomorphism can be easily extended To the structure of hypernear-rings,since the product operation is classical.The formal definition is given below:

Definition 3.2. [1] *Let H and H' be two hypernear-rings. The mapping $f : H \longrightarrow H'$ is a homomorphism of hypernear-rings, if holds:*

(i) $f(x + y) = f(x) + f(y)$;

(ii) $f(x \cdot y) = f(x) \cdot f(y)$;

(iii) $f(0) = 0$;

for all $x, y \in H$.

Definition 3.3. [1] *A normal subhypergroup A of hypergroup $(R, +)$ is*

$-$ *a left hyperideal of R if $x \cdot A \in A$, for all $x \in R$, $a \in A$;*

$-$ *a right hyperideal of R if $(x + A) \cdot y - xy \subseteq A$, for all $x, y \in R$;*

$-$ *a bilaterally hyperideal of R if $(x + A) \cdot y - xy \cup z \cdot A \subseteq A$, for all $x, y, z \in R$.*

Finally,we recall below the notion of congruence relation on a hypernear-ring that we will extend.

Definition 3.4. *Let $(H, +, .)$ be a hypernear-ring.A congruence on H is an equiva-lence relation $\sim$ which for all $a, b, c, d \in H$ satisfies that if $a \sim b$ and $c \sim d$ then*

• *for all $x \in a + c$ there exists $y \in b + d$ such that $x \sim y$,*

• *for all $y \in b + d$ there exists $x \in a + c$ such that $x \sim y$, and*

• *$ac \sim bd$.*

In classical settings,it is usual to consider the kernel relation associated to a homomorphism ;this idea has been used in the framework of hyperoperations as follows:

Definition 3.5. *Any hypernear-ring homomorphism $f : H \longrightarrow H'$ defines a con-gruence relation, namely kernel relation $\sim_f$,defined as*

$a \sim_f b$ if and only if $f(a) = f(b)$.

The relationship between the concepts of homomorphism,congruence and ideals in the framework of hyperstructures is similar to that in the classical case.

4   Fuzzy homomorphisms on hypernear-rings

A fuzzy relation is a mapping φ from $H_1 \times H_2$ into $[0, 1]$, that is to say, any fuzzy subset of $H_1 \times H_2$. The powerset extension of a fuzzy relation is defined as, $\hat{\varphi} : 2^{H_1} \times 2^{H_2} \to [0, 1]$ with

$$\hat{\varphi}(X, Y) = (\bigwedge_{x \in X} \bigvee_{y \in Y} \varphi(x, y)) \wedge (\bigwedge_{y \in Y} \bigvee_{x \in X} \varphi(x, y)).$$

The composition of fuzzy relations φ and ϕ is defined As follows:

$$(\phi \circ \varphi)(a, c) = \bigvee_{b \in H_2} [\varphi(a, b) \wedge \phi(b, c)].$$

A fuzzy relation μ on $H \times H$ is said to be

- **reflexive** if $\mu(x, x) = 1$, for all $x \in H$;

- **symmetric** if $\mu(x, y) = \mu(y, x)$,for all $x, y \in H$;

- **transitive** if $\mu(x, a) \wedge \mu(a, y) \leq \mu(x, y)$,for all $x, a, y \in H$.

A reflexive,symmetric and transitive fuzzy relation on H is called a **fuzzy equivalence**.A fuzzy equivalence μ on H is called a **fuzzy equality** if for any $x, y \in H$, $\mu(x, y) = 1$ implies $x = y$.

Definition 4.1. *A fuzzy equivalence relation μ on a hypernear-ring $(H, +, .)$ is said to be* **fuzzy congruence relation** *,if for all $a, b, c, d \in H$;*

- $\mu(a, b) \wedge \mu(c, d) \leq \hat{\mu}(a + c, b + d)$;

- $\mu(a, b) \wedge \mu(c, d) \leq \mu(ac, bd)$.

The fuzzification of the concept of function that we adopt has been introduced in [16],also studied in [12, 13, 14], and more recently in [7].We will introduce the extension of the notion of perfect fuzzy function.

Definition 4.2. *Let μ and λ be two fuzzy equalities defined on the sets H_1 and H_2, respectively.A* **partial Fuzzy function** *φ from H_1 to H_2 is a mapping φ : $H_1 \times H_2 \to [0, 1]$ satisfying the following conditions for all $a, a' \in H_1$ and $b, b' \in H_2$:*

ext1 $\varphi(a, b) \wedge \mu(a, a') \leq \varphi(a', b)$,

ext2 $\varphi(a,b) \wedge \lambda(b,b') \leq \varphi(a,b')$,

part $\varphi(a,b) \wedge \varphi(a,b') \leq \lambda(b,b')$.

If, in addition, the following condition hold:

f1 *for all $a \in H_1$ there is $b \in H_2$ such that $\varphi(a,b) = 1$*

then we say that φ is a **perfect fuzzy function**

It is not difficult to show that the element b in condition $(f1)$ above is unique.As a result,every perfect fuzzy function defines a crisp mapping from H_1 to H_2 Called ***crisp description of*** φ.

Definition 4.3. *Let $(H_1, +, .)$ and $(H_2, +, .)$ be two hypernear-rings endowed with fuzzy equalities μ and λ, respectively, such that $\lambda(a,b) = \lambda(-a,-b)$*

A perfect fuzzy function $\varphi \in [0,1]^{H_1 \times H_2}$ is said to be a **fuzzy homomorphism** *if for all $a_1, a_2 \in H_1$ and $b_1, b_2 \in H_2$ the following conditions hold:*

- $\varphi(a_1, b_1) \wedge \varphi(a_2, b_2) \leq \hat{\varphi}(a_1 + a_2, b_1 + b_2)$
- $\varphi(a_1, b_1) \wedge \varphi(b_1, b_2) \leq \varphi(a_1 a_2, b_1 b_2)$
- $\varphi(0,0) = 1$

Moreover, φ is said to be **complete** *if the following conditions hold:*

1. *if $\bigvee_{y \in Y} \varphi(a,y) = 1$, then there exists $y \in H_2$ such that $\varphi(a,y) = 1$,*
2. *if $\bigvee_{x \in X} \varphi(x,b) = 1$, then there exists $x \in H_1$ such that $\varphi(x,b) = 1$.*

Remark:Hereafter,unless stated otherwise, we will always consider that we are working with a complete fuzzy homomorphism φ between hypernear-rings H_1 and H_2 And fuzzy equalities μ and λ,respectively.

Proposition 4.4. *Given φ between H_1 and H_2, the crisp description f of φ is a hypernear-ring homomorphism.*

The notion of fuzzy homomorphism between hypernear-rings behaves properly with respect to the composition of fuzzy relations,in that the composition of fuzzy homomorphisms is a fuzzy homomorphism.Furthermore,the composition is associative and there exists an identity for this composition.As a result,the class of hypernear-rings together with the fuzzy homomorphisms between them forms a category.

Let us concentrate now on the relationship between fuzzy homomorphism and congruences.

Definition 4.5. *the **fuzzy kernel relation** induced by φ in $H_1$is defined as $\mu_\varphi(a, a') = \varphi(a, f(a'))$.*

We adopt here the term kernel as an extension of the crisp case because of the inequality

$$\varphi(a, b) \wedge \varphi(a', b) \leq \mu_\varphi(a, a').$$

Moreover, in [7], the authers prove the following result.

Proposition 4.6. *Let φ be a perfect fuzzy function from H_1 to H_2. For all $a, a' \in H_1$, we have:*

$$\mu_\varphi(a, a') = \bigvee_{b \in H_2} \varphi(a, b) \wedge \varphi(a', b).$$

In the case of fuzzy homomorphisms between hypernear-rings,we prove that this fuzzy equivalence relations also a fuzzy congruence on the initial hypernear-ring.

Theorem 4.7. *Consider φ between H_1 and H_2. The fuzzy kernel relation μ_φ is a fuzzy congruence relation Which includes the fuzzy equality μ in H_1.*

Proof. Let us see that μ_φ is compatible with the sum operation,then we have:

$$\hat{\mu}_\varphi(a_1 + a_3, a_2 + a_4) =$$
$$= (\bigwedge_{a \in a_1+a_3} \bigvee_{a' \in a_2+a_4} [\mu_\varphi(a, a')]) \wedge (\bigwedge_{a' \in a_2+a_4} \bigvee_{a \in a_1+a_3} [\mu_\varphi(a, a')])$$
$$= \hat{\mu}_\varphi(a_1 + a_3, f(a_2 + a_4)) \quad \text{as} \quad \mu_\varphi(a, a') = \varphi(a, f(a'))$$
$$= \hat{\mu}_\varphi(a_1 + a_3, f(a_2) + f(a_4))$$
$$\geq \varphi(a_1, f(a_2)) \wedge \varphi(a_3, f(a_4))$$
$$= \mu_\varphi(a_1, a_2) \wedge \mu_\varphi(a_3, a_4)$$

The compatibility with the multiplication follows from

$$\mu_\varphi(a_1 a_3, a_2 a_4) = \varphi(a_1 a_3, f(a_2 a_4))$$
$$= \varphi(a_1 a_3, f(a_2) f(a_4)) \quad \text{by Prop. 12}$$
$$\geq \varphi(a_1, f(a_2) \wedge \varphi(a_3, f(a_4))$$
$$= \mu_\varphi(a_1, a_2) \wedge \mu_\varphi(a_3, a_4)$$

Now,let us show that $\mu \leq \mu_\varphi$:

$$\mu(a, a') = \mu(a, a') \wedge \varphi(a', f(a'))$$
$$\leq \varphi(a, f(a')) = \mu_\varphi(a, a') \quad \text{by } (ext1) \qquad \square$$

In the rest of this section we will show the canonical Decomposition theorem for a complete fuzzy homomorphism and a fuzzy congruence relation. For suitable extensions on the notions of injectivity and surjectivity we will rely on the definitions given in [12].

Definition 4.8. *A perfect fuzzy function $\varphi \in [0,1]^{H_1 \times H_2}$ is said to be:*

- **surjective** *if for all $b \in H_2$ there exists $a \in H_1$ such that $\varphi(a, b) = 1$.*

- **injective** *if $\varphi(a, b) \wedge \varphi(a', b) \leq \mu(a, a')$ for all $a, a' \in H_1$ and $b \in H_2$.*

- **bijective** *if it is injective and surjective.*

The image set is $Im\varphi = \{b \in H_2 | there exists a \in H_1 with \varphi(a, b) = 1\}$.

In order to define the different homomorphisms involved in the decomposition theorem, we have to introduce the quotient set associated to a fuzzy equivalence relation.

Definition 4.9. *Let $(H, +, .)$ be a hypernear-ring and μ be a fuzzy equivalence relation in H. An **equivalence Class** of an element $a \in H$ is defined as:*

$$\mu(a) \in [0,1]^H \quad with \quad \mu(a)(a') = \mu(a, a')$$

The quotient set is defined as $H/\mu = \{\mu(a) | a \in H\}$ And a fuzzy equality $\bar{\mu}$ can be defined in H/μ as:

$$\bar{\mu}(\mu(a), \mu(a')) = \mu(a, a').$$

*The **fuzzy projection** τ from H to H/μ is defined as $\tau(a, \mu(a')) = \mu(a, a')$.*

Proposition 4.10. *Let $(H, +, .)$ be a hypernear-ring and μ be a fuzzy equality in H and μ_H be a fuzzy congruence relation in H that includes μ. The fuzzy projection τ from H to H/μ_H is a surjective fuzzy homomorphism where the hyperoperations in H/μ_H are given by*

$$\mu_H(a_1) + \mu_H(a_2) = \{\mu_H(c) \mid c \in a_1 + a_2\}$$
$$\mu_H(a_1).\mu_H(a_2) = \{\mu_H(c) \mid c \in a_1 a_2\}$$

The zero element is $\mu_H(0)$ and the fuzzy equality is $\bar{\mu}_H$.

Remark. In order to prove that the canonical inclusion from the image of a homomorphism is an injective fuzzy homomorphism,were call the following result from [12]:given φ between H_1 and H_2,there exists a unique crisp function h such that $\varphi(a,b) = \lambda(h(a),b)$. This h actually coincides with the crisp description f Of φ, which satisfies $\varphi(a,f(a)) = 1$.

Lemma 4.11. *Given φ between H_1 and H_2,then the inclusion ι from $Im\varphi$ to H_2 defined as $\iota(b,b') = \lambda(b,b')$ is an injective fuzzy homomorphism.*

Theorem 4.12. *Any complete fuzzy homomorphism φ from H_1 to H_2 can be canonically decomposed as $\varphi = \iota \circ \bar{\varphi} \circ \tau$ where τ is the fuzzy projection from H_1 to $H_1/\mu\varphi$, and ι is the inclusion from $Im\varphi$ to H_2, and $\bar{\varphi}$ Is the isomorphism from $H_1/\mu\varphi$ to $Im\varphi$ defined as $\bar{\varphi}(\mu\varphi(a),b) = \varphi(a,b)$, and the operations and the fuzzy equality in $Im\varphi$ being the corresponding restrictions of those in H_2.*

Proof. We will prove $(ext1),(inj)$ and $(surj)$ for $\bar{\varphi}$ Since the rest of properties are straightforward:

$(ext1)$ $\bar{\varphi}(\mu_\varphi(a),b) \wedge \bar{\mu}_\varphi(\mu_\varphi(a),\mu_\varphi(a'))$

$\qquad = \varphi(a,b) \wedge \mu_\varphi(a,a')$

$\qquad = \varphi(a,b) \wedge \varphi(a,f(a'))$

$\qquad = \varphi(a,b) \wedge \varphi(a,f(a')) \wedge \varphi(a',f(a'))$

$\qquad \leq \lambda(b,f(a')) \wedge \varphi(a',f(a'))$

$\qquad \leq \varphi(a',b) = \bar{\varphi}(\mu_\varphi(a'),b)$

(inj) $\bar{\varphi}(\mu_\varphi(a),b) \wedge \bar{\varphi}(\mu_\varphi(a'),b)$

$\qquad = \varphi(a,b) \wedge \varphi(a',b)$

$\qquad \leq \mu_\varphi(a,a') = \bar{\mu}_\varphi(\mu_\varphi(a),\mu_\varphi(a'))$

$(surj)$ For all $b \in Im\varphi$ there exists $a \in H_1$ such that $\varphi(a,b) = 1$ and then $\bar{\varphi}(\mu_\varphi(a),b) = 1$

Now,let us check that $\varphi = \iota \circ \bar{\varphi} \circ \tau$:

$(\varphi = \iota \circ \bar{\varphi} \circ \tau)(a,b) =$

$\qquad \bigvee_{\mu_\varphi(a') \in H_1/\mu\varphi, b' \in im\varphi}[\tau(a,\mu_\varphi(a') \wedge \bar{\varphi}(\mu_\varphi(a'),b') \wedge \iota(b',b))]$

$\qquad = \bigvee_{a' \in H_1, b' \in im\varphi}[\mu_\varphi(a,a') \wedge \varphi(a',b') \wedge \lambda(b',b)]$

$$\leq \bigvee_{a' \in H_1} [\mu_\varphi(a, a') \wedge \varphi(a', b)]$$
$$= \bigvee_{a' \in H_1} [\varphi(a, f(a') \wedge \varphi(a', b)]$$
$$= \bigvee_{a' \in H_1} [\varphi(a, f(a') \wedge \varphi(a', f(a')) \wedge \varphi(a', b)]$$
$$\leq \bigvee_{a' \in H_1} [\varphi(a, f(a') \wedge \lambda(f(a'), b)]$$
$$\leq \varphi(a, b).$$

Conversely, $\varphi(a, b) = \lambda(f(a), b) = \tau(a, \mu_\varphi(a)) \wedge \bar{\varphi}(\mu_\varphi(a), f(a)) \wedge \lambda(f(a), b) \leq (\iota \circ \bar{\varphi} \circ \tau)(a, b)$. $\qquad \square$

5 Fuzzy hyperideals and homomorphisms

This section is devoted to the fuzzy extension of the Classical relation between crisp ideals and homomorphisms. First of all,we adopt the definition of fuzzy hyperideal defined. Nevertheless,the superfluous conditions have been removed.

Definition 5.1. *Let* $(H, +, .)$ *be a hypernear-ring.A fuzzy subset* ρ *of* H *is a fuzzy hyperideal if it satisfies,for All* $a, b \in H$ *we have:*

(1) $\quad \rho(a) \wedge \rho(b) \leq \rho(x) \quad$ *for all* $\quad x \in a - b,$

(2) $\quad \rho(a) \vee \rho(b) \leq \rho(ab).$

The kernel relation in a hypernear-ring can be expressed in terms of the kernel of the corresponding homomorphism.In this section we prove that the same occurs in the fuzzy case,due to the convenient properties including in our definition of fuzzy homomorphism.

Let $(H_1, +, .)$ *and* $(H_2, +, .)$ *be two hypernear-rings endowed with fuzzy equalities* μ *and* λ,*respectively,such that* $\lambda(a, b) = \lambda(-a, -b)$.

Let us consider now the fuzzy kernel relation induced By φ *in* H_1, $\mu_\varphi \in [0, 1]^{H_1 \times H_1}$ *,defined as* $\mu_\varphi(a, a') = \varphi(a, f(a'))$,*where h is the crisp description of* φ.

Note that $\mu_\varphi(0) \in [0, 1]^{H_1}$ is a fuzzy subset of H_1 and,

$$\mu_\varphi(0)(a) = \mu_\varphi(0, a) = \varphi(0, f(a)) = \lambda(0, f(a)) = \lambda(f(a), 0) = \varphi(a, 0)$$

Theorem 5.2. *Consider* φ *between* H_1 *and* H_2 *and the fuzzy kernel relation* μ_φ.*Then* $\mu_\varphi(0)$ *is a fuzzy hyperideal.*

Proof. Firstly,observe that $\varphi(a,b) = \lambda(f(a),b) = \lambda(-f(a),-b) = \lambda(f(-a),-b) = \varphi(-a,-b)$ for all $a \in H_1$ and $b \in H_2$.On the other hand,

$$\mu_\varphi(0)(a) \wedge \mu_\varphi(0)(b) =$$

$$\varphi(0,f(a)) \wedge \varphi(0,f(b))$$
$$= \varphi(0,f(a)) \wedge \varphi(0,-f(b))$$
$$\leq \hat{\varphi}(0,f(a)-f(b))$$
$$= \hat{\varphi}(0,f(a-b))$$
$$= \bigwedge_{y \in f(a-b)} \varphi(0,y)$$
$$= \bigwedge_{x \in a-b} \varphi(0,f(x))$$
$$= \bigwedge_{x \in a-b} \mu_\varphi(0)(x)$$

So $\mu_\varphi(0)(a) \wedge \mu_\varphi(0)(b) \leq \mu_\varphi(0)(x)$ for al $x \in a-b$.

Related to the multiplication,

$$\mu_\varphi(0)(a) = \mu_\varphi(0,a) = \mu_\varphi(0,a) \wedge \mu_\varphi(b,b) \leq \mu_\varphi(0,a.b) = \mu_\varphi(0)(a.b). \qquad \square$$

Once proven that the kernel of a fuzzy homomorphism between hypernear-rings is a fuzzy hyperideal,we consider whether the existing relationship between the fuzzy hyperideal and the congruence defined by the fuzzy homomorphisms is what one could expect.

In the crisp case,given an ideal I,a congruence is defined [11] by;

$$a \sim b \quad (mod I) \quad \text{if and only if} \quad (a-b) \cap I \neq \emptyset$$

The natural form of fuzzifying this construction would be the following:if ρ is a fuzzy hyperideal of H, the fuzzy relation $\mu_\rho \in [0,1]^{H \times H}$ should be;

$$\mu_\rho(a,a') = \bigvee_{x \in a-a'} \rho(x)$$

In a nutshell,the fuzzy subset $\mu_\varphi(0)$ plays the role of the kernel of the homomorphism in the crisp case.

Theorem 5.3. *Consider φ between H_1 and H_2 and the fuzzy kernel relation μ_φ.Then*

$$\mu_\varphi(a,a') = \bigvee_{x \in a-a'} \mu_\varphi(0)(x)$$

Proof. Firstly we prove $\mu_\varphi(a,a') \leq \bigvee_{x \in a-a'} \mu_\varphi(0)(x)$.

From Proposition 4.6 ,

$$\mu_\varphi(a, a') = \varphi(a, f(a')) = \bigvee_{b \in H_2} \varphi(a, b) \wedge \varphi(a', b) \qquad (1)$$

On the other hand, for all $b \in H_2$, we have:

$$\varphi(a, b) \wedge \varphi(a', b) \leq \hat\varphi(a - a', b - b)$$
$$= [\bigwedge_{x \in a-a'} \bigvee_{y \in b-b} \varphi(x, y)] \wedge [\bigwedge_{y \in b-b} \bigvee_{x \in a-a'} \varphi(x, y)]$$
$$\leq \bigwedge_{y \in b-b} \bigvee_{x \in a-a'} \varphi(x, y)]$$
$$\leq \bigvee_{x \in a-a'} \varphi(x, 0)$$
$$= \bigvee_{x \in a-a'} \mu_\varphi(0)(x)$$

Thus, $\bigvee_{b \in H_2} \varphi(a, b) \wedge \varphi(a', b) \leq \bigvee_{x \in a-a'} \mu_\varphi(0)(x)$ and by (1),one obtains the inequality required.Now,we check that $\bigvee_{x \in a-a'} \varphi(x, 0) \leq \mu_\varphi(a, a')$. It suffices to prove that $\mu_\varphi(a, a')$ is an upper bound.

$$\mu_\varphi(0)(x) = \mu_\varphi(x, 0) = \mu_\varphi(x, 0) \wedge \mu_\varphi(a', a') \leq \hat\mu_\varphi(x + a', a') = \bigwedge_{z \in x+a'} \mu_\varphi(z, a') \leq$$
$\mu_\varphi(a, a')$. $\qquad\qquad\qquad \square$

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