J. Math. Computer Sci., 40 (2026), 1–21

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Quantum symmetric analysis of interval-valued mappings based on generalized Hukuhara differences

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Abstract

The primary aim of this investigation is to examine the quantum symmetric differentiability and anti-derivative characteristics of interval-valued (I.V.) mappings utilizing generalized Hukuhara differences. Initially, we present the concepts of the I.V. left quantum symmetric derivative operator and offer its characterization. We present the left quantum symmetric integral operator and its essential properties, grounded in the newly proposed derivative operator. Subsequently, we examine their various essential properties. Finally, we present the applications of our proposed operators to integral inequalities concerning I.V. convex mappings and totally ordered convex mappings. Moreover, the validity of our results is corroborated by numerical and graphical representations.

Keywords: Convex mapping, interval-valued mapping, Hukuhara difference, symmetric quantum, center-radius, Hermite-Hadamard inequality.

2020 MSC: 26A51, 26D07, 26D10, 26D15, 26D20.

1. Introduction and preliminaries

Let us recall the notion of convex mapping.

Definition 1.1. Any mapping $\Phi : [\rho_3, \rho_4] \to \mathbb{R}$ is referred to be a convex mapping if

$$(1-\xi)\Phi(\rho_3) + \xi\Phi(\rho_4) \geqslant \Phi((1-\xi)\rho_3 + \xi\rho_4),$$

where $\xi \in [0, 1]$.

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doi: 10.22436/jmcs.040.01.01

Received: 2024-12-03 *Revised:* 2025-02-12 *Accepted:* 2025-03-11

Convex mappings are well-known in mathematical analysis due to their distinctive properties. Especially their role in the development of inequalities is very crucial. Several fundamental inequalities are celebrated by the notion of convex mapping like Jensen's inequality, Young's inequality and its related inequalities and Hermite-Hadamard (H-H) type inequalities. Here, we provide the trapezium inequality involving convex mappings. Let $\Phi : [\rho_3, \rho_4] \to \mathbb{R}$ be a convex mapping, then

$$\Phi\left(\frac{\rho_3+\rho_4}{2}\right) \leqslant \frac{1}{\rho_4-\rho_3}\int_{\rho_3}^{\rho_2} \Phi(\mathfrak{z}_1)d\mathfrak{z}_1 \leqslant \frac{\Phi(\rho_3)+\Phi(\rho_4)}{2}.$$

This inequality is one of the interesting characteristics of convex mappings because it is often reported as the criteria for convex mappings. Moreover, by implementing the convex mappings on Hermite-Hadamard's inequality, we can restore the relations between means and special mappings. Also, this inequality predicts the lower and upper bounds of the average mean integral. Numerous works have been carried out about (H-H) inequality, consult [15, 28]. Now we recollect some essential concepts related to interval analysis. Suppose that the set of all non-empty compact intervals on real line $\mathbb{R}$ is represented by $\mathbb{R}_I$. For any $\mathcal{X}_1, \mathcal{X}_2 \in \mathbb{R}_I$ such that $\mathcal{X}_1 = [\rho_{3*}, \rho_3{}^*]$ and $\mathcal{X}_2 = [\rho_{4*}, \rho_4{}^*]$ and $\lambda \in \mathbb{R}$, then

$$\mathcal{X}_1 + \mathcal{X}_2 = [\rho_{3*}, \rho_3{}^*] + [\rho_{4*}, \rho_4{}^*] = [\rho_{3*} + \rho_{4*}, \rho_3{}^* + \rho_4{}^*],$$

and

$$\lambda[\rho_{3*}, \rho_3{}^*] = \begin{cases} [\lambda\rho_{3*}, \lambda\rho_3{}^*], & \lambda > 0, \\ [\lambda\rho_3{}^*, \lambda\rho_{3*}], & \lambda < 0, \\ 0, & \lambda = 0. \end{cases}$$

The generalized Hukuhara (gh) difference between intervals was introduced in [32]. Let $\mathcal{X}_1, \mathcal{X}_2 \in \mathbb{R}_I$, then gh difference is described as follows:

$$\mathcal{X}_1 \ominus_g \mathcal{X}_2 = [\min\{\rho_{3*} - \rho_{4*}, \rho_3{}^* - \rho_4{}^*\}, \max\{\rho_{3*} - \rho_{4*}, \rho_3{}^* - \rho_4{}^*\}].$$

Also for $\mathcal{X}_1 \in \mathbb{R}_I$, the length of interval is computed by $w(\rho_3) = \rho_3{}^* - \rho_{3*}$. Then for all $\mathcal{X}_1, \mathcal{X}_2 \in \mathbb{R}_I$, we have

$$\mathcal{X}_1 \ominus_g \mathcal{X}_2 = \begin{cases} [\rho_{3*} - \rho_{4*}, \rho_3{}^* - \rho_4{}^*], & w(\mathcal{X}_1) \geqslant w(\mathcal{X}_2), \\ [\rho_3{}^* - \rho_4{}^*, \rho_{3*} - \rho_{4*}], & w(\mathcal{X}_1) \leqslant w(\mathcal{X}_2). \end{cases}$$

Some properties linked with gh-difference are described as follows:

1. $\mathcal{X}_1 \ominus_g \mathcal{X}_1 = \{0\}$, $\mathcal{X}_1 \ominus_g \{0\} = \mathcal{X}_1$, $\{0\} \ominus_g \mathcal{X}_1 = -\mathcal{X}_1$;
2. $\mathcal{X}_1 \ominus_g \mathcal{X}_2 = (-\mathcal{X}_2) \ominus_g (-\mathcal{X}_1) = -(\mathcal{X}_2 \ominus_g \mathcal{X}_1)$;
3. $\mathcal{X}_1 \ominus_g (-\mathcal{X}_2) = \mathcal{X}_2 \ominus_g (-\mathcal{X}_1) = -(\mathcal{X}_2 \ominus_g \mathcal{X}_1)$;
4. $(\mathcal{X}_1 + \mathcal{X}_2) \ominus_g \mathcal{X}_2 = \mathcal{X}_1$;
5. $\lambda\mathcal{X}_1 \ominus_g \lambda\mathcal{X}_2 = \lambda(\mathcal{X}_1 \ominus_g \mathcal{X}_2)$.

Any mapping $\Phi : [\rho_3, \rho_4] \to \mathbb{R}_I$ is said to be interval-valued (I.V.) mapping if $\Phi(\mathfrak{z}_1) = [\Phi_*(\mathfrak{z}_1), \Phi^*(\mathfrak{z}_1)]$ such that $\Phi_*(\mathfrak{z}_1) \leqslant \Phi^*(\mathfrak{z}_1), \forall \mathfrak{z}_1 \in [\rho_3, \rho_4]$. It is important to note that $\lim_{\mathfrak{z}_1 \to \mathfrak{z}_\circ} \Phi(\mathfrak{z}_1)$ exists $\Leftrightarrow \lim_{\mathfrak{z}_1 \to \mathfrak{z}_\circ} \Phi_*(\mathfrak{z}_1)$ and $\lim_{\mathfrak{z}_1 \to \mathfrak{z}_\circ} \Phi^*(\mathfrak{z}_1)$. More specifically, an I.V. mapping Φ is continuous $\Leftrightarrow$ both Φ_* and Φ^* are continuous. Suppose that $\Phi, \Psi : [\rho_3, \rho_4] \to \mathbb{R}_I$, then $(\Phi \ominus_g \Psi)(\mathfrak{z}_1) = \Phi(\mathfrak{z}_1) \ominus_g \Psi(\mathfrak{z}_1) : [\rho_3, \rho_4] \to \mathbb{R}_I$ as:

$$\lim_{\mathfrak{z}_1 \to \mathfrak{z}_\circ} (\Phi \ominus_g \Psi)(\mathfrak{z}_1) = \mathcal{X}_1 \ominus_g \mathcal{X}_2$$

exists, if $\lim_{\mathfrak{z}_1 \to \mathfrak{z}_\circ} \Phi(\mathfrak{z}_1) = \mathcal{X}_1$ and $\lim_{\mathfrak{z}_1 \to \mathfrak{z}_\circ} \Psi(\mathfrak{z}_1) = \mathcal{X}_2$. One can observe that $\Phi, \Psi : [\rho_3, \rho_4] \to \mathbb{R}_I$ are continuous mappings, then $\Phi \ominus_g \Psi$ is also continuous mapping.

Now let us provide the differentiability concepts based on the $\ominus_g$ difference.

Definition 1.2. Suppose $\Phi : [\rho_3, \rho_4] \to \mathbb{R}$ is an I.V. mapping, then

$$\Phi'(\mathfrak{z}_1) = \lim_{h \to 0} \frac{\Phi(\mathfrak{z}_1 + h) \ominus_g \Phi(\mathfrak{z}_1)}{h}$$

is known as $\ominus_g$ derivative at $\mathfrak{z}_1 \in [\rho_3, \rho_4]$. Φ is differentiable on (ρ_3, ρ_4) if it is differentiable at all the interior points of the domain.

Next, we present the I.V. convex mappings.

Definition 1.3. Let $\Phi : [\rho_3, \rho_4] \to \mathbb{R}$ be an I.V. mapping, such that $\Phi(\mathfrak{z}_1) = [\Phi_*(\mathfrak{z}_1), \Phi^*(\mathfrak{z}_1)]$. Then it is referred to be a I.V. convex mapping if

$$\Phi((1-\xi)\rho_3 + \xi\rho_4) \supseteq (1-\xi)\Phi(\rho_3) + \xi\Phi(\rho_4), \quad \xi \in [0,1].$$

Now we present the criteria to assess the convexity of an I.V. mapping.

Theorem 1.4. *Let $\Phi : [\rho_3, \rho_4] \to \mathbb{R}$ be an I.V. mapping, such that $\Phi(\mathfrak{z}_1) = [\Phi_*(\mathfrak{z}_1), \Phi^*(\mathfrak{z}_1)]$. Then Φ is I.V. convex mapping $\Leftrightarrow$ Φ_* is convex mapping and Φ^* is concave mapping.*

Now, we present the I.V. variant of (H-H) inequality. Let $\Phi : [\rho_3, \rho_4] \to \mathbb{R}$ be an I.V. convex mapping, then

$$\Phi\left(\frac{\rho_3 + \rho_4}{2}\right) \supseteq \frac{1}{\rho_4 - \rho_3} \int_{\rho_3}^{\rho_2} \Phi(\mathfrak{z}_1) d\mathfrak{z}_1 \supseteq \frac{\Phi(\rho_3) + \Phi(\rho_4)}{2}.$$

For comprehensive review, see [26].

q-symmetric calculus. Throughout the paper, $I = [\rho_3, \rho_4]$ is any arbitrary subset of $\mathbb{R}$ containing 0 and $q \in (0, 1]$. Then the q-geometric set is described as: $I_q = \{q\mathfrak{z}_1 : \mathfrak{z}_1 \in I\}$. Based on this set the classical symmetric quantum difference operator, left and right quantum symmetric derivatives are explained as follows.

Definition 1.5 ([19])**.** Let $\Phi : I \to \mathbb{R}$ be an I.V. mapping. Then symmetric q-derivative operator is described as:

$$D_q^s \Phi(\xi) = \frac{\Phi(q^{-1}\xi) - \Phi(q\xi)}{(q^{-1} - q)\xi}, \quad \xi \neq 0,$$

and $D_q^s \Phi(0) = \Phi'(0)$, $\xi = 0$ provided that Φ is differentiable at $\xi = 0$. If Φ is differentiable mapping at $\xi \in I_q$, then $\lim_{q \to 1} D_q^s \Phi(\xi) = \Phi'(\xi)$.

Also, the q-symmetric number is given as $n_{q,s} = \frac{q^{-n} - q^n}{q^{-1} - q}$. Moreover, in [5] was explored the concept of left q-symmetric derivatives and integrals over finite intervals. Following the reasoning from [5], let's assume that $J = [\rho_3, \rho_4] \subset \mathbb{R}$, $0 \in J$, and $0 < q < 1$, then the left q-symmetric derivative is described as follows.

Definition 1.6 ([5])**.** Let $\Phi : J \to \mathbb{R}$ be a continuous mapping, then

$${}_{\rho_3}D_q^s \Phi(\xi) = \frac{\Phi(q^{-1}\xi + (1 - q^{-1})\rho_3) - \Phi(q\xi + (1 - q)\rho_3)}{(q^{-1} - q)(\xi - \rho_3)}, \quad \xi \neq \rho_3,$$

and ${}_{\rho_3}D_q^s \Phi(\rho_3) = \lim_{q \to 1} {}_{\rho_3}D_q^s \Phi(\xi)$, if limit exist. If $\rho_3 = 0$, then ${}_{\rho_3}D_q^s \Phi = D_q^s \Phi$.

In [5] the corresponding q-symmetric integral was also investigated, stated as follows.

Definition 1.7. Suppose $\Phi : J \to \mathbb{R}$ be a continuous mapping, then

$$\begin{aligned}\int_{\rho_3}^{\rho_4} \Phi(\xi)_{\rho_3} d_q^s \xi &= (\rho_4 - \rho_3)(q^{-1} - q) \sum_{n=0}^{\infty} q^{2n+1} \Phi(q^{2n+1}\rho_4 + (1 - q^{2n+1})\rho_3) \\ &= (\rho_4 - \rho_3)(1 - q^2) \sum_{n=0}^{\infty} q^{2n} \Phi(q^{2n+1}\rho_4 + (1 - q^{2n+1})\rho_3).\end{aligned}$$

If $\rho_3 = 0$, it reduces to the classical symmetric q-integrals given in [19]. For more information about q-symmetric differences one can consult [38]. It is important to note that these symmetric operators cannot be reduced to Jackson q-operators. By applying $q \to 1$, classical symmetric concepts can be restored.

For the purpose of studying impulsive difference equations, Tariboon and Ntouyas [34] introduced the terms q-derivative and integral operator over finite intervals in 2013. The q-analogues of classical inequalities defined over finite intervals were created using them. The q-counterparts of multiple known inequalities were computed by the authors in [33] using the left q-derivative. The corrected version of trapezium inequality, taking differentiable mappings into account, was studied by Alp et al. (2018) [3]. Within the framework of q_1-calculus, the authors created Milne-Mercer type inequalities in [8]. A few estimates for Ostrowski's inequality were established by Nosheen et al. [27] using the concept of left symmetric q-derivative, integral operators, and s-convexity. Kunt et al. [22] investigated and characterized right-sided q-operators in 2022. See [1, 16, 18, 23, 24, 30, 35] for detailed information.

The error bounds for Ostrowski's quadrature rule were determined in [12, 13] by means of I.V. mappings and applications to numerical integration, respectively. Additionally, the method used in this paper opened the door to new discoveries in the fields, as Costa and Roman Flores [14] demonstrated in 2017 using fuzzy mappings, which led to some intriguing inequalities. Trapezoid type inequalities were studied by Sharma et al. [31] using generalized convexity based on an invex set. Research on Jensen and Hadamard type inequalities through unified I.V. convex mappings and I.V. of Chebyshev type inequalities, respectively, was conducted in 2018 by Zhao and colleagues [36, 37]. In order to find new fuzzy versions of (H-H)-type inequality, Khan et al. [21] investigated new viewpoints on coordinated convexity in fuzzy sense. In order to report new generic inequalities within a fractional framework, the author [10] presented a unified class of convexity. Budak et al. [11] investigated fractional (H-H) type containments using I.V. convex mappings. The authors of [6] discussed the concept of coordinated harmonic convex mappings and used Raina's mapping as the kernel to derive some double fractional integral inequalities. The concept of I.V. quantum derivatives and integrals involving gh-differences was introduced by Lou et al. [25], who also obtained new versions of the (H-H) inequality. On the other hand, the post q-derivative and integral operators in the I.V. sense were studied by Kalsoom et al. [20], who gave the (H-H)-type inclusions. Amir et al. [2] expanded the idea of quantum calculus and presented the I.V. (p, q)-(H-H) type inequalities. Working on bi-convex sets, Bin-Mohsin et al. [9] presented a new class of mappings and examined their potential uses in proving (H-H) type inequalities. Using convex mappings and integral operators with an exponential mapping as the kernel, Duo and Zhou derived the two-dimensional inequalities in 2022 [17]. The new general fractional operators in the AB sense and their application to derive Trapezium type inequalities were discussed in [7].

The primary objective of the present study is to investigate the symmetric q-derivative and integral operators within the context of interval analysis. Initially, we revisit fundamental concepts pertaining to interval analysis and symmetric calculus. In the following segment of the investigation, we present the concept of I.V. left quantum symmetric derivative and integral operators over finite intervals that incorporate gh-differences, along with several essential properties.

2. Main results

2.1. ${}_{\rho_3}d^{i}_{q,s}$-*operator and its properties*

The current study focusses on I.V. quantum symmetric operators using $\ominus_g$ differences. We first discuss left I.V. quantum symmetric difference and integral operators and their properties. The left q-symmetric differentiable operators are represented by ${}_{\rho_3}d^{i}_{q,s}$.

Definition 2.1. Suppose $\Phi : I \to \mathbb{R}_I$ be continuous I.V. mapping, then the left I.V. q-symmetric difference operator is defined as:

$$ {}_{\rho_3}D^{i}_{q,s} = \begin{cases} \frac{\Phi(q^{-1}\mathfrak{z}_1+(1-q^{-1})\rho_3)\ominus_g \Phi(q\mathfrak{z}_1+(1-q)\rho_3)}{(q^{-1}-q)(\mathfrak{z}_1-\rho_3)}, & \mathfrak{z}_1 \neq \rho_3, \\ \lim_{\mathfrak{z}_1\to\rho_3} {}_{\rho_3}D^{i}_{q,s} = \frac{\Phi(q^{-1}\mathfrak{z}_1)\ominus_g \Phi(q\mathfrak{z}_1)}{\mathfrak{z}_1(q^{-1}-q)}, & \mathfrak{z}_1 = \rho_3. \end{cases} $$

Example 2.2. Assume that $\Phi : [0,1] \to \mathbb{R}_I$ be an I.V. mapping such that $\Phi(\mathfrak{z}_1) = [-2\mathfrak{z}_1, 2\mathfrak{z}_1]$, then we compute ${}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1)$ by implementing Definition 2.1,

$$ \begin{aligned} {}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1) &= \frac{[-2q^{-1}\mathfrak{z}_1 - 2(1-q^{-1})\rho_3,\ 2q^{-1}\mathfrak{z}_1 + 2(1-q^{-1})\rho_3] \ominus_g [-2q\mathfrak{z}_1 - 2(1-q)\rho_3,\ 2q\mathfrak{z}_1 + 2(1-q)\rho_3]}{(q^{-1}-q)(\mathfrak{z}_1-\rho_3)} \\ &= \frac{[-2q^{-1}\mathfrak{z}_1 - 2(1-q^{-1})\rho_3 + 2q\mathfrak{z}_1 + 2(1-q)\rho_3,\ 2q^{-1}\mathfrak{z}_1 + 2(1-q^{-1})\rho_3 - 2q\mathfrak{z}_1 - 2(1-q)\rho_3]}{(q^{-1}-q)(\mathfrak{z}_1-\rho_3)} \\ &= \frac{[-2(q^{-1}-q)(\mathfrak{z}_1-\rho_3),\ 2(q^{-1}-q)(\mathfrak{z}_1-\rho_3)]}{(q^{-1}-q)(\mathfrak{z}_1-\rho_3)} = [-2, 2]. \end{aligned} $$

Theorem 2.3. *A mapping $\Phi : [\rho_3, \rho_4] \to \mathbb{R}_I$ is a left I.V. q-symmetric differentiable at $\mathfrak{z}_1 \in [\rho_3, \rho_4] \Leftrightarrow \Phi_*$ and Φ^* are q-symmetric differentiable at $\mathfrak{z}_1 \in [\rho_3, \rho_4]$ and*

$$ {}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1) = [\min\{{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)\},\ \max\{{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)\}]. \tag{2.1} $$

Proof. Assume that Φ is I.V. q-symmetric differentiable mapping, then there exists Ψ_* and Ψ^* such that ${}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1) = [\Psi_*, \Psi^*]$, then by considering the Definition 2.1, we have

$$ \Psi_*(\mathfrak{z}_1) = \min\left\{ \frac{\Phi_*(q^{-1}\mathfrak{z}_1 + (1-q^{-1})\rho_3) - \Phi_*(q\mathfrak{z}_1 + (1-q)\rho_3)}{(q^{-1}-q)(\mathfrak{z}_1-\rho_3)}, \frac{\Phi^*(q^{-1}\mathfrak{z}_1 + (1-q^{-1})\rho_3) - \Phi^*(q\mathfrak{z}_1 + (1-q)\rho_3)}{(q^{-1}-q)(\mathfrak{z}_1-\rho_3)} \right\}, $$

and

$$ \Psi^*(\mathfrak{z}_1) = \max\left\{ \frac{\Phi_*(q^{-1}\mathfrak{z}_1 + (1-q^{-1})\rho_3) - \Phi_*(q\mathfrak{z}_1 + (1-q)\rho_3)}{(q^{-1}-q)(\mathfrak{z}_1-\rho_3)}, \frac{\Phi^*(q^{-1}\mathfrak{z}_1 + (1-q^{-1})\rho_3) - \Phi^*(q\mathfrak{z}_1 + (1-q)\rho_3)}{(q^{-1}-q)(\mathfrak{z}_1-\rho_3)} \right\}, $$

exist. Then ${}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1)$ and ${}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)$ exist and (2.1) is straight forward. Conversely, suppose Φ_* and Φ^* are left q-symmetric differentiable at $\mathfrak{z}_1$. If ${}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1) \leqslant {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)$, then

$$ \begin{aligned} &[{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)] \\ &= \left[\frac{\Phi_*(q^{-1}\mathfrak{z}_1 + (1-q^{-1})\rho_3) - \Phi_*(q\mathfrak{z}_1 + (1-q)\rho_3)}{(q^{-1}-q)(\mathfrak{z}_1-\rho_3)}, \frac{\Phi^*(q^{-1}\mathfrak{z}_1 + (1-q^{-1})\rho_3) - \Phi^*(q\mathfrak{z}_1 + (1-q)\rho_3)}{(q^{-1}-q)(\mathfrak{z}_1-\rho_3)} \right] \\ &= \frac{\Phi(q^{-1}\mathfrak{z}_1 + (1-q^{-1})\rho_3) \ominus_g \Phi(q\mathfrak{z}_1 + (1-q)\rho_3)}{(q^{-1}-q)(\mathfrak{z}_1-\rho_3)}. \end{aligned} $$

Similarly, if ${}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1) \leqslant {}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1)$, then ${}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1) = [{}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1)]$. □

Example 2.4. Suppose that $\Phi[\rho_3, \rho_4] \to \mathbb{R}^+_I$ is given by $[-|\mathfrak{z}_1|, |\mathfrak{z}_1|]$, then, for $\rho_3 < 0$,

$$ \begin{aligned} {}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1) &= \frac{[-(1-q^{-1})\rho_3,\ (1-q^{-1})\rho_3] \ominus_g [(1-q)\rho_3, (1-q)\rho_3]}{-(q^{-1}-q)\rho_3} \\ &= \min\left\{ \frac{2q-1-q^2}{1-q^2}, \frac{-(2q-1-q^2)}{1-q^2} \right\}, \max\left\{ \frac{2q-1-q^2}{1-q^2}, \frac{-(2q-1-q^2)}{1-q^2} \right\} \\ &= \min\left\{ \frac{q-1}{1+q}, \frac{1-q}{1+q} \right\}, \max\left\{ \frac{q-1}{1+q}, \frac{1-q}{1+q} \right\} = \left[\frac{q-1}{1+q}, \frac{1-q}{1+q} \right]. \end{aligned} $$

Similarly, for $\rho_3 > 0$,

$$
\begin{aligned}
{}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1) &= \frac{\left[(1-q^{-1})\rho_3, -(1-q^{-1})\rho_3\right] \ominus_g [-(1-q)\rho_3, (1-q)\rho_3]}{-(q^{-1}-q)\rho_3} \\
&= \min\left\{\frac{(1-q^{-1})\rho_3 + (1-q)\rho_3}{-(q^{-1}-q)\rho_3}, \frac{-(1-q^{-1})\rho_3 - (1-q)\rho_3}{-(q^{-1}-q)\rho_3}\right\}, \\
&\quad \max\left\{\frac{(1-q^{-1})\rho_3 + (1-q)\rho_3}{-(q^{-1}-q)\rho_3}, \frac{-(1-q^{-1})\rho_3 - (1-q)\rho_3}{-(q^{-1}-q)\rho_3}\right\} \\
&= \min\left\{\frac{1-q}{1+q}, \frac{q-1}{1+q}\right\}, \max\left\{\frac{1-q}{1+q}, \frac{q-1}{1+q}\right\} = \left[\frac{q-1}{1+q}, \frac{1-q}{1+q}\right].
\end{aligned}
$$

Similarly, for $\rho_3 = 0$, one can easily obtain.

Now we provide another characterization of left q-symmetric derivative based on the monotonic property of mappings.

Theorem 2.5. *Let $\Phi : [\rho_3, \rho_4] \to \mathbb{R}_I$ be a left I.V. q-symmetric differentiable on (ρ_3, ρ_4), then*

1. ${}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1) = [{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)]$, *if Φ is $\mathfrak{l}$-increasing;*
2. ${}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1) = [{}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1)]$, *if Φ is $\mathfrak{l}$-decreasing.*

Proof. Suppose that Φ is $\mathfrak{l}$-increasing and left q-symmetric differentiable mapping on $[\rho_3, \rho_4]$, and we observe that $q^{-1}\mathfrak{z}_1 + (1-q^{-1})\rho_3 > q\mathfrak{z}_1 + (1-q)\rho_3$ for $q \in (0,1)$. Since $\mathfrak{l}(\Phi) = \Phi^* - \Phi_*$ is increasing, so

$$
\begin{aligned}
&\left[\Phi^*(q^{-1}\mathfrak{z}_1 + (1-q^{-1})\rho_3) - \Phi_*(q^{-1}\mathfrak{z}_1 + (1-q^{-1})\rho_3)\right] - [\Phi^*(q\mathfrak{z}_1 + (1-q)\rho_3) - \Phi_*(q\mathfrak{z}_1 + (1-q)\rho_3)] > 0, \\
&\Phi^*(q^{-1}\mathfrak{z}_1 + (1-q^{-1})\rho_3) - \Phi^*(q\mathfrak{z}_1 + (1-q)\rho_3) > \Phi_*(q^{-1}\mathfrak{z}_1 + (1-q^{-1})\rho_3) - \Phi_*(q\mathfrak{z}_1 + (1-q)\rho_3).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&{}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1) \\
&= \left[\frac{[\Phi_*(q^{-1}\mathfrak{z}_1 + (1-q^{-1})\rho_3),\ \Phi^*(q^{-1}\mathfrak{z}_1 + (1-q^{-1})\rho_3)] \ominus_g [\Phi_*(q\mathfrak{z}_1 + (1-q)\rho_3) - \Phi^*(q\mathfrak{z}_1 + (1-q)\rho_3)]}{(q^{-1}-q)(\mathfrak{z}_1 - \rho_3)}\right] \\
&= \left[\frac{\Phi_*(q^{-1}\mathfrak{z}_1 + (1-q^{-1})\rho_3) - \Phi_*(q\mathfrak{z}_1 + (1-q)\rho_3)}{(q^{-1}-q)(\mathfrak{z}_1 - \rho_3)}, \frac{\Phi^*(q^{-1}\mathfrak{z}_1 + (1-q^{-1})\rho_3) - \Phi^*(q\mathfrak{z}_1 + (1-q)\rho_3)}{(q^{-1}-q)(\mathfrak{z}_1 - \rho_3)}\right] \\
&= [{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)].
\end{aligned}
$$

Hence the required result is achieved. By a similar process, we can prove the second result. $\square$

Remark 2.6. If $c \in (\rho_3, \rho_4)$ and Φ is $\mathfrak{l}$-increasing on $[\rho_3, c)$ and $\mathfrak{l}$-decreasing on $(c, \rho_4]$, then

$$
{}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1) = [{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)],
$$

on $[\rho_3, c)$ and

$$
{}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1) = [{}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1)],
$$

on $(c, \rho_4]$.

Example 2.7. Let $\Phi : [0,1] \to \mathbb{R}^+_I$ be a I.V. mapping such that $\Phi(\mathfrak{z}_1) = [-\mathfrak{z}_1{}^2 - 1,\ \mathfrak{z}_1{}^2 - 2\mathfrak{z}_1]$ and $\mathfrak{l}(\mathfrak{z}_1) = 2\mathfrak{z}_1{}^2 - 2\mathfrak{z}_1 + 1$. It is clear that $\mathfrak{l}(\mathfrak{z}_1)$ is decreasing on $\left(0, \frac{1}{2}\right)$ and increasing on $\left(\frac{1}{2}, 1\right]$, so

$$
{}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1) = \begin{cases}
\left[-(q^{-1}+q)\mathfrak{z}_1,\ (q^{-1}+q)\mathfrak{z}_1 - 2\right], & \mathfrak{z}_1 \in \left(0, \frac{1}{2}\right), \\
\{-1\}, & \mathfrak{z}_1 = \frac{1}{2}, \\
\left[-(q^{-1}+q)\mathfrak{z}_1 - \frac{2-q-q^{-1}}{2},\ (q^{-1}+q)\mathfrak{z}_1 + \frac{2-q-q^{-1}}{2} - 2\right], & \mathfrak{z}_1 \in \left(\frac{1}{2}, 1\right].
\end{cases}
$$

Theorem 2.8. *Let $\Phi : [\rho_3, \rho_4] \to \mathbb{R}_I$ be a left symmetric $\mathfrak{q}$-differentiable mapping. Then for any $c = [c_*, c^*] \in \mathbb{R}_I$ and $\alpha \in \mathbb{R}$, then mappings $\Phi + c$ and $\alpha\Phi$ are also left $\mathfrak{q}$-symmetric differentiable on $[\rho_3, \rho_4]$. Then*

1. ${}_{\rho_3}D^{i}_{\mathfrak{q},s}(\Phi + c)(\mathfrak{z}_1) = {}_{\rho_3}D^{i}_{\mathfrak{q},s}\Phi(\mathfrak{z}_1)$;
2. ${}_{\rho_3}D^{i}_{\mathfrak{q},s}(\alpha\Phi)(\mathfrak{z}_1) = \alpha\,{}_{\rho_3}D^{i}_{\mathfrak{q},s}\Phi(\mathfrak{z}_1)$.

Proof. For $\mathfrak{z}_1 \in [\rho_3, \rho_4]$ and from Definition 2.1, we have

$$\begin{aligned}{}_{\rho_3}D^{i}_{\mathfrak{q},s}(\Phi + c)(\mathfrak{z}_1) &= \frac{\Phi(\mathfrak{q}^{-1}\mathfrak{z}_1 + (1-\mathfrak{q}^{-1})\rho_3) + c \ominus_g \Phi(\mathfrak{q}\mathfrak{z}_1 + (1-\mathfrak{q})\rho_3) + c}{(\mathfrak{q}^{-1} - \mathfrak{q})(\mathfrak{z}_1 - \rho_3)} \\ &= \frac{\Phi(\mathfrak{q}^{-1}\mathfrak{z}_1 + (1-\mathfrak{q}^{-1})\rho_3) \ominus_g \Phi(\mathfrak{q}\mathfrak{z}_1 + (1-\mathfrak{q})\rho_3)}{(\mathfrak{q}^{-1} - \mathfrak{q})(\mathfrak{z}_1 - \rho_3)} = {}_{\rho_3}D^{i}_{\mathfrak{q},s}\Phi(\mathfrak{z}_1).\end{aligned}$$

We left the second proof for interested readers. □

Theorem 2.9. *Let $\Phi : [\rho_3, \rho_4] \to \mathbb{R}_I$ be a left $\mathfrak{q}$-symmetric differentiable mapping. For any $c = [c_*, c^*] \in \mathbb{R}_I$, if $\mathfrak{l}(\Phi) - \mathfrak{l}(c)$ have constant sign over $[\rho_3, \rho_4]$, then $\Phi \ominus_g c$ is left $\mathfrak{q}$-symmetric differentiable and ${}_{\rho_3}D^{i}_{\mathfrak{q},s}(\Phi \ominus_g c)(\mathfrak{z}_1) = {}_{\rho_3}D^{i}_{\mathfrak{q},s}\Phi(\mathfrak{z}_1)$.*

Proof. Take $\mathfrak{z}_1 \in [\rho_3, \rho_4]$, then

$$\begin{aligned}{}_{\rho_3}D^{i}_{\mathfrak{q},s}(\Phi \ominus_g c)(\mathfrak{z}_1) &= \frac{(\Phi(\mathfrak{q}^{-1}\mathfrak{z}_1 + (1-\mathfrak{q}^{-1})\rho_3) \ominus_g c) \ominus_g (\Phi(\mathfrak{q}\mathfrak{z}_1 + (1-\mathfrak{q})\rho_3) \ominus_g c)}{(\mathfrak{q}^{-1} - \mathfrak{q})(\mathfrak{z}_1 - \rho_3)} \\ &= \frac{\Phi(\mathfrak{q}^{-1}\mathfrak{z}_1 + (1-\mathfrak{q}^{-1})\rho_3) \ominus_g \Phi(\mathfrak{q}\mathfrak{z}_1 + (1-\mathfrak{q})\rho_3)}{(\mathfrak{q}^{-1} - \mathfrak{q})(\mathfrak{z}_1 - \rho_3)} = {}_{\rho_3}D^{i}_{\mathfrak{q},s}\Phi(\mathfrak{z}_1).\end{aligned}$$

Hence the proof is completed. □

Theorem 2.10. *Let $\Phi, \Psi : [\rho_3, \rho_4] \to \mathbb{R}$ are left $\mathfrak{q}$-symmetric differentiable mappings, then sum $\Phi + \Psi$ is a left $\mathfrak{q}$-symmetric differentiable, if one of the following cases holds.*

1. *If Φ, Ψ are equally $\mathfrak{l}$-monotonic on $[\rho_3, \rho_4]$, then ${}_{\rho_3}D^{i}_{\mathfrak{q},s}(\Phi(\mathfrak{z}_1) + \Psi(\mathfrak{z}_1)) = {}_{\rho_3}D^{i}_{\mathfrak{q},s}\Phi(\mathfrak{z}_1) + {}_{\rho_3}D^{i}_{\mathfrak{q},s}\Psi(\mathfrak{z}_1)$.*
2. *If Φ, Ψ are differently $\mathfrak{l}$-monotonic on $[\rho_3, \rho_4]$, then*

$${}_{\rho_3}D^{i}_{\mathfrak{q},s}(\Phi(\mathfrak{z}_1) + \Psi(\mathfrak{z}_1)) = {}_{\rho_3}D^{i}_{\mathfrak{q},s}\Phi(\mathfrak{z}_1) \ominus_g (-1)\,{}_{\rho_3}D^{i}_{\mathfrak{q},s}\Psi(\mathfrak{z}_1).$$

Moreover, in both cases, ${}_{\rho_3}D^{i}_{\mathfrak{q},s}(\Phi(\mathfrak{z}_1) + \Psi(\mathfrak{z}_1)) \supseteq {}_{\rho_3}D^{i}_{\mathfrak{q},s}\Phi(\mathfrak{z}_1) + {}_{\rho_3}D^{i}_{\mathfrak{q},s}\Psi(\mathfrak{z}_1)$.

Proof. Suppose Φ, Ψ are I.V. left $\mathfrak{q}$-symmetric differentiable and $\mathfrak{l}$-increasing on $[\rho_3, \rho_4]$. Then Φ_*, Φ^*, Ψ_* and Ψ^* are left $\mathfrak{q}$-symmetric differentiable, and ${}_{\rho_3}D_{\mathfrak{q},s}\Phi_*(\mathfrak{z}_1) \leqslant {}_{\rho_3}D_{\mathfrak{q},s}\Phi^*(\mathfrak{z}_1)$, ${}_{\rho_3}D_{\mathfrak{q},s}\Psi_*(\mathfrak{z}_1) \leqslant {}_{\rho_3}D_{\mathfrak{q},s}\Psi^*(\mathfrak{z}_1)$. Then $\Phi_* + \Psi_*$ and $\Phi^* + \Psi^*$ are $\mathfrak{q}$-symmetric and

$$\begin{aligned}{}_{\rho_3}D^{i}_{\mathfrak{q},s}(\Phi(\mathfrak{z}_1) + \Psi(\mathfrak{z}_1)) &= [\min\{{}_{\rho_3}D_{\mathfrak{q},s}\Phi_*(\mathfrak{z}_1) + {}_{\rho_3}D_{\mathfrak{q},s}\Psi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{\mathfrak{q},s}\Phi^*(\mathfrak{z}_1) + {}_{\rho_3}D_{\mathfrak{q},s}\Psi^*(\mathfrak{z}_1)\}, \\ &\qquad \max\{{}_{\rho_3}D_{\mathfrak{q},s}\Phi_*(\mathfrak{z}_1) + {}_{\rho_3}D_{\mathfrak{q},s}\Psi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{\mathfrak{q},s}\Phi^*(\mathfrak{z}_1) + {}_{\rho_3}D_{\mathfrak{q},s}\Psi^*(\mathfrak{z}_1)\}] \\ &= [{}_{\rho_3}D_{\mathfrak{q},s}\Phi_*(\mathfrak{z}_1) + {}_{\rho_3}D_{\mathfrak{q},s}\Psi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{\mathfrak{q},s}\Phi^*(\mathfrak{z}_1) + {}_{\rho_3}D_{\mathfrak{q},s}\Psi^*(\mathfrak{z}_1)] \\ &= {}_{\rho_3}D^{i}_{\mathfrak{q},s}\Phi(\mathfrak{z}_1) + {}_{\rho_3}D^{i}_{\mathfrak{q},s}\Psi(\mathfrak{z}_1).\end{aligned}$$

By similar proceedings, we can obtain the result for both Φ and Ψ are $\mathfrak{l}$-decreasing. Now, suppose that Φ is $\mathfrak{l}$ increasing and Ψ is $\mathfrak{l}$-decreasing, then ${}_{\rho_3}D_{\mathfrak{q},s}\Phi_*(\mathfrak{z}_1) \leqslant {}_{\rho_3}D_{\mathfrak{q},s}\Phi^*(\mathfrak{z}_1)$, ${}_{\rho_3}D_{\mathfrak{q},s}\Psi^*(\mathfrak{z}_1) \leqslant {}_{\rho_3}D_{\mathfrak{q},s}\Psi_*(\mathfrak{z}_1)$. Also,

$$\begin{aligned}{}_{\rho_3}D^{i}_{\mathfrak{q},s}(\Phi(\mathfrak{z}_1) + \Psi(\mathfrak{z}_1)) &= [\min\{{}_{\rho_3}D_{\mathfrak{q},s}\Phi_*(\mathfrak{z}_1) + {}_{\rho_3}D_{\mathfrak{q},s}\Psi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{\mathfrak{q},s}\Phi^*(\mathfrak{z}_1) + {}_{\rho_3}D_{\mathfrak{q},s}\Psi^*(\mathfrak{z}_1)\}, \\ &\qquad \max\{{}_{\rho_3}D_{\mathfrak{q},s}\Phi_*(\mathfrak{z}_1) + {}_{\rho_3}D_{\mathfrak{q},s}\Psi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{\mathfrak{q},s}\Phi^*(\mathfrak{z}_1) + {}_{\rho_3}D_{\mathfrak{q},s}\Psi^*(\mathfrak{z}_1)\}],\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
&{}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1)\ominus_g(-1)_{\rho_3}D^{i}_{q,s}\Psi(\mathfrak{z}_1)\\
&\quad=[{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)]\ominus_g(-1)\,[{}_{\rho_3}D_{q,s}\Psi^*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Psi_*(\mathfrak{z}_1)]\\
&\quad=[{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)]\ominus_g[-{}_{\rho_3}D_{q,s}\Psi_*(\mathfrak{z}_1),\ -{}_{\rho_3}D_{q,s}\Psi^*(\mathfrak{z}_1)]\\
&\quad=[\min\{{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1)+{}_{\rho_3}D_{q,s}\Psi_*(\mathfrak{z}_1)\},\ \max\{{}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)+{}_{\rho_3}D_{q,s}\Psi^*(\mathfrak{z}_1)\}]\,.
\end{aligned}\tag{2.3}$$

By comparing (2.2) and (2.3), we concluded that ${}_{\rho_3}D^{i}_{q,s}(\Phi(\mathfrak{z}_1)+\Psi(\mathfrak{z}_1))={}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1)\ominus_g(-1)_{\rho_3}D^{i}_{q,s}\Psi(\mathfrak{z}_1)$. Since $\Phi+\Psi$ is $\mathfrak{l}$-increasing and decreasing, we have ${}_{\rho_3}D^{i}_{q,s}(\Phi(\mathfrak{z}_1)+\Psi(\mathfrak{z}_1))\subseteq{}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1)+{}_{\rho_3}D^{i}_{q,s}\Psi(\mathfrak{z}_1)$. Hence the required results are achieved. □

Theorem 2.11. *Let $\Phi,\Psi:[\rho_3,\rho_4]\to\mathbb{R}$ are left q-symmetric differentiable mappings and $\mathfrak{l}(\Phi)-\mathfrak{l}(\Psi)$ has constant sign over domain, then $\Phi\ominus_g\Psi$ is a left q-symmetric differentiable, if one of the case holds:*

1. *If Φ,Ψ are equally $\mathfrak{l}$-monotonic on $[\rho_3,\rho_4]$, then ${}_{\rho_3}D^{i}_{q,s}(\Phi(\mathfrak{z}_1)\ominus_g\Psi(\mathfrak{z}_1))={}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1)\ominus_g{}_{\rho_3}D^{i}_{q,s}\Psi(\mathfrak{z}_1)$.*
2. *If Φ,Ψ are differently $\mathfrak{l}$-monotonic on $[\rho_3,\rho_4]$, then*

$${}_{\rho_3}D^{i}_{q,s}(\Phi(\mathfrak{z}_1)\ominus_g\Psi(\mathfrak{z}_1))={}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1)+(-1)_{\rho_3}D^{i}_{q,s}\Psi(\mathfrak{z}_1).$$

Proof. Assume that $\mathfrak{l}(\Phi)\geqslant\mathfrak{l}(\Psi)$ and $\Phi\ominus_g\Psi=[\Phi_*-\Psi_*,\ \Phi^*-\Psi^*]$. Suppose Φ,Ψ are I.V. left q-symmetric differentiable and $\mathfrak{l}$-increasing on $[\rho_3,\rho_4]$. Then Φ_*,Φ^*,Ψ_* and Ψ^* are left q-symmetric differentiable, and ${}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1)\leqslant{}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)$, ${}_{\rho_3}D_{q,s}\Psi_*(\mathfrak{z}_1)\leqslant{}_{\rho_3}D_{q,s}\Psi^*(\mathfrak{z}_1)$. Then $\Phi_*-\Psi_*$ and $\Phi^*-\Psi^*$ are q-symmetric differentiable, thus $\Phi\ominus_g\Psi$ is left q-symmetric differentiable mapping such that

$$\begin{aligned}
&{}_{\rho_3}D^{i}_{q,s}(\Phi(\mathfrak{z}_1)\ominus_g\Psi(\mathfrak{z}_1))\\
&\quad=[\min\{{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1)-{}_{\rho_3}D_{q,s}\Psi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)-{}_{\rho_3}D_{q,s}\Psi^*(\mathfrak{z}_1)\},\\
&\qquad\max\{{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1)-{}_{\rho_3}D_{q,s}\Psi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)-{}_{\rho_3}D_{q,s}\Psi^*(\mathfrak{z}_1)\}]\\
&\quad=[{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1)-{}_{\rho_3}D_{q,s}\Psi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)-{}_{\rho_3}D_{q,s}\Psi^*(\mathfrak{z}_1)]\\
&\quad=[{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)]\ominus_g[{}_{\rho_3}D_{q,s}\Psi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Psi^*(\mathfrak{z}_1)]\\
&\quad={}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1)\ominus_g{}_{\rho_3}D^{i}_{q,s}\Psi(\mathfrak{z}_1).
\end{aligned}$$

By similar proceedings, we can obtain the result for both Φ and Ψ are $\mathfrak{l}$-decreasing. Now, suppose that Φ is $\mathfrak{l}$ increasing and Ψ is $\mathfrak{l}$-decreasing, then ${}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1)\leqslant{}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)$, ${}_{\rho_3}D_{q,s}\Psi^*(\mathfrak{z}_1)\leqslant{}_{\rho_3}D_{q,s}\Psi_*(\mathfrak{z}_1)$. Also,

$$\begin{aligned}
{}_{\rho_3}D^{i}_{q,s}(\Phi(\mathfrak{z}_1)\ominus_g\Psi(\mathfrak{z}_1))&=[\min\{{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1)-{}_{\rho_3}D_{q,s}\Psi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)-{}_{\rho_3}D_{q,s}\Psi^*(\mathfrak{z}_1)\},\\
&\qquad\max\{{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1)-{}_{\rho_3}D_{q,s}\Psi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)-{}_{\rho_3}D_{q,s}\Psi^*(\mathfrak{z}_1)\}]\\
&=[{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1)-{}_{\rho_3}D_{q,s}\Psi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)-{}_{\rho_3}D_{q,s}\Psi^*(\mathfrak{z}_1)]
\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}
&{}_{\rho_3}D^{i}_{q,s}\Phi(\mathfrak{z}_1)+(-1)_{\rho_3}D^{i}_{q,s}\Psi(\mathfrak{z}_1)\\
&\quad=[{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)]+(-1)\,[{}_{\rho_3}D_{q,s}\Psi^*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Psi_*(\mathfrak{z}_1)]\\
&\quad=[{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)]+[-{}_{\rho_3}D_{q,s}\Psi_*(\mathfrak{z}_1),\ -{}_{\rho_3}D_{q,s}\Psi^*(\mathfrak{z}_1)]\\
&\quad=[\min\{{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1)-{}_{\rho_3}D_{q,s}\Psi_*(\mathfrak{z}_1)\},\ \max\{{}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)-{}_{\rho_3}D_{q,s}\Psi^*(\mathfrak{z}_1)\}]\\
&\quad=[{}_{\rho_3}D_{q,s}\Phi_*(\mathfrak{z}_1)-{}_{\rho_3}D_{q,s}\Psi_*(\mathfrak{z}_1),\ {}_{\rho_3}D_{q,s}\Phi^*(\mathfrak{z}_1)-{}_{\rho_3}D_{q,s}\Psi^*(\mathfrak{z}_1)]\,.
\end{aligned}\tag{2.5}$$

Comparison of (2.4) and (2.5) yields the required result. □

2.2. ${}_{\rho_3}I^{i}_{q,s}$*-integral operator and its properties*

In the following part of the study, we present the concept of I.V. left q-symmetric integral operator and its essential characterization. For our convenience, we specify the space of I.V. left q-symmetric integrable mappings and space of all continuous I.V. mappings by ${}_{\rho_3}I^{i}_{q,s}$ and $C([\rho_3,\rho_4],\mathbb{R}_I)$, respectively.

Definition 2.12. Let $\Phi \in C([\rho_3,\rho_4],\mathbb{R}_I)$. Then ${}_{\rho_3}I^{i}_{q,s}$-integral operator is described as:

$$\begin{aligned}\int_{\rho_3}^{\rho_4} \Phi(\mathfrak{z}_1) d_q^s \mathfrak{z}_1 &= (q^{-1}-q)(\rho_4-\rho_3)\sum_{n=0}^{\infty} q^{2n+1}\Phi(q^{2n+1}\rho_4+(1-q^{2n+1})\rho_3)\\ &= (1-q^2)(\rho_4-\rho_3)\sum_{n=0}^{\infty} q^{2n}\Phi(q^{2n+1}\rho_4+(1-q^{2n+1})\rho_3).\end{aligned}$$

One can easily observe that a mapping is considered to be I.V. left q-symmetric integrable if $\sum_{n=0}^{\infty} q^{2n}\Phi(q^{2n+1}\rho_4+(1-q^{2n+1})\rho_3)$ converges.

Theorem 2.13. *Let* $\Phi \in C([\rho_3,\rho_4],\mathbb{R}_I)$ *and* $c \in (\rho_3,\mathfrak{z}_1) \subseteq [\rho_3,\rho_4]$. *Then*

$$\int_{\rho_3}^{\mathfrak{z}_1} \Phi(\xi) d_q^s \xi = \int_{\rho_3}^{c} \Phi(\xi) d_q^s \xi + \int_{c}^{\mathfrak{z}_1} \Phi(\xi) d_q^s \xi.$$

Proof. From Definition 2.12, we can write

$$\begin{aligned}&\int_{\rho_3}^{c} \Phi(\xi) d_q^s \xi + \int_{c}^{\mathfrak{z}_1} \Phi(\xi) d_q^s \xi\\ &= (1-q^2)(c-\rho_3)\sum_{n=0}^{\infty} q^{2n}\Phi(q^{2n+1}c+(1-q^{2n+1})\rho_3) + (1-q^2)(\mathfrak{z}_1-c)\sum_{n=0}^{\infty} q^{2n}\Phi(q^{2n+1}\mathfrak{z}_1+(1-q^{2n+1})c)\\ &= \left[(1-q^2)(c-\rho_3)\sum_{n=0}^{\infty} q^{2n}\Phi_*(q^{2n+1}c+(1-q^{2n+1})\rho_3),\ (1-q^2)(c-\rho_3)\sum_{n=0}^{\infty} q^{2n}\Phi^*(q^{2n+1}c+(1-q^{2n+1})\rho_3)\right]\\ &\quad + \left[(1-q^2)(\mathfrak{z}_1-c)\sum_{n=0}^{\infty} q^{2n}\Phi_*(q^{2n+1}\mathfrak{z}_1+(1-q^{2n+1})c),\ (1-q^2)(\mathfrak{z}_1-c)\sum_{n=0}^{\infty} q^{2n}\Phi^*(q^{2n+1}\mathfrak{z}_1+(1-q^{2n+1})c)\right]\\ &= \left[(1-q^2)(\mathfrak{z}_1-\rho_3)\sum_{n=0}^{\infty} q^{2n}\Phi_*(q^{2n+1}\mathfrak{z}_1+(1-q^{2n+1})\rho_3),\ (1-q^2)(\mathfrak{z}_1-\rho_3)\sum_{n=0}^{\infty} q^{2n}\Phi^*(q^{2n+1}\mathfrak{z}_1+(1-q^{2n+1})\rho_3)\right]\\ &= \int_{\rho_3}^{\mathfrak{z}_1} \Phi(\xi) d_q^s \xi.\end{aligned}$$

Hence, the result is proved. □

Theorem 2.14. *Let* $\Phi \in C([\rho_3,\rho_4],\mathbb{R}_I)$ *and* $(\rho_3,\mathfrak{z}_1) \subseteq [\rho_3,\rho_4]$. *Then* $\Phi \in_{\rho_3} I^{i}_{q,s} \Leftrightarrow$ *both* Φ_* *and* Φ^* *are left* q*-symmetric integrable mappings. Also*

$$\int_{\rho_3}^{\mathfrak{z}_1} \Phi(\xi) d_q^s \xi = \left[\int_{\rho_3}^{\mathfrak{z}_1} \Phi_*(\xi) d_q^s \xi,\ \int_{\rho_3}^{\mathfrak{z}_1} \Phi^*(\xi) d_q^s \xi\right].$$

Proof. Consider $\Phi \in_{\rho_3} I^{i}_{q,s}$, then

$$\begin{aligned}\int_{\rho_3}^{\mathfrak{z}_1} \Phi(\xi) d_q^s \xi &= (1-q^2)(\mathfrak{z}_1-\rho_3)\sum_{n=0}^{\infty} q^{2n}\Phi(q^{2n+1}\mathfrak{z}_1+(1-q^{2n+1})\rho_3)\\ &= \left[(1-q^2)(\mathfrak{z}_1-\rho_3)\sum_{n=0}^{\infty} q^{2n}\Phi_*(q^{2n+1}\mathfrak{z}_1+(1-q^{2n+1})\rho_3)\right.\end{aligned}$$

$$(1-q^2)(\mathfrak{z}_1-\rho_3)\sum_{n=0}^{\infty} q^{2n}\Phi^*(q^{2n+1}\mathfrak{z}_1+(1-q^{2n+1})\rho_3)\Big].$$

This implies that both Φ_* and Φ^* are q-symmetric integrable mappings. Conversely, suppose that Φ_* and Φ^* are q-symmetric integrable mappings and $\Phi_* \leqslant \Phi^*$, then the result is obvious. So, the result is proven. □

Example 2.15. Let $\Phi : [0,2] \to \mathbb{R}_I$ such that $\Phi(\xi) = [2\xi^2, 3\xi]$, then

$$\begin{aligned}\int_0^2 \Phi(\xi) d_q^s\xi &= \left[\int_0^2 2\xi^2 d_q^s\xi,\ \int_0^2 3\xi d_q^s\xi\right]\\ &= \left[4(1-q^2)\sum_{n=0}^{\infty} q^{2n}(2q^{2n+1})^2,\ 6(1-q^2)\sum_{n=0}^{\infty} q^{2n}(2q^{2n+1})\right]\\ &= \left[16(1-q^2)q^2\sum_{n=0}^{\infty} q^{6n},\ 12(1-q^2)q\sum_{n=0}^{\infty} q^{4n}\right] = \left[\frac{16q^2}{1+q^2+q^4}, \frac{12q}{1+q^2}\right].\end{aligned}$$

Theorem 2.16. *Let $\Phi, \Psi \in C([\rho_3, \rho_4], \mathbb{R}_I)$ and $(\rho_3, \mathfrak{z}_1) \subseteq [\rho_3, \rho_4]$. Then for $\alpha \in \mathbb{R}_I$,*

1. $\int_{\rho_3}^{\rho_4} [\Phi(\xi) + \Psi(\xi)]_{\rho_3} d_q^s\xi = \int_{\rho_3}^{\rho_4} \Phi(\xi)_{\rho_3} d_q^s\xi + \int_{\rho_3}^{4} \Psi(\xi)_{\rho_3} d_q^s\xi$;
2. $\int_{\rho_3}^{\rho_4} (\alpha\Phi)(\xi)_{\rho_3} d_q^s\xi = \alpha\int_{\rho_3}^{\rho_4} \Phi(\xi)_{\rho_3} d_q^s\xi$.

Proof. From Definition of 2.12, we have

$$\begin{aligned}&\int_{\rho_3}^{\rho_4} [\Phi(\xi) + \Psi(\xi)]_{\rho_3} d_q^s\xi\\ &= (1-q^2)(\rho_4-\rho_3)\sum_{n=0}^{\infty} q^{2n}\left[\Phi_*(q^{2n+1}\rho_4+(1-q^{2n+1})\rho_3)+\Psi_*(q^{2n+1}\rho_4+(1-q^{2n+1})\rho_3),\right.\\ &\quad \left.\Phi^*(q^{2n+1}\rho_4+(1-q^{2n+1})\rho_3)+\Psi^*(q^{2n+1}\rho_4+(1-q^{2n+1})\rho_3)\right]\\ &= (1-q^2)(\rho_4-\rho_3)\sum_{n=0}^{\infty} q^{2n}\left[\Phi_*(q^{2n+1}\rho_4+(1-q^{2n+1})\rho_3),\ \Phi^*(q^{2n+1}\rho_4+(1-q^{2n+1})\rho_3)\right]\\ &\quad + (1-q^2)(\rho_4-\rho_3)\sum_{n=0}^{\infty} q^{2n}\left[\Psi_*(q^{2n+1}\rho_4+(1-q^{2n+1})\rho_3),\ \Psi^*(q^{2n+1}\rho_4+(1-q^{2n+1})\rho_3)\right]\\ &= \int_{\rho_3}^{\rho_4} \Phi(\xi)_{\rho_3} d_q^s\xi + \int_{\rho_3}^{4} \Psi(\xi)_{\rho_3} d_q^s\xi.\end{aligned}$$

Hence the first prove is obtained. The proof of second result is obvious. □

Theorem 2.17. *Let $\Phi, \Psi \in C([\rho_3, \rho_4], \mathbb{R}_I)$ and $(\rho_3, \mathfrak{z}_1) \subseteq [\rho_3, \rho_4]$, then*

$$\int_{\rho_3}^{\rho_4} \Phi(\xi)_{\rho_3} d_q^s\xi \ominus_g \int_{\rho_3}^{4} \Psi(\xi)_{\rho_3} d_q^s\xi \subseteq \int_{\rho_3}^{\rho_4} \Phi(\xi) \ominus_g \Psi(\xi)_{\rho_3} d_q^s\xi.$$

Furthermore, $l(\Phi) - l(\Psi)$ has constant sign on $[\rho_3, \rho_4]$, then

$$\int_{\rho_3}^{\rho_4} \Phi(\xi)_{\rho_3} d_q^s\xi \ominus_g \int_{\rho_3}^{4} \Psi(\xi)_{\rho_3} d_q^s\xi = \int_{\rho_3}^{\rho_4} \Phi(\xi) \ominus_g \Psi(\xi)_{\rho_3} d_q^s\xi.$$

Proof. We observe that

$$\begin{aligned}\int_{\rho_3}^{\rho_4} \min\{\Phi_* - \Psi_*, \Phi^* - \Psi^*\}_{\rho_3} d_q^s \xi &\leqslant \min \int_{\rho_3}^{\rho_4} \{\Phi_* - \Psi_*, \Phi^* - \Psi^*\}_{\rho_3} d_q^s \xi \\ &\leqslant \max \int_{\rho_3}^{\rho_4} \{\Phi_* - \Psi_*, \Phi^* - \Psi^*\}_{\rho_3} d_q^s \xi \\ &\leqslant \int_{\rho_3}^{\rho_4} \max\{\Phi_* - \Psi_*, \Phi^* - \Psi^*\}_{\rho_3} d_q^s \xi.\end{aligned} \tag{2.6}$$

Also, we have

$$\begin{aligned}&\int_{\rho_3}^{\rho_4} \Phi(\xi)_{\rho_3} d_q^s \xi \ominus_g \int_{\rho_3}^{4} \Psi(\xi)_{\rho_3} d_q^s \xi \\ &= \Bigg[\min\left\{\int_{\rho_3}^{\rho_4} (\Phi_* - \Psi_*)_{\rho_3} d_q^s \xi, \int_{\rho_3}^{\rho_4} (\Phi^* - \Psi^*)_{\rho_3} d_q^s \xi\right\}, \\ &\quad \max\left\{\int_{\rho_3}^{\rho_4} (\Phi_* - \Psi_*)_{\rho_3} d_q^s \xi, \int_{\rho_3}^{\rho_4} (\Phi^* - \Psi^*)_{\rho_3} d_q^s \xi\right\}\Bigg] \\ &\subseteq \left[\int_{\rho_3}^{\rho_4} \min\{(\Phi_* - \Psi_*), \Phi^* - \Psi^*\}_{\rho_3} d_q^s \xi, \int_{\rho_3}^{\rho_4} \max\{(\Phi_* - \Psi_*), \Phi^* - \Psi^*\}_{\rho_3} d_q^s \xi\right] \\ &= \int_{\rho_3}^{\rho_4} \Phi(\xi) \ominus_g (\xi)_{\rho_3} d_q^s \xi.\end{aligned} \tag{2.7}$$

Comparison of (2.6) and (2.7) results desired containment. By the notion of generalized Hukuhara difference, if $l(\Phi) \geqslant l(\Psi)$, then $\Phi \ominus_g \Psi = [\Phi_* - \Psi_*, \Phi^* - \Psi^*]$ and if $l(\Phi) \leqslant l(\Psi)$, then $[\Phi^* - \Psi^*, \Phi_* - \Psi_*]$. We suppose that $l(\Phi) \geqslant l(\Psi)$ on $[\rho_3, \rho_4]$ such that $\Phi \ominus_g \Psi = [\Phi_* - \Psi_*, \Phi^* - \Psi^*]$. This implies that $\int_{\rho_3}^{\rho_4} (\Phi_* - \Psi_*)_{\rho_3} d_q^s \xi \leqslant \int_{\rho_3}^{\rho_4} (\Phi^* - \Psi^*)_{\rho_3} d_q^s \xi$. Now

$$\begin{aligned}\int_{\rho_3}^{\rho_4} \Phi \ominus_g \Psi_{\rho_3} d_q^s \xi &= \left[\int_{\rho_3}^{\rho_4} \min\{\Phi_* - \Psi_*, \Phi^* - \Psi^*\}, \int_{\rho_3}^{\rho_4} \max\{\Phi_* - \Psi_*, \Phi^* - \Psi^*\}\right] \\ &= \left[\int_{\rho_3}^{\rho_4} \Phi_{*\rho_3} d_q^s \xi, \int_{\rho_3}^{\rho_4} \Phi^*{}_{\rho_3} d_q^s \xi\right] \ominus_g \left[\int_{\rho_3}^{\rho_4} \Psi_{*\rho_3} d_q^s \xi, \int_{\rho_3}^{\rho_4} \Psi^*{}_{\rho_3} d_q^s \xi\right] \\ &= \int_{\rho_3}^{\rho_4} \Phi(\xi)_{\rho_3} d_q^s \xi \ominus_g \int_{\rho_3}^{\rho_4} \Psi(\xi)_{\rho_3} d_q^s \xi.\end{aligned}$$

Hence, the desired result is obtained. □

Theorem 2.18. *Let $\Phi : [\rho_3, \rho_4] \to \mathbb{R}_I$ be left q-symmetric differentiable mapping on (ρ_3, ρ_4) and ${}_{\rho_3}D_{q,s}^{i}(\Phi(\mathfrak{Z}_1) \in C([\rho_3, \rho_4], \mathbb{R}_I)$. If Φ is l-monotone on $[s, \xi]$, then*

$$\Phi(\rho_4) \ominus_g \Phi(c) = \int_{c}^{\rho_4} {}_{\rho_3}D_{q,s}^{i} \Phi(\xi)_{\rho_3} d_q^s \xi, \quad c \in (\rho_3, \rho_4). \tag{2.8}$$

Proof. If Φ is left I.V. q-symmetric differentiable mapping, then Φ_* and Φ^* are left q-symmetric differentiable mappings, so ${}_{\rho_3}D_{q,s}\Phi_*$ and ${}_{\rho_3}D_{q,s}\Phi^*$ are left q-symmetric integrable mappings. Therefore Φ is also left I.V. q-symmetric integrable mapping. If Φ is l-increasing on $[\rho_3, \rho_4]$, then ${}_{\rho_3}D_{q,s}^{i}(\Phi(\xi) = \left[{}_{\rho_3}D_{q,s}^{i}\Phi_*(\xi), {}_{\rho_3}D_{q,s}^{i}\Phi^*(\xi)\right]$, then

$$\Phi_*(\rho_4) - \Phi_*(c) = \int_{c}^{\rho_4} {}_{\rho_3}D_{q,s}\Phi_*(\xi)_{\rho_3} d_q^s \xi, \quad \Phi^*(\rho_4) - \Phi^*(c) = \int_{c}^{\rho_4} {}_{\rho_3}D_{q,s}\Phi^*(\xi)_{\rho_3} d_q^s \xi. \tag{2.9}$$

Since $\Phi_* \leqslant \Phi^*$ and from (2.9), we acquire

$$\Phi(\rho_4) = \Phi(c) + \int_c^{\rho_4} {}_{\rho_3}D^i_{q,s}\Phi(\xi)_{\rho_3}d^s_q\xi.$$

Now from the notion of gh-difference, then

$$\Phi(\mathfrak{z}_1) \ominus_g \Phi(c) = \int_c^{\rho_4} {}_{\rho_3}D_{q,s}\Phi(\xi)_{\rho_3}d^s_q\xi.$$

If Φ is $\mathfrak{l}$-decreasing on $[\rho_3, \rho_4]$, then ${}_{\rho_3}D^i_{q,s}(\Phi(\xi) = \left[{}_{\rho_3}D^i_{q,s}\Phi^*(\xi),\ {}_{\rho_3}D^i_{q,s}\Phi_*(\xi)\right]$, then

$$\begin{aligned}\int_c^{\rho_4} {}_{\rho_3}D^i_{q,s}\Phi(\xi)_{\rho_3}d^s_q\xi &= \left[\int_c^{\rho_4} {}_{\rho_3}D_{q,s}\Phi^*(\xi)_{\rho_3}d^s_q\xi,\ \int_c^{\rho_4} {}_{\rho_3}D_{q,s}\Phi_*(\xi)_{\rho_3}d^s_q\xi\right]\\ &= [\Phi^*(\rho_4) - \Phi^*(c),\ \Phi^*(\rho_4) - \Phi**(c)]\\ &= [\Phi_*(\mathfrak{z}_1),\ \Phi^*(\mathfrak{z}_1)] \ominus_g [\Phi_*(c),\ \Phi^*(c)] = \Phi(\mathfrak{z}_1) \ominus_g \Phi(c).\end{aligned}$$

Hence, the required result is acquired. □

Remark 2.19. If Φ is $\mathfrak{l}$-increasing, then (2.8) can be interpreted as:

$$\Phi(\mathfrak{z}_1) = \Phi(c) + \int_c^{\rho_4} {}_{\rho_3}D_{q,s}\Phi(\xi)_{\rho_3}d^s_q\xi.$$

If Φ is $\mathfrak{l}$-decreasing, then (2.8) can be interpreted as:

$$\Phi(\mathfrak{z}_1) = \Phi(c) \ominus_g (-1)\int_c^{\rho_4} {}_{\rho_3}D_{q,s}\Phi(\xi)_{\rho_3}d^s_q\xi.$$

It is interesting to observe that the above result does not hold, if Φ is not $\mathfrak{l}$-monotone.

2.3. *Applications to Hermite-Hadamard's inequality*

In the subsequent part, we utilize the I.V. left q-symmetric integral operator for the development of (H-H) type inequalities.

Theorem 2.20. *Suppose $\Phi : [\rho_3, \rho_4] \to \mathbb{R}_I$ be I.V. left q-symmetric differentiable mapping, then*

$$\begin{aligned}&\Phi\left(\frac{\rho_3 q^2 + \rho_4}{1+q^2}\right) - \frac{(1-q)(\rho_4-\rho_3)}{1+q^2}\,{}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3 q^2 + \rho_4}{1+q^2}\right)\\ &\supseteq \frac{1}{\rho_4-\rho_3}\int_{\rho_3}^{\rho_4}\Phi(\mathfrak{z}_1)_{\rho_3}d_q(\mathfrak{z}_1) \supseteq \frac{(1+q^2-q)\Phi(\rho_3) + q\Phi(\rho_4)}{1+q^2}.\end{aligned}$$

Proof. Since Φ is I.V. left q-symmetric differentiable mapping on (ρ_3, ρ_4), then there exist two tangents at $\frac{\rho_3 q^2+\rho_4}{1+q^2}$ and given as:

$$h_*(\mathfrak{z}_1) = \Phi_*\left(\frac{\rho_3 q^2 + \rho_4}{1+q^2}\right) + {}_{\rho_3}D_{q,s}\Phi_*\left(\frac{\rho_3 q^2 + \rho_4}{1+q^2}\right)\left(\mathfrak{z}_1 - \frac{\rho_3 q^2 + \rho_4}{1+q^2}\right)$$

and

$$h^*(\mathfrak{z}_1) = \Phi^*\left(\frac{\rho_3 q^2 + \rho_4}{1+q^2}\right) + {}_{\rho_3}D_{q,s}\Phi^*\left(\frac{\rho_3 q^2 + \rho_4}{1+q^2}\right)\left(\mathfrak{z}_1 - \frac{\rho_3 q^2 + \rho_4}{1+q^2}\right).$$

Since Φ is I.V. convex mapping, then $H_1(\mathfrak{z}_1) \supseteq \Phi(\mathfrak{z}_1)$ and then applying the left I.V. q-symmetric integration, we have

$$\begin{aligned}
&\int_{\rho_3}^{\rho_4} H_1(\rho_3,\rho_4)\,{}_{\rho_3}d_q(\mathfrak{z}_1)\\
&= \int_{\rho_3}^{\rho_4}\left[\Phi\left(\frac{\rho_3 q^2+\rho_4}{1+q^2}\right) + {}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3 q^2+\rho_4}{1+q^2}\right)\left(\mathfrak{z}_1 - \frac{\rho_3 q^2+\rho_4}{1+q^2}\right)\right]{}_{\rho_3}d_q(\mathfrak{z}_1)\\
&= (\rho_4-\rho_3)\Phi\left(\frac{\rho_3 q^2+\rho_4}{1+q^2}\right) + {}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3 q^2+\rho_4}{1+q^2}\right)\left(\int_{\rho_3}^{b} x\,{}_{\rho_3}d_q(\mathfrak{z}_1) - (\rho_4-\rho_3)\frac{\rho_3 q^2+\rho_4}{1+q^2}\right)\\
&= (\rho_4-\rho_3)\Phi\left(\frac{\rho_3 q^2+\rho_4}{1+q^2}\right) + (\rho_4-\rho_3)\,{}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3 q^2+\rho_4}{1+q^2}\right)\\
&\quad\times\left((1-q^2)\sum_{n=0}^{\infty} q^{2n}(q^{2n+1}\rho_4 + (1-q^{2n+1}\rho_3)) - \frac{\rho_3 q^2+\rho_4}{1+q^2}\right)\\
&= (\rho_4-\rho_3)\left[\Phi\left(\frac{\rho_3 q^2+\rho_4}{1+q^2}\right) + {}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3 q^2+\rho_4}{1+q^2}\right)\left(\frac{q(\rho_4-\rho_3)+\rho_3(1+q^2)}{1+q^2} - \frac{\rho_3 q^2+\rho_4}{1+q^2}\right)\right]\\
&= (\rho_4-\rho_3)\left[\Phi\left(\frac{\rho_3 q^2+\rho_4}{1+q^2}\right) - \frac{(1-q)(\rho_4-\rho_3)}{1+q^2}\,{}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3 q^2+\rho_4}{1+q^2}\right)\right] \supseteq \int_{\rho_3}^{\rho_4}\Phi(\mathfrak{z}_1)\,{}_{\rho_3}d_q(\mathfrak{z}_1).
\end{aligned}$$

Now we establish the prove of second containment. Moreover, the secant line through $(\rho_3, \Phi(\rho_3))$ and $(\rho_4, \Phi(\rho_4))$ can be expressed as:

$$Y_*(\mathfrak{z}_1) = \Phi_*(\rho_3) + \frac{\Phi_*(\rho_4)-\Phi_*(\rho_3)}{\rho_4-\rho_3}(\xi-\rho_3)$$

and

$$Y^*(\mathfrak{z}_1) = \Phi^*(\rho_3) + \frac{\Phi^*(\rho_4)-\Phi^*(\rho_3)}{\rho_4-\rho_3}(\xi-\rho_3).$$

Since $\Phi(\mathfrak{z}_1) = [\Phi_*(\mathfrak{z}_1),\ \Phi^*(\mathfrak{z}_1)]$ is I.V. convex mappings and $Y(\mathfrak{z}_1) \subseteq \Phi(\mathfrak{z}_1)$, then

$$\begin{aligned}
\int_{\rho_3}^{\rho_4} Y(\mathfrak{z}_1)\,{}_{\rho_3}d_q(\mathfrak{z}_1) &= \int_{\rho_3}^{\rho_4}\left[\Phi(\rho_3) + \frac{\Phi(\rho_4)-\Phi(\rho_3)}{\rho_4-\rho_3}(\xi-\rho_3)\right]{}_{\rho_3}d_q(\mathfrak{z}_1)\\
&= (\rho_4-\rho_3)\Phi(\rho_3) + \frac{\Phi(\rho_4)-\Phi(\rho_3)}{\rho_4-\rho_3}\left(\int_{\rho_3}^{\rho_4}\xi\,{}_{\rho_3}d_q(\mathfrak{z}_1) - (\rho_4-\rho_3)\rho_3\right)\\
&= (\rho_4-\rho_3)\Phi(\rho_3) + (\Phi(\rho_4)-\Phi(\rho_3))\left((1-q^2)\sum_{n=0}^{\infty} q^{2n}(q^{2n+1}\rho_4 + (1-q^{2n+1}\rho_3)) - \rho_3\right)\\
&= (\rho_4-\rho_3)\Phi(\rho_3) + (\Phi(\rho_4)-\Phi(\rho_3))\left(\frac{q(\rho_4-\rho_3)+(1+q^2)\rho_3}{1+q^2} - \rho_3\right)\\
&= (\rho_4-\rho_3)\left[\frac{(1+q^2-q)\Phi(\rho_3)+q\Phi(\rho_4)}{1+q^2}\right] \subseteq \int_{\rho_3}^{\rho_4}\Phi(\mathfrak{z}_1)\,{}_{\rho_3}d_q(\mathfrak{z}_1).
\end{aligned}$$

This completes the proof. □

Example 2.21. Let $\Phi : [0,2] \to \mathbb{R}_I$ be I.V. right q-symmetric differentiable mapping such that $\Phi(\mathfrak{z}_1) = [2{\mathfrak{z}_1}^2, -2{\mathfrak{z}_1}^2 + 20]$. Then from Theorem 2.20, we obtain

$$\Phi\left(\frac{\rho_3 q^2+\rho_4}{1+q^2}\right) = \Phi\left(\frac{2}{1+q^2}\right) = \left[\frac{8}{(1+q^2)^2}, -\frac{8}{(1+q^2)^2} + 20\right].$$

Next, I.V. left q-symmetric derivative of Φ is given as ${}_0D_{q,s}\Phi\left(\frac{2}{1+q^2}\right) = \left[\frac{-4}{q}, \frac{4}{q}\right]$. Moreover, I.V. left q-symmetric integral of Φ is given as:

$$\frac{1}{2}\int_0^2 \Phi(\mathfrak{z}_1)_0 d_q^s x = \int_0^2 [2\mathfrak{z}_1{}^2, -2\mathfrak{z}_1{}^2+20]_0 d_q^s x = \left[\frac{8q^2}{1+q^2+q^4}, -\frac{8q^2}{1+q^2+q^4}+20\right]. \tag{2.10}$$

From above computations, we have following containment

$$\begin{aligned}&\left[\frac{8}{(1+q^2)^2}-\frac{8(1-q)}{q(1+q^2)}, 20-\frac{8}{(1+q^2)^2}+\frac{8(1-q)}{q*(1+q^2)}\right]\\ &\supseteq \left[\frac{8q^2}{1+q^2+q^4}, 20-\frac{8q^2}{1+q^2+q^4}\right] \supseteq \left[\frac{8q}{1+q^2}, \frac{20q^2+20-8q}{1+q^2}\right].\end{aligned} \tag{2.11}$$

For $q = \frac{1}{3}$ in (2.11), we have

$$[-7.92, 27.92] \supseteq [0.791209, 19.2088] \supseteq [2.4, 17.6].$$

For graphical validation, we take $q \in (0,1)$ in (2.11) shown in Figure 1.

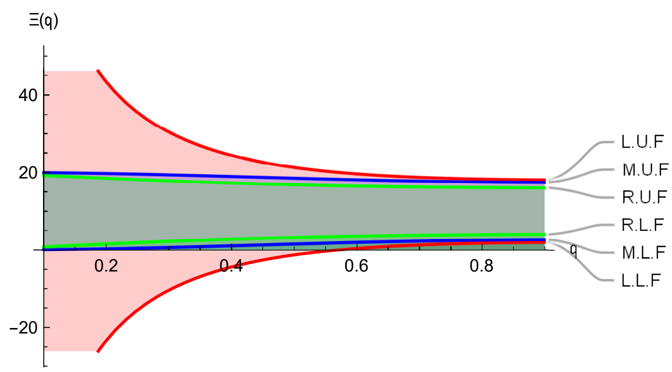

Figure 1: Validating the accuracy of Theorem 2.20.

Theorem 2.22. *Suppose $\Phi : [\rho_3, \rho_4] \to \mathbb{R}_I$ be I.V. left q-symmetric differentiable mapping, then*

$$\begin{aligned}&\Phi\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right) + \frac{q(1-q)(\rho_4-\rho_3)}{1+q^2}\,{}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right)\\ &\supseteq \frac{1}{\rho_4-\rho_3}\int_{\rho_3}^{\rho_4}\Phi(\mathfrak{z}_1)_{\rho_3} d_q(\mathfrak{z}_1) \supseteq \frac{(1+q^2-q)\Phi(\rho_3)+q\Phi(\rho_4)}{1+q^2}.\end{aligned}$$

Proof. Since Φ is I.V. left q-symmetric differentiable mapping on (ρ_3, ρ_4), then there exist two tangents at $\frac{\rho_3+\rho_4 q^2}{1+q^2}$ and given as:

$${}_2h_*(\mathfrak{z}_1) = \Phi_*\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right) + {}_{\rho_3}D_{q,s}\Phi_*\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right)\left(\mathfrak{z}_1 - \frac{\rho_3+\rho_4 q^2}{1+q^2}\right)$$

and

$${}_2h^*(\mathfrak{z}_1) = \Phi^*\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right) + {}_{\rho_3}D_{q,s}\Phi^*\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right)\left(\mathfrak{z}_1 - \frac{\rho_3+\rho_4 q^2}{1+q^2}\right).$$

Since Φ is I.V. convex mapping, thus $H_2(\mathfrak{z}_1) \supseteq \Phi(\mathfrak{z}_1)$ and then applying the left I.V. q-symmetric integration, we have

$$\begin{aligned}
&\int_{\rho_3}^{\rho_4} H_2(\rho_3,\rho_4)\,_{\rho_3}d_q(\mathfrak{z}_1)\\
&= \int_{\rho_3}^{\rho_4}\left[\Phi\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right) + {}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right)\left(\mathfrak{z}_1 - \frac{\rho_3+\rho_4 q^2}{1+q^2}\right)\right]{}_{\rho_3}d_q(\mathfrak{z}_1)\\
&= (\rho_4-\rho_3)\Phi\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right) + {}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right)\left(\int_{\rho_3}^{b} x\,_{\rho_3}d_q(\mathfrak{z}_1) - (\rho_4-\rho_3)\frac{\rho_3+\rho_4 q^2}{1+q^2}\right)\\
&= (\rho_4-\rho_3)\Phi\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right) + (\rho_4-\rho_3)\,_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right)\\
&\quad\times\left((1-q^2)\sum_{n=0}^{\infty} q^{2n}(q^{2n+1}\rho_4 + (1-q^{2n+1}\rho_3)) - \frac{\rho_3+\rho_4 q^2}{1+q^2}\right)\\
&= (\rho_4-\rho_3)\left[\Phi\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right) + {}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right)\left(\frac{q(\rho_4-\rho_3)+\rho_3(1+q^2)}{1+q^2} - \frac{\rho_3+\rho_4 q^2}{1+q^2}\right)\right]\\
&= (\rho_4-\rho_3)\left[\Phi\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right) + \frac{q(1-q)(\rho_4-\rho_3)}{1+q^2}\,_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right)\right] \supseteq \int_{\rho_3}^{\rho_4}\Phi(\mathfrak{z}_1)\,_{\rho_3}d_q(\mathfrak{z}_1).
\end{aligned}$$

This completes the proof. □

Example 2.23. Let $\Phi : [0,2] \to \mathbb{R}_I$ be I.V. right q-symmetric differentiable mapping such that $\Phi(\mathfrak{z}_1) = [2{\mathfrak{z}_1}^2, -2{\mathfrak{z}_1}^2 + 20]$. Then from Theorem 2.22, we obtain:

$$\Phi\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right) = \Phi\left(\frac{2q^2}{1+q^2}\right) = \left[\frac{8q^4}{(1+q^2)^2}, -\frac{8q^4}{(1+q^2)^2} + 20\right]. \tag{2.12}$$

From (2.10) and (2.12), we have following containment

$$\begin{aligned}
&\left[\frac{8q^4}{(1+q^2)^2} - \frac{8(1-q)}{(1+q^2)}, 20 - \frac{8q^4}{(1+q^2)^2} + \frac{8(1-q)}{(1+q^2)}\right]\\
&\supseteq \left[\frac{8q^2}{1+q^2+q^4}, 20 - \frac{8q^2}{1+q^2+q^4}\right] \supseteq \left[\frac{8q}{1+q^2}, \frac{20q^2+20-8q}{1+q^2}\right].
\end{aligned} \tag{2.13}$$

For $q = \frac{1}{3}$ in (2.13), we have

$$[-4.72, 24.72] \supseteq [0.791209, 19.2088] \supseteq [2.4, 17.6].$$

For graphical validation, we take $q \in (0,1)$ in (2.13) shown in Figure 2.

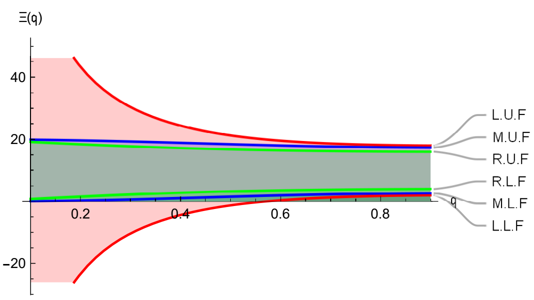

Figure 2: Validating the accuracy of Theorem 2.22.

Theorem 2.24. *Suppose $\Phi : [\rho_3, \rho_4] \to \mathbb{R}_I$ be I.V. left q-symmetric differentiable mapping, then*

$$\Phi\left(\frac{\rho_3+\rho_4}{2}\right)+\frac{(2q-1-q^2)(\rho_4-\rho_3)}{2(1+q^2)}\,{}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3+\rho_4}{2}\right) \supseteq \frac{1}{\rho_4-\rho_3}\int_{\rho_3}^{\rho_4}\Phi(\mathfrak{z}_1)\,{}_{\rho_3}d_q(\mathfrak{z}_1) \supseteq \frac{(1+q^2-q)\Phi(\rho_3)+q\Phi(\rho_4)}{1+q^2}.$$

Proof. Since Φ is I.V. left q-symmetric differentiable mapping on (ρ_3, ρ_4), then there exist two tangents at $\frac{\rho_3+\rho_4}{2}$ and given as:

$${}_3h_*(\mathfrak{z}_1)=\Phi_*\left(\frac{\rho_3+\rho_4}{2}\right)+{}_{\rho_3}D_{q,s}\Phi_*\left(\frac{\rho_3+\rho_4}{2}\right)\left(\mathfrak{z}_1-\frac{\rho_3+\rho_4}{2}\right)$$

and

$${}_3h^*(\mathfrak{z}_1)=\Phi^*\left(\frac{\rho_3+\rho_4}{2}\right)+{}_{\rho_3}D_{q,s}\Phi^*\left(\frac{\rho_3+\rho_4}{2}\right)\left(\mathfrak{z}_1-\frac{\rho_3+\rho_4}{2}\right).$$

Since Φ is I.V. convex mapping, then $H_3(\mathfrak{z}_1) \subseteq \Phi(\mathfrak{z}_1)$ and then applying the left I.V. q-symmetric integration, we have

$$\begin{aligned}
&\int_{\rho_3}^{\rho_4} H_3(\rho_3,\rho_4)\,{}_{\rho_3}d_q(\mathfrak{z}_1)\\
&=\int_{\rho_3}^{\rho_4}\left[\Phi\left(\frac{\rho_3+\rho_4}{2}\right)+{}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3+\rho_4}{2}\right)\left(\mathfrak{z}_1-\frac{\rho_3+\rho_4}{2}\right)\right]{}_{\rho_3}d_q(\mathfrak{z}_1)\\
&=(\rho_4-\rho_3)\Phi\left(\frac{\rho_3+\rho_4}{2}\right)+{}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3+\rho_4}{2}\right)\left(\int_{\rho_3}^{b}x\,{}_{\rho_3}d_q(\mathfrak{z}_1)-(\rho_4-\rho_3)\frac{\rho_3+\rho_4}{2}\right)\\
&=(\rho_4-\rho_3)\Phi\left(\frac{\rho_3+\rho_4}{2}\right)+(\rho_4-\rho_3)\,{}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3+\rho_4}{2}\right)\\
&\quad\times\left((1-q^2)\sum_{n=0}^{\infty}q^{2n}(q^{2n+1}\rho_4+(1-q^{2n+1}\rho_3))-\frac{\rho_3+\rho_4}{2}\right)\\
&=(\rho_4-\rho_3)\left[\Phi\left(\frac{\rho_3+\rho_4}{2}\right)+{}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3+\rho_4}{2}\right)\left(\frac{q(\rho_4-\rho_3)+\rho_3(1+q^2)}{1+q^2}-\frac{\rho_3+\rho_4}{2}\right)\right]\\
&=(\rho_4-\rho_3)\left[\Phi\left(\frac{\rho_3+\rho_4}{2}\right)+\frac{(2q-1-q^2)(\rho_4-\rho_3)}{2(1+q^2)}\,{}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3+\rho_4}{2}\right)\right]\supseteq\int_{\rho_3}^{\rho_4}\Phi(\mathfrak{z}_1)\,{}_{\rho_3}d_q(\mathfrak{z}_1).
\end{aligned}$$

□

Example 2.25. Let $\Phi : [0, 2] \to \mathbb{R}_I$ be I.V. right q-symmetric differentiable mapping such that $\Phi(\mathfrak{z}_1) = [2\mathfrak{z}_1{}^2, -2\mathfrak{z}_1{}^2 + 20]$. Then from Theorem 2.24, we obtain

$$\Phi\left(\frac{\rho_3+\rho_4}{2}\right)=\Phi(1)=[2,\ 18]. \tag{2.14}$$

From (2.10) and (2.14), we have following containment

$$\left[\frac{6q-2-2q^2}{q},\ \frac{14q+2+2q^2}{q}\right]\supseteq\left[\frac{8q^2}{1+q^2+q^4},\ 20-\frac{8q^2}{1+q^2+q^4}\right]\supseteq\left[\frac{8q}{1+q^2},\ \frac{20q^2+20-8q}{1+q^2}\right]. \tag{2.15}$$

For $q = \frac{1}{3}$ in (2.15), we have

$$[-0.666667, 20.6667] \supseteq [0.791209, 19.2088] \supseteq [2.4, 17.6].$$

For graphical validation, we take $q \in (0, 1)$ in (2.15) shown in Figure 3.

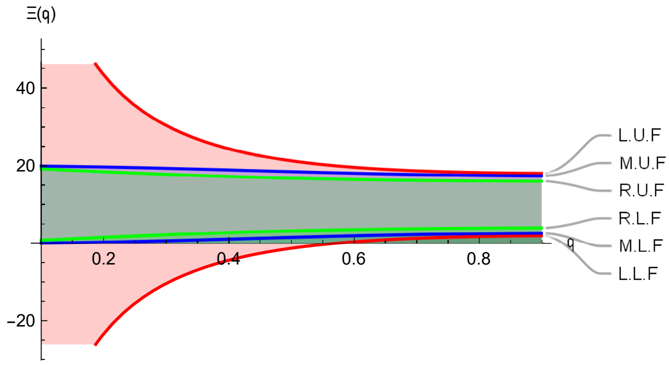

Figure 3: Validating the accuracy of Theorem 2.24.

2.4. *Symmetric quantum analogues of Hermite-Hadamard's inequality via totaly ordered convex mapping*

This part of study contains Hadamard's type inequalities utilizing center-radius ordering relation. First, we mention the cr relation from [4] and which is given as follows. Let $w = [\underline{w}, \overline{w}] = \langle w_c, \overline{w_r} \rangle$, $v = [\underline{v}, \overline{v}] = \langle v_c, \overline{v_r} \rangle \in \mathbb{R}_I$, the cr ranking of interval is demonstrated as:

$$w \preceq_{cr} v \Longleftrightarrow \begin{cases} w_c < v_c, & \text{if } w_c \neq v_c, \\ w_r \leqslant v_r, & \text{if } w_= v_c. \end{cases}$$

Based on this ranking relation, the analogue of convexity presented in [29], which is given as follows. Any I.V. mapping $\Phi : [\rho_3, \rho_4] \to \mathbb{R}_I^+$ such that $[\Phi_*, \Phi^*] = \langle \Phi_c, \Phi_r \rangle$ is said to be a cr I.V. convex, if

$$\Phi(\xi\rho_3 + (1-\xi)\rho_4) \preceq_{cr} \xi\Phi(\rho_3) + (1-\xi)\Phi(\rho_4), \quad \xi \in [0,1].$$

Theorem 2.26. *Suppose* $\Phi : [\rho_3, \rho_4] \to \mathbb{R}_I$ *be I.V. left* q*-symmetric differentiable* cr *convex mapping such that* $[\Phi_*, \Phi^*] = \langle \Phi_c, \Phi_r \rangle$, *then*

$$\begin{aligned} &\Phi\left(\frac{\rho_3 + \rho_4 q^2}{1+q^2}\right) + \frac{q(1-q)(\rho_4 - \rho_3)}{1+q^2} {}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3 + \rho_4 q^2}{1+q^2}\right) \\ &\preceq_{cr} \frac{1}{\rho_4 - \rho_3} \int_{\rho_3}^{\rho_4} \Phi(\mathfrak{z}_1)\, {}_{\rho_3}d_q(\mathfrak{z}_1) \preceq_{cr} \frac{(1+q^2-q)\Phi(\rho_3) + q\Phi(\rho_4)}{1+q^2}. \end{aligned}$$

Proof. Since Φ is I.V. left q-symmetric differentiable cr-convex mapping on (ρ_3, ρ_4), then there exist two tangents at $\frac{\rho_3+\rho_4 q^2}{1+q^2}$ and given as:

$${}_2h_c(\mathfrak{z}_1) = \Phi_c\left(\frac{\rho_3 + \rho_4 q^2}{1+q^2}\right) + {}_{\rho_3}D_{q,s}\Phi_c\left(\frac{\rho_3 + \rho_4 q^2}{1+q^2}\right)\left(\mathfrak{z}_1 - \frac{\rho_3 + \rho_4 q^2}{1+q^2}\right)$$

and

$${}_2h_r(\mathfrak{z}_1) = \Phi_r\left(\frac{\rho_3 + \rho_4 q^2}{1+q^2}\right) + {}_{\rho_3}D_{q,s}\Phi_r\left(\frac{\rho_3 + \rho_4 q^2}{1+q^2}\right)\left(\mathfrak{z}_1 - \frac{\rho_3 + \rho_4 q^2}{1+q^2}\right).$$

Since Φ is I.V. cr convex mapping, thus $\langle {}_2h_c(\mathfrak{z}_1), {}_2h_r(\mathfrak{z}_1) \rangle \preceq_{cr} \langle \Phi_c(\mathfrak{z}_1), \Phi_r(\mathfrak{z}_1) \rangle$ and then applying the left I.V. q-symmetric integration, we have

$$\begin{aligned} \int_{\rho_3}^{\rho_4} {}_2h_c(\mathfrak{z}_1)\, {}_{\rho_3}d_q(\mathfrak{z}_1) &= \Phi_c\left(\frac{\rho_3 + \rho_4 q^2}{1+q^2}\right) + \frac{q(1-q)(\rho_4 - \rho_3)}{1+q^2} {}_{\rho_3}D_{q,s}\Phi_c\left(\frac{\rho_3 + \rho_4 q^2}{1+q^2}\right) \\ &\leqslant \frac{1}{(\rho_4 - \rho_3)} \int_{\rho_3}^{\rho_4} \Phi_c(\mathfrak{z}_1)\, {}_{\rho_3}d_q(\mathfrak{z}_1) \end{aligned} \tag{2.16}$$

and

$$\begin{aligned}\int_{\rho_3}^{\rho_4} {}_2h_r(\mathfrak{z}_1)\,{}_{\rho_3}d_q(\mathfrak{z}_1) &= \Phi_r\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right) + \frac{q(1-q)(\rho_4-\rho_3)}{1+q^2}\,{}_{\rho_3}D_{q,s}\Phi_r\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right)\\ &\leqslant \frac{1}{(\rho_4-\rho_3)}\int_{\rho_3}^{\rho_4}\Phi_r(\mathfrak{z}_1)\,{}_{\rho_3}d_q(\mathfrak{z}_1).\end{aligned} \tag{2.17}$$

Combining (2.16) and (2.17), we have

$$\begin{aligned}&\left\langle \Phi_c\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right) + \frac{q(1-q)(\rho_4-\rho_3)}{1+q^2}\,{}_{\rho_3}D_{q,s}\Phi_c\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right),\right.\\ &\qquad\left.\Phi_r\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right) + \frac{q(1-q)(\rho_4-\rho_3)}{1+q^2}\,{}_{\rho_3}D_{q,s}\Phi_r\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right)\right\rangle\\ &\qquad\preceq_{cr}\left\langle \frac{1}{(\rho_4-\rho_3)}\int_{\rho_3}^{\rho_4}\Phi_c(\mathfrak{z}_1)\,{}_{\rho_3}d_q(\mathfrak{z}_1),\ \frac{1}{(\rho_4-\rho_3)}\int_{\rho_3}^{\rho_4}\Phi_r(\mathfrak{z}_1)\,{}_{\rho_3}d_q(\mathfrak{z}_1)\right\rangle.\end{aligned}$$

This implies that

$$\Phi\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right) + \frac{q(1-q)(\rho_4-\rho_3)}{1+q^2}\,{}_{\rho_3}D_{q,s}\Phi\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right) \preceq_{cr} \frac{1}{(\rho_4-\rho_3)}\int_{\rho_3}^{\rho_4}\Phi(\mathfrak{z}_1)\,{}_{\rho_3}d_q(\mathfrak{z}_1).$$

To establish second inequality, we utilize the concept of secant line through $(\rho_3, \Phi(\rho_3))$ and $(\rho_4, \Phi(\rho_4))$, which can be expressed as

$$Y_c(\mathfrak{z}_1) = \Phi_c(\rho_3) + \frac{\Phi_c(\rho_4)-\Phi_c(\rho_3)}{\rho_4-\rho_3}(\xi-\rho_3) \quad \text{and} \quad Y_r(\mathfrak{z}_1) = \Phi_r(\rho_3) + \frac{\Phi_r(\rho_4)-\Phi_r(\rho_3)}{\rho_4-\rho_3}(\xi-\rho_3).$$

Since $\Phi(\mathfrak{z}_1) = \langle \Phi_c(\mathfrak{z}_1), \Phi_r(\mathfrak{z}_1)\rangle$ is I.V. cr-convex mappings and $\langle \Phi_c(\mathfrak{z}_1), \Phi_r(\mathfrak{z}_1)\rangle \preceq_{cr} \langle Y_c(\mathfrak{z}_1), Y_r(\mathfrak{z}_1)\rangle$, then

$$\frac{1}{(\rho_4-\rho_3)}\int_{\rho_3}^{\rho_4}\Phi_c(\mathfrak{z}_1)\,{}_{\rho_3}d_q(\mathfrak{z}_1) \leqslant \frac{1}{(\rho_4-\rho_3)}\int_{\rho_3}^{\rho_4}Y_c(\mathfrak{z}_1)\,{}_{\rho_3}d_q(\mathfrak{z}_1) = \frac{(1+q^2-q)\Phi_c(\rho_3)+q\Phi_c(\rho_4)}{1+q^2} \tag{2.18}$$

and

$$\frac{1}{(\rho_4-\rho_3)}\int_{\rho_3}^{\rho_4}\Phi_r(\mathfrak{z}_1)\,{}_{\rho_3}d_q(\mathfrak{z}_1) \leqslant \frac{1}{(\rho_4-\rho_3)}\int_{\rho_3}^{\rho_4}Y_r(\mathfrak{z}_1)\,{}_{\rho_3}d_q(\mathfrak{z}_1) = \frac{(1+q^2-q)\Phi_r(\rho_3)+q\Phi_r(\rho_4)}{1+q^2}. \tag{2.19}$$

Combining (2.18) and (2.19) via cr relation, we get

$$\begin{aligned}&\left\langle \frac{1}{(\rho_4-\rho_3)}\int_{\rho_3}^{\rho_4}\Phi_c(\mathfrak{z}_1)\,{}_{\rho_3}d_q(\mathfrak{z}_1),\ \frac{1}{(\rho_4-\rho_3)}\int_{\rho_3}^{\rho_4}\Phi_r(\mathfrak{z}_1)\,{}_{\rho_3}d_q(\mathfrak{z}_1)\right\rangle\\ &\qquad\preceq_{cr}\left\langle \frac{(1+q^2-q)\Phi_c(\rho_3)+q\Phi_c(\rho_4)}{1+q^2},\ \frac{(1+q^2-q)\Phi_r(\rho_3)+q\Phi_r(\rho_4)}{1+q^2}\right\rangle.\end{aligned}$$

This completes the proof. □

Example 2.27. Let $\Phi : [0,2] \to \mathbb{R}_I$ be I.V. right q-symmetric differentiable mapping such that $\Phi(\mathfrak{z}_1) = [-2{\mathfrak{z}_1}^2+8,\ 2{\mathfrak{z}_1}^2+10]$. Then from Theorem 2.22, we obtain

$$\Phi\left(\frac{\rho_3+\rho_4 q^2}{1+q^2}\right) = \Phi\left(\frac{2q^2}{1+q^2}\right) = \left[-\frac{8q^4}{(1+q^2)^2}+8,\ \frac{8q^4}{(1+q^2)^2}+10\right]. \tag{2.20}$$

From (2.10) and (2.20), we have following containment:

$$\begin{aligned}&\left[-\frac{8(1-q)q^2}{q^2+1}-\frac{8q^4}{(q^2+1)^2}+8,\ \frac{8(1-q)q^2}{q^2+1}+\frac{8q^4}{(q^2+1)^2}+10\right]\\&\quad\preceq_{cr}\left[8-\frac{8q^2}{q^4+q^2+1},\ \frac{8q^2}{q^4+q^2+1}+10\right]\preceq_{cr}\left[\frac{8\left(q^2-q+1\right)}{q^2+1},\ \frac{10q^2+8q+10}{q^2+1}\right].\end{aligned}\tag{2.21}$$

For $q=\frac{1}{3}$ in (2.21), we have

$$[7.38667, 10.6133]\preceq_{cr}[7.20879,\ 10.7912]\preceq_{cr}[5.6,\ 12.4].$$

For graphical validation, we take $q\in(0,1)$ in (2.13) shown in Figure 4.

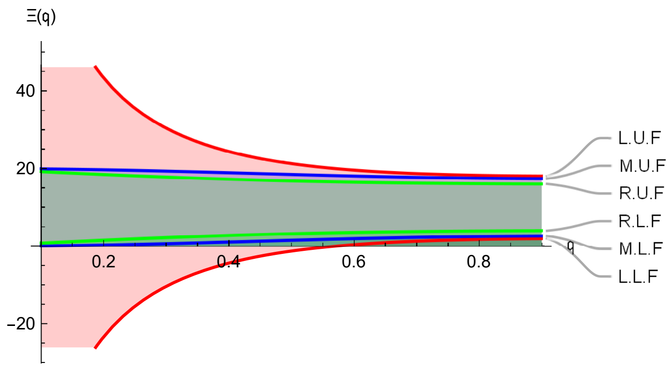

Figure 4: Validating the accuracy of Theorem 2.26.

3. Conclusion

In the present study, we have introduced the idea of I.V. symmetric quantum derivative and integral operators. Also, we have provided a detailed characterization of newly proposed operators. From an application point of view, we have proved some new (H-H) inequalities. The primary outcomes are supported by numerical and graphical illustrations. It is important to note that the derivative of I.V. mappings involving absolute mappings can be computed from our proposed operators. We hope these operators and techniques will create new venues for the research. Based on these operators several kinds of inequalities can be obtained.

Acknowledgment

The authors are thankful to the editor and the anonymous reviewers for their valuable comments and suggestions.

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