J. Math. Computer Sci., 40 (2026), 22–37

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Extended Grüss type inequalities for generalized $(\mathfrak{l}, \mathfrak{m})$-fractional integrals with applications

Muhammad Yousaf[a], Sajid Iqbal[a], Muhammad Samraiz[b], Miguel Vivas-Cortez[c,*]

[a]Department of Mathematics & Statistics, University of Southern Punjab, Bosan Road, Multan, Pakistan.

[b]Department of Mathematics, University of Sargodha, Sargodha, Pakistan.

[c]Pontificia Universidad Católica del Ecuador, Quito, Ecuador.

Abstract

The main motivation of this paper is to establish certain Grüss type inequalities by using the new generalized $(\mathfrak{l}, \mathfrak{m})$-Riemman-Liouville fractional integrals. We obtain the related results for the geometric, arithmetic, and harmonic $(\mathfrak{l}, \mathfrak{m})$-Riemman-Liouville fractional integrals as special cases of our general results. Also, we apply the Young's and Cauchy-Schwarz inequalities to obtain the variety of some related estimates. Our findings have potential applications in various fields of mathematical analysis and its related disciplines.

Keywords: Inequalities, fractional integrals, means, kernel, Young's inequality.

2020 MSC: 26D10, 26D15, 26A33.

1. Introduction

The study of integral inequalities has been a cornerstone in the field of mathematical analysis, offering critical insights and tools for a variety of applications. Among these, fractional integral inequalities have gained significant attention due to their ability to generalize classical integral inequalities and their applicability in various scientific and engineering disciplines (see [16, 18, 22, 26, 29]). The analysis of fractional order integral and derivative operators is the focus of fractional calculus. Fractional calculus drew the attention of multiple researchers in the past few years, who did outstanding work (see, for example, [1, 4, 7, 13–17, 20, 21, 31]). We focus on a generalized type of integral operator, which is a tool that helps us to deal with functions in a more flexible way. By exploring Grüss-type inequalities in this context, we can gain new insights and develop more powerful mathematical tools. For more related literature about the mathematical inequalities that involve the fractional calculus and the unified version of continuous and discrete inequalities, we refer to [2, 3, 5, 6, 23–25].

*Corresponding author

Email addresses: myousif31730938@gmail.com (Muhammad Yousaf), sajid_uos2000@yahoo.com; sajidiqbal@isp.edu.pk (Sajid Iqbal), muhammad.samraiz@uos.edu.pk; msamraizuos@gmail.com (Muhammad Samraiz), mjvivas@puce.edu.ec (Miguel Vivas-Cortez)

doi: 10.22436/jmcs.040.01.02

Received: 2024-08-27 *Revised:* 2025-03-04 *Accepted:* 2025-04-04

The Grüss inequality is one of the most intriguing of these kinds of inequalities. The product of two functions and the generalized integral of their product are related by the Grüss inequality. The role of continuous and discrete examples of Grüss type modifications in analyzing the qualitative behavior of integral and differential equations has been well-documented. We aim to demonstrate modified variants of Grüss inequality through the use of the generalized integrals operator with some application. Due to the belief that these variations are essential, research into these types of variants has continued to progress. Many academic articles have highlighted their applications and importance (see [19, 27–30, 32]). This inequality motivates us to examine the behavior of Grüss type inequalities that involve the generalized integral operator with some application of mean fractional integrals. Therefore, several fractional integral inequalities are obtained; these inequalities are valuable in the domain of fractional integral inequalities. Additionally, we utilize these fractional inequalities to discover novel estimates related to the generalized Riemann-Liouville fractional integral. To support our primary results later, we give some preliminary data in this sequel. The Gruss type inequalities become more fascinating when studying the deviation between the integral of product and product of integrals. In the last two decades, the work on Gruss type inequalities get considerable attention by the numerous research to give the generalizations and imports for different fractional integrals, which include Riemman-Liuville's fractional integrals, Weyl's fractional integrals, Katugampola fractional integrals, Haadmard fractional integrals, Caputo-Fabrizio fractional integrals, conformable fractional integrals, and proportional fractional integrals. The most interesting inequality is the Grüss inequality, which is expressed in the following theorem (see [12]).

Theorem 1.1. *Let $\mathfrak{R}$ be a set of real numbers, $f, F, t, T \in \mathfrak{R}$ and $\Upsilon, \gamma : [r, s] \to \mathfrak{R}$ be two positive functions such that $f \leqslant \Upsilon(\mu) \leqslant F$ and $t \leqslant \gamma(\mu) \leqslant T$ for all $\mu \in [r, s]$. Then*

$$\left| \frac{1}{s-r} \int_r^s \gamma(\mu)\Upsilon(\mu)d\mu - \frac{1}{(s-r)^2} \int_r^s \gamma(\mu)d\mu \int_r^s \Upsilon(\mu)d\mu \right| \leqslant \frac{1}{4}(F-f)(T-t),$$

where the constant $\frac{1}{4}$ cannot be improved.

This paper is structured as follows. After the introduction, in Section 2, we provide the necessary background and preliminaries of fractional integrals. We shall also introduce the (l, m)-Riemann-Liouville fractional integral. In Section 3, we derive the new Grüss-type inequalities using the established framework. We shall derive the results for geometric, arithmetic and harmonic (l, m)-Riemann-Liouville fractional integrals. A wide range of related inequalities shall be derived using the Young's and Cauchy-Schwarz inequalities.

2. The general (l, m)-Riemann-Liouville fractional integrals

Firstly, we shall describe the general (l, m)-Riemann-Liouville fractional integrals of order s with two exponential parameters, l and m. These integrals generalize the Riemann-Liouville fractional integrals and are motivated by the works mentioned above. We shall emulate the actions of the authors in [10, 11] as part of the investigation.

Let $\varphi : (0, +\infty) \times (0, +\infty) \to [0, +\infty]$ be a map satisfying the condition $\varphi(k, k) = k$, for example:

1. the arithmetic mean $\varphi_1(l, m) = \frac{l+m}{2}$;
2. the geometric mean $\varphi_2(l, m) = \sqrt{l.m}$;
3. $\varphi_3(l, m) = \frac{k^2}{l}$ called the inverse of harmonic case.

Diaz and Pariguan gave the definition of k-gamma function in [9] and later on Kuldeep defined the (l, m)-Gamma function in [11] in 2017.

Definition 2.1. Let $\beta \in C/lZ^{-}; l, m \in \mathbb{R}^{+} - \{0\}$ and $Re(\beta) > 0$, where $Re(\beta)$ denotes the real part of β. Then the integral representation of (l, m)-Gamma function is given by

$$\Gamma_{(l,m)}(\beta) = \int_{0}^{\infty} x^{\beta-1} e^{\frac{-x^{l}}{m}} dx. \tag{2.1}$$

Definition 2.2 ([8])**.** Let $[c, d] \subseteq [0, +\infty]$, $c < d$, $g \in L_1[c, d]$, and $l, m > 0$. The left and right-sided general (l, m)-Riemann-Liouville fractional integral of order $\lambda > 0$ are define as

$$_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)} g(y) = \frac{1}{l\Gamma_{(l,m)}(\frac{l\lambda}{\varphi(l,m)})} \int_{c}^{y} (y-x)^{\frac{\lambda}{\varphi(l,m)}-1} g(x) dx, \ \ c < y \leqslant d.$$

$$_{d^-}\mathfrak{I}^{\lambda}_{\varphi(l,m)} g(y) = \frac{1}{l\Gamma_{(l,m)}(\frac{l.\lambda}{\varphi(l,m)})} \int_{y}^{d} (x-y)^{\frac{\lambda}{\varphi(l,m)}-1} g(x) dx, \ \ c \leqslant y < d,$$

where $\Gamma_{(l,m)}$ is defined in (2.1).

Here we shall extract the definitions of the arithmetic, geometric and harmonic (l, m)-Riemman-Liouville fractional integrals operators from the Definition 2.2 as special cases.

Case 2.3. Choose $\varphi(l, m) = \frac{l+m}{2}$ in Definition 2.2, we obtain left and right sided the arithmetic mean (l, m)-Riemann-Liouville fractional integrals, respectively, as

$$_{c^+}A^{\lambda}_{\varphi(l,m)} g(y) = \frac{1}{l\Gamma_{(l,m)}(\frac{2l\lambda}{l+m})} \int_{c}^{y} (y-x)^{\frac{2\lambda}{l+m}-1} g(x) dx, \ \ c < y \leqslant d,$$

$$_{d^-}A^{\lambda}_{\varphi(l,m)} g(y) = \frac{1}{l\Gamma_{(l,m)}(\frac{2l\lambda}{l+m})} \int_{y}^{d} (x-y)^{\frac{2\lambda}{l+m}-1} g(x) dx, \ \ c \leqslant y < d,$$

where $\Gamma_{(l,m)}$ is defined in (2.1).

Case 2.4. By taking $\varphi(l, m) = \sqrt{l.m}$, the Definition 2.2 becomes for the geometric (l, m)-Riemann-Liouville fractional integrals, i.e.,

$$_{c^+}G^{\lambda}_{\varphi(l,m)} g(y) = \frac{1}{l\Gamma_{(l,m)}\left(\sqrt{\frac{l}{m}}\lambda\right)} \int_{c}^{y} (y-x)^{\frac{\lambda}{\sqrt{l.m}}-1} g(x) dx, \ \ c < y \leqslant d,$$

$$_{d^-}G^{\lambda}_{\varphi(l,m)} g(y) = \frac{1}{l\Gamma_{(l,m)}\left(\sqrt{\frac{l}{m}}\lambda\right)} \int_{y}^{d} (x-y)^{\frac{\lambda}{\sqrt{l.m}}-1} g(x) dx, \ \ c \leqslant y < d,$$

where $\Gamma_{(l,m)}$ is defined in (2.1).

Case 2.5. As special case of Definition 2.2, we can write it for harmonic (l, m)-Riemann-Liouville fractional integrals by taking $\varphi(l, m) = \frac{l^2}{m}$, we have that

$$_{c^+}H^{\lambda}_{\varphi(l,m)} g(y) = \frac{1}{l\Gamma_{(l,m)}(\frac{m}{l}s)} \int_{c}^{y} (y-x)^{\frac{m\lambda}{l^2}-1} g(x) dx, \ \ c < y \leqslant d,$$

$$ {}_{d^-}H^{\lambda}_{\varphi(l,m)}g(y) = \frac{1}{l\Gamma_{(l,m)}(\frac{m}{l}s)} \int_{y}^{d} (x-y)^{\frac{m\lambda}{l^2}-1} g(x)dx, \; c \leqslant y < d, $$

where $\Gamma_{(l,m)}$ is defined in (2.1).

The results given in this paper generalize the results given in [22]. If we choose $l = m = 1$, then (l, m)-Riemann-Liouville fractional integrals given in Definition 2.2 reduce to the of Riemannn-Liouville's fractional integrals, and defined as follows. Let $[c, d] \subseteq [0, +\infty]$, $c < d$, $g \in L_1[c, d]$. The left and right-sided Riemann-Liouville fractional integral of order $\lambda > 0$ are:

$$ {}_{c^+}\mathfrak{I}^{\lambda}g(y) = \frac{1}{\Gamma(\lambda)} \int_{c}^{y} (y-x)^{\lambda-1} g(x)dx, \; c < y \leqslant d, \quad {}_{d^-}\mathfrak{I}^{\lambda}g(y) = \frac{1}{\Gamma(\lambda)} \int_{y}^{d} (x-y)^{\lambda-1} g(x)dx, \; c \leqslant y < d, $$

where $\Gamma(\beta)$ is defined as $\Gamma(\beta) = \int\limits_{0}^{\infty} x^{\beta-1}e^{-x}dx$.

3. The main results

Our first result is given in upcoming theorem.

Theorem 3.1. *Let $[r, s] \subseteq [0, \infty)$, where $r < s$, $\Upsilon \in L_1[r, s]$, and $l, m > 0$, and let ${}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}$ denote the (l, m)-Riemann-Liouville fractional integrals of order $\lambda > 0$. Suppose that η_1, η_2 are integrable functions define on $[0, \infty)$ such that*

$$ \eta_1(\xi) \leqslant \Upsilon(\xi) \leqslant \eta_2(\xi), \tag{3.1} $$

for all $\xi \in (0, \infty)$, then the following inequality holds:

$$ \begin{aligned} &{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\,{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi) + {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\,{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi) \\ &\quad \geqslant {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)\,{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi) + {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\,{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi). \end{aligned} \tag{3.2} $$

Proof. By using (3.1) for all $r, s \geqslant 0$, we have

$$ (\eta_2(r) - \Upsilon(r))(\Upsilon(s) - \eta_1(s)) \geqslant 0, $$

and can be written as

$$ \eta_2(r)\Upsilon(s) + \Upsilon(r)\eta_1(s) \geqslant \eta_2(r)\eta_1(s) + \Upsilon(r)\Upsilon(s). \tag{3.3} $$

Multiplying (3.3) by $\frac{1}{(l.\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - r)^{\frac{\lambda}{\varphi(l,m)}-1}$ on both sides and integrating from c to ξ with respect to r, we obtain

$$ \begin{aligned} &\int_{c}^{\xi} \frac{1}{(l.\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - r)^{\frac{\lambda}{\varphi(l,m)}-1}\eta_2(r)\Upsilon(s)dr + \int_{c}^{\xi} \frac{1}{(l.\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - r)^{\frac{\lambda}{\varphi(l,m)}-1}\Upsilon(r)\eta_1(s)dr \\ &\geqslant \int_{c}^{\xi} \frac{1}{(l.\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - r)^{\frac{\lambda}{\varphi(l,m)}-1}\eta_2(r)\eta_1(s)dr + \int_{c}^{\xi} \frac{1}{(l.\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - r)^{\frac{\lambda}{\varphi(l,m)}-1}\Upsilon(r)\Upsilon(s)dr. \end{aligned} \tag{3.4} $$

Applying the Definition 2.2 of the left sided (l, m)-Riemann-Liouville fractional integral on (3.4), we have

$$ {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)\Upsilon(s) + {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\eta_1(s) \geqslant {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)\eta_1(s) + {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\Upsilon(s). \tag{3.5} $$

Again multiplying (3.5) by $\frac{1}{(l.\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - s)^{\frac{s}{\varphi(l,m)}-1}$ on both sides and integrating from c to ξ with respect to s, we get

$$\begin{aligned}
&\int_c^{\xi} \frac{1}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - s)_{c^+}^{\frac{\lambda}{\varphi(l,m)}-1}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)\Upsilon(s)ds \\
&\quad + \int_c^{\xi} \frac{1}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - s)_{c^+}^{\frac{\lambda}{\varphi(l,m)}-1}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\eta_1(s)ds \\
&\quad \geqslant \int_c^{\xi} \frac{1}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - s)^{\frac{\lambda}{\varphi(l,m)}-1}\eta_1(s)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)ds \\
&\quad + \int_c^{\xi} \frac{1}{(l\Gamma_{(l,m)})\left(\frac{ls}{\varphi(l,m)}\right)}(\xi - s)^{\frac{\lambda}{\varphi(l,m)}-1}\Upsilon(s)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)ds.
\end{aligned} \tag{3.6}$$

Applying the Definition 2.2 of the left sided (l, m)-Riemann-Liouville fractional integral on (3.6), we have

$$\begin{aligned}
&{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi) +_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi) \\
&\quad \geqslant_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi) +_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi).
\end{aligned}$$

This completes the proof. □

Corollary 3.2. *We will now determine the result for the geometric (l, m)-Riemann-Liouville fraction integral. We choose $\varphi(l, m) = \sqrt{l.m}$, then ${}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}$ reduces to the ${}_{c^+}G^{\lambda}_{\varphi(l,m)}$ and the inequality (3.2) in Theorem 3.1 becomes*

$$\begin{aligned}
&{}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_2(\xi) +_{c^+} G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_1(\xi) \\
&\quad \geqslant_{c^+} G^{\lambda}_{\varphi(l,m)}\eta_2(\xi)_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_1(\xi) +_{c^+} G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi).
\end{aligned}$$

Corollary 3.3. *If we choose $\varphi(l, m) = \frac{l+m}{2}$ in Theorem 3.1, then ${}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}$ reduces to the ${}_{c^+}A^{\lambda}_{\varphi(l,m)}$ and we get the result for arithmetic (l, m)-Riemann-Liouville fraction integral, and the inequality (3.2) becomes*

$$\begin{aligned}
&{}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_2(\xi) +_{c^+} A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_1(\xi) \\
&\quad \geqslant_{c^+} A^{\lambda}_{\varphi(l,m)}\eta_2(\xi)_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_1(\xi) +_{c^+} A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi).
\end{aligned}$$

Corollary 3.4. *If we take $\varphi(l, m) = \frac{l^2}{m}$ in Theorem 3.1, then ${}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}$ reduces to the ${}_{c^+}H^{\lambda}_{\varphi(l,m)}$, and we get the inequality for the harmonic (l, m)-Riemann-Liouville fraction integral and the inequality (3.2) takes the form*

$$\begin{aligned}
&{}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_2(\xi) +_{c^+} H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_1(\xi) \\
&\quad \geqslant_{c^+} H^{\lambda}_{\varphi(l,m)}\eta_2(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_1(\xi) +_{c^+} H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi).
\end{aligned}$$

If we consider the boundedness of the function Υ in Theorem 3.1, then we can have the next corollary.

Corollary 3.5. *Let $t, T \in \mathfrak{R}$ and $\xi \geqslant 0$, let Υ be a positive function such that $t \leqslant \Upsilon(\xi) \leqslant T$, then, it satisfies*

$$Tk(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi) + tk(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi) \geqslant Ttk(\xi)k(\xi) +_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi),$$

where

$$k(\xi) = \frac{\varphi(l,m)}{l\lambda\Gamma_{(l,m)}\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - a)^{\frac{\lambda}{\varphi(l,m)}}. \tag{3.7}$$

Now we shall establish the inequality for geometric (l, m)-Riemann-Liouville fraction integral.

Corollary 3.6. *Choose $\varphi(l, m) = \sqrt{lm}$ in Corollary 3.5 and get*

$$Tk(\xi)_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi) + tk(\xi)_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi) \geqslant Ttk(\xi)k(\xi) +_{c^+} G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi).$$

The inequality for arithmetic (l, m)-Riemann-Liouville fractional integral is given in next corollary.

Corollary 3.7. *Using $\varphi(l, m) = \frac{l+m}{2}$ in Corollary 3.5, we get*

$$Tk(\xi)_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi) + tk(\xi)_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi) \geqslant Ttk(\xi)k(\xi) +_{c^+} A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi).$$

The inequality for harmonic (l, m)-Riemann-Liouville fractional integral is given in upcoming result.

Corollary 3.8. *We use $\varphi(l, m) = \frac{l^2}{m}$ in Corollary 3.5 and obtain*

$$Tk(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi) + tk(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi) \geqslant Ttk(\xi)k(\xi) +_{c^+} H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi).$$

Theorem 3.9. *Suppose that the assumptions of the Theorem 3.1 holds. Also suppose that there are integrable functions μ_1, μ_2 on $[0, \infty]$ and γ is a positive function on $[0, \infty)$ such that $\mu_1(\xi) \leqslant \gamma(\xi) \leqslant \mu_2(\xi)$ for all $\xi \in (0, \infty)$, satisfying the following four inequalities:*

$$\begin{aligned}
&{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi) +_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_1(\xi)\\
&\quad \geqslant_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_1(\xi) +_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi), \qquad \text{(a)}\\
&{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_2(\xi) +_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi)\\
&\quad \geqslant_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_2(\xi) +_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi), \qquad \text{(b)}\\
&{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_2(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi) +_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi)\\
&\quad \geqslant_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi) +_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_2(\xi), \qquad \text{(c)}\\
&{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_1(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi) +_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi)\\
&\quad \geqslant_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi) +_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_1(\xi). \qquad \text{(d)}
\end{aligned}$$

Proof. For all $r, s \geqslant 0$, we have $(\eta_2(r) - \Upsilon(r))(\gamma(s) - \mu_1(s)) \geqslant 0$, which can be rewritten as

$$\eta_2(r)\gamma(s) + \Upsilon(r)\mu_1(s) \geqslant \eta_2(r)\mu_1(s) + \Upsilon(r)\gamma(s). \tag{3.8}$$

Multiplying the inequality (3.8) by $\frac{1}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - s)^{\frac{\lambda}{\varphi(l,m)}-1}$ on both sides and integrating from c to ξ, with respect to s, we obtain

$$\begin{aligned}
&\int_c^{\xi} \frac{1}{(l\Gamma_{(l,m)})\left(\frac{ls}{\varphi(l,m)}\right)}(\xi - s)^{\frac{\lambda}{\varphi(l,m)}-1}\eta_2(r)\gamma(s)ds + \int_c^{\xi} \frac{1}{(l\Gamma_{(l,m)})\left(\frac{ls}{\varphi(l,m)}\right)}(\xi - s)^{\frac{\lambda}{\varphi(l,m)}-1}\Upsilon(r)\mu_1(s)ds\\
&\geqslant \int_c^{\xi} \frac{1}{(l\Gamma_{(l,m)})\left(\frac{ls}{\varphi(l,m)}\right)}(\xi - s)^{\frac{\lambda}{\varphi(l,m)}-1}\eta_2(r)\mu_1(s)ds + \frac{1}{(l\Gamma_{(l,m)})\left(\frac{ls}{\varphi(l,m)}\right)}\int_c^{\xi}(\xi - s)^{\frac{\lambda}{\varphi(l,m)}-1}\Upsilon(r)\gamma(s)ds.
\end{aligned} \tag{3.9}$$

Applying the Definition 2.2 of the (l, m)-Riemann-Liouville fractional integral on (3.9), we have

$${}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi)\eta_2(r) + \Upsilon(r)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_1(\xi) \geqslant \eta_2(r)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_1(\xi) + \Upsilon(r)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi). \tag{3.10}$$

Again multiplying (3.10) with $\frac{1}{(\mathfrak{l}\Gamma_{(\mathfrak{l},\mathfrak{m})})\left(\frac{\mathfrak{l}\lambda}{\varphi(\mathfrak{l},\mathfrak{m})}\right)}(\xi-r)^{\frac{s}{\varphi(\mathfrak{l},\mathfrak{m})}-1}$ on both sides and integrating from c to ξ with respect to r, we get some necessary settings

$$\begin{aligned}&{}_{c^+}\mathfrak{J}^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi)_{c^+}\mathfrak{J}^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_2(\xi)+{}_{c^+}\mathfrak{J}^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}\mathfrak{J}^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_1(\xi)\\&\quad\geqslant{}_{c^+}\mathfrak{J}^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_2(\xi)_{c^+}\mathfrak{J}^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_1(\xi)+{}_{c^+}\mathfrak{J}^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}\mathfrak{J}^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi).\end{aligned}$$

This is complete proof of part (a). □

To prove it's other parts (b)-(d), we shall take following inequalities:

(b) $(\mu_2(r)-\gamma(r))(\Upsilon(s)-\eta_1(s))\geqslant 0$;
(c) $(\eta_2(r)-\Upsilon(r))(\mu_2(s)-\gamma(s))\geqslant 0$;
(d) $(\eta_1(r)-\Upsilon(r))(\mu_1(s)-\gamma(s))\geqslant 0$.

Since the proof is similar to the proof the part (a), therefore we omit the details.

Corollary 3.10. *Now we shall establish the inequalities for the geometric $(\mathfrak{l},\mathfrak{m})$-Riemann-Liouville fractional integral, for this we use of $\varphi(\mathfrak{l},\mathfrak{m})=\sqrt{\mathfrak{l}\mathfrak{m}}$ in Theorem 3.9 to get*

$$\begin{aligned}&{}_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi)_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_2(\xi)+{}_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_1(\xi)\\&\quad\geqslant{}_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_2(\xi)_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_1(\xi)+{}_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi), \qquad \text{(a)}\\&{}_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_2(\xi)+{}_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi)_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_1(\xi)\\&\quad\geqslant{}_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_1(\xi)_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_2(\xi)+{}_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi), \qquad \text{(b)}\\&{}_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_2(\xi)_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_2(\xi)+{}_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi)\\&\quad\geqslant{}_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_2(\xi)_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi)+{}_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_2(\xi), \qquad \text{(c)}\\&{}_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_1(\xi)_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_1(\xi)+{}_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi)\\&\quad\geqslant{}_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_1(\xi)_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi)+{}_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}G^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_1(\xi). \qquad \text{(d)}\end{aligned}$$

Corollary 3.11. *The arithmetic $(\mathfrak{l},\mathfrak{m})$-Riemann-Liouville will now be determined. For this we put $\varphi(\mathfrak{l},\mathfrak{m})=\frac{\mathfrak{l}+\mathfrak{m}}{2}$ in Theorem 3.9 and get*

$$\begin{aligned}&{}_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi)_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_2(\xi)+{}_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_1(\xi)\\&\quad\geqslant{}_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_2(\xi)_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_1(\xi)+{}_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi), \qquad \text{(a)}\\&{}_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_2(\xi)+{}_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi)_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_1(\xi)\\&\quad\geqslant{}_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_1(\xi)_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_2(\xi)+{}_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi), \qquad \text{(b)}\\&{}_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_2(\xi)_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_2(\xi)+{}_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi)\\&\quad\geqslant{}_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_2(\xi)_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi)+{}_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_2(\xi), \qquad \text{(c)}\\&{}_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_1(\xi)_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_1(\xi)+{}_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi)\\&\quad\geqslant{}_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_1(\xi)_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi)+{}_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}A^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_1(\xi). \qquad \text{(d)}\end{aligned}$$

Corollary 3.12. *Now we shall establish the result for harmonic $(\mathfrak{l},\mathfrak{m})$-Riemann-Liouville fraction integral, for this we choose $\varphi(\mathfrak{l},\mathfrak{m})=\frac{\mathfrak{l}^2}{\mathfrak{m}}$ in Theorem 3.9 and get*

$$\begin{aligned}&{}_{c^+}H^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi)_{c^+}H^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_2(\xi)+{}_{c^+}H^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}H^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_1(\xi)\\&\quad\geqslant{}_{c^+}H^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\eta_2(\xi)_{c^+}H^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\mu_1(\xi)+{}_{c^+}H^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\Upsilon(\xi)_{c^+}H^{\lambda}_{\varphi(\mathfrak{l},\mathfrak{m})}\gamma(\xi), \qquad \text{(a)}\end{aligned}$$

$$\begin{aligned}&{}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\mu_2(\xi)+{}_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_1(\xi)\\&\quad\geqslant{}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_1(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\mu_2(\xi)+{}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma(\xi),\end{aligned}\tag{b}$$

$$\begin{aligned}&{}_{c^+}H^{\lambda}_{\varphi(l,m)}\mu_2(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_2(\xi)+{}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma(\xi)\\&\quad\geqslant{}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_2(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma(\xi)+{}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\mu_2(\xi),\end{aligned}\tag{c}$$

$$\begin{aligned}&{}_{c^+}H^{\lambda}_{\varphi(l,m)}\mu_1(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_1(\xi)+{}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma(\xi)\\&\quad\geqslant{}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_1(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma(\xi)+{}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\mu_1(\xi).\end{aligned}\tag{d}$$

Lemma 3.13. *Suppose that the assumptions of the Theorem 3.1 hold. Also suppose that there are integrable functions μ_1,μ_2 on $[0,\infty]$ and γ is a positive function on $[0,\infty)$ such that $\mu_1(\xi)\leqslant\gamma(\xi)\leqslant\mu_2(\xi)$ for all $\xi\in(0,\infty)$ satisfying the following:*

$$\begin{aligned}&k(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^2(\xi)-({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))^2\\&\quad=({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_2(\xi)-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_1(\xi))\\&\qquad-k(\xi)({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_2(\xi)-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_1(\xi))\\&\qquad+k(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi).\mu_1(\xi)-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_1(\xi)+k(\ell)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\mu_2(\xi)\\&\qquad-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_2(\xi)-k(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_1(\xi).\mu_2(\xi)+{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_1(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_2(\xi),\end{aligned}$$

where $k(\xi)$ is given from (3.7).

Proof. Since $r,s\geqslant 0$, we write

$$\begin{aligned}&(\eta_2(s)-\Upsilon(s))(\Upsilon(r)-\eta_1(r))+(\eta_2(r)-\Upsilon(r))(\Upsilon(s)-\eta_1(s))\\&\quad-(\eta_2(r)-\Upsilon(r))(\Upsilon(r)-\eta_1(r))-(\eta_2(s)-\Upsilon(s))(\Upsilon(s)-\eta_1(s))\\&=\Upsilon(r)^2+(\Upsilon(s))^2-2\Upsilon(r)\Upsilon(s)+\eta_2(s)\Upsilon(r)+\eta_1(r)\Upsilon(s)-\eta_1(r)\eta_2(s)\\&\quad+\eta_2(s)\Upsilon(s)+\eta_1(s)\Upsilon(r)-\eta_1(s)\eta_2(r)-\Upsilon(r)\eta_2(s)+\eta_1(r)\eta_2(r)\\&\quad-\eta_1(r)\Upsilon(r)-\eta_2(s)\Upsilon(s)+\eta_1(s)\eta_2(s)-\eta_1(s).\Upsilon(s).\end{aligned}\tag{3.11}$$

Multiplying (3.11) by $\frac{1}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi-r)^{\frac{\lambda}{\varphi(l,m)}-1}$ on both sides and integrating from c to ξ with respect to r,

$$\begin{aligned}&(\eta_2(s)-\Upsilon(s))({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi))+({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))(\Upsilon(s)-\eta_1(s))\\&\quad-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\eta_2(\xi)-\Upsilon(\xi))(\Upsilon(\xi)-\eta_1(\xi))-k(\xi)(\eta_2(s)-\Upsilon(s))(\Upsilon(s)-\eta_1(s))\\&={}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)^2+k(\xi)(\Upsilon(s))^2-2{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi).\Upsilon(s)+\eta_2(s)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\\&\quad+{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi)\Upsilon(s)-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi)\eta_2(s)+k(\xi)\eta_2(s)\Upsilon(s)+\eta_1(s)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\\&\quad-\eta_1(s)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\eta_2(s)+{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi)\eta_2(\xi)\\&\quad-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi)\Upsilon(\xi)-k(\xi)\eta_2(s)\Upsilon(s)+k(\xi)\eta_1(s)\eta_2(s)-k(\xi)\eta_1(s)\Upsilon(s).\end{aligned}\tag{3.12}$$

Now multiplying (3.12) with $\frac{1}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi-s)^{\frac{\lambda}{\varphi(l,m)}-1}$ on both sides and integrating from c to ξ with respect to s,

$$\begin{aligned}&({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi))\\&\quad+({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi))\\&\quad-k(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\eta_2(\xi)-\Upsilon(\xi))(\Upsilon(\xi)-\eta_1(\xi))-k(\xi)({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)\end{aligned}$$

$$
\begin{aligned}
&-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi))\\
&= k(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)^2+k(\xi)({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))^2-2{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\\
&\quad+{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)+{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\\
&\quad-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)+k(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\eta_2(\xi)+{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\\
&\quad-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)-{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)\\
&\quad+k(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi)\eta_2(\xi)-k(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi)\Upsilon(\xi)-k(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\eta_2(\xi)\\
&\quad+k(\xi)\eta_1(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)-k(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\eta_1(\xi).
\end{aligned}
$$

This completes proof of Lemma 3.13. □

Corollary 3.14. *Putting $\varphi(l,m)=\sqrt{l.m}$ in Lemma 3.13 we get the result for the Geometric (l,m)-Riemann-Liouville integral, i.e.,*

$$
\begin{aligned}
&({}_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_2(\xi)-{}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))({}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)-{}_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_1(\xi))\\
&\quad+({}_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_2(\xi)-{}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))({}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)-{}_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_1(\xi))\\
&\quad-k(\xi){}_{c^+}G^{\lambda}_{\varphi(l,m)}(\eta_2(\xi)-\Upsilon(\xi))(\Upsilon(\xi)-\eta_1(\xi))-k(\xi)({}_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_2(\xi)\\
&\quad-{}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))({}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)-{}_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_1(\xi))\\
&= k(\xi){}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)^2+k(\xi)({}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))^2-2{}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi){}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\\
&\quad+{}_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_2(\xi){}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)+{}_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_1(\xi){}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)-{}_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_1(\xi){}_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_2(\xi)\\
&\quad+k(\xi){}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\eta_2(\xi)+{}_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_1(\xi){}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\\
&\quad-{}_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_1(\xi){}_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_2(\xi)-{}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi){}_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_2(\xi)\\
&\quad+k(\xi){}_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_1(\xi)\eta_2(\xi)-k(\xi){}_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_1(\xi)\Upsilon(\xi)-k(\xi){}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\eta_2(\xi)\\
&\quad+k(\xi)\eta_1(\xi){}_{c^+}G^{\lambda}_{\varphi(l,m)}\eta_2(\xi)-k(\xi){}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\eta_1(\xi).
\end{aligned}
$$

Corollary 3.15. *Here we shall deduce the result for the arithmetic (l,m)-Riemann-Liouville fractional integral. For this we put $\varphi(l,m)=\frac{l+m}{2}$ in Lemma 3.13 and get*

$$
\begin{aligned}
&({}_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_2(\xi)-{}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))({}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)-{}_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_1(\xi))\\
&\quad+({}_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_2(\xi)-{}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))({}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)-{}_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_1(\xi))\\
&\quad-k(\xi){}_{c^+}A^{\lambda}_{\varphi(l,m)}(\eta_2(\xi)-\Upsilon(\xi))(\Upsilon(\xi)-\eta_1(\xi))-k(\xi)({}_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_2(\xi)\\
&\quad-{}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))({}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)-{}_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_1(\xi))\\
&= k(\xi){}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)^2+k(\xi)({}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))^2-2{}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi){}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\\
&\quad+{}_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_2(\xi){}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)+{}_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_1(\xi){}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\\
&\quad-{}_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_1(\xi){}_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_2(\xi)+k(\xi){}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\eta_2(\xi)+{}_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_1(\xi){}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\\
&\quad-{}_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_1(\xi){}_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_2(\xi)-{}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi){}_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_2(\xi)\\
&\quad+k(\xi){}_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_1(\xi)\eta_2(\xi)-k(\xi){}_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_1(\xi)\Upsilon(\xi)-k(\xi){}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\eta_2(\xi)\\
&\quad+k(\xi)\eta_1(\xi){}_{c^+}A^{\lambda}_{\varphi(l,m)}\eta_2(\xi)-k(\xi){}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\eta_1(\xi).
\end{aligned}
$$

Corollary 3.16. *We choose $\varphi(l,m)=\frac{l^2}{m}$ in Lemma 3.13 to get the result for harmonic (l,m)-Riemann-Liouville fraction integral, i.e.,*

$$
({}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_2(\xi)-{}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))({}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)-{}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_1(\xi))
$$

$$
\begin{aligned}
&+ ({}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_2(\xi) - {}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))({}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi) - {}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_1(\xi))\\
&- k(\xi){}_{c^+}H^{\lambda}_{\varphi(l,m)}(\eta_2(\xi) - \Upsilon(\xi))(\Upsilon(\xi) - \eta_1(\xi)) - k(\xi)({}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_2(\xi)\\
&- {}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))({}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi) - {}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_1(\xi))\\
= {}& k(\xi){}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)^2 + k(\xi)({}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))^2 - 2{}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi){}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\\
&+ {}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_2(\xi){}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi) + {}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_1(\xi){}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\\
&- {}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_1(\xi){}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_2(\xi) + k(\xi){}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\eta_2(\xi) + {}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_1(\xi){}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\\
&- {}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_1(\xi){}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_2(\xi) - {}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi){}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_2(\xi)\\
&+ k(\xi){}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_1(\xi)\eta_2(\xi) - k(\xi){}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_1(\xi)\Upsilon(\xi) - k(\xi){}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\eta_2(\xi)\\
&+ k(\xi)\eta_1(\xi){}_{c^+}H^{\lambda}_{\varphi(l,m)}\eta_2(\xi) - k(\xi){}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\eta_1(\xi).
\end{aligned}
$$

Corollary 3.17. *Suppose that the assumptions of the Corollary 3.5 hold. Also suppose that $t, T \in \Re$ $t < T, k > 0$ and let Υ be a positive function on $[0, \infty)$ such that $t \leqslant \Upsilon(\xi) \leqslant T$ holds, then the following inequality holds*

$$
\begin{aligned}
&k(\xi)({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^2(\xi) - ({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))^2\\
&\quad = (T.k(\xi) - {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi) - t.k(\xi)) - k(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}((T - \Upsilon(\xi))(\Upsilon(\xi) - t)),
\end{aligned}
$$

where $k(\xi)$ is given by (3.7).

Remark 3.18. The results for the geometric, arithmetic and harmonic (l, m)-Riemann-Liouville fraction integral can by obtained by substituting $\varphi(l, m) = \sqrt{l.m}$, $\varphi(l, m) = \frac{l+m}{2}$ and $\varphi(l, m) = \frac{l^2}{m}$ in Corollary 3.17 respectively.

Theorem 3.19. *Let $r, s > 0$, $\Upsilon, \eta_1, \eta_2, \chi, \mu_1$ and μ_2 be six integrable functions on $[0, \infty)$. If conditions $\eta_1(\xi) \leqslant \Upsilon(\xi) \leqslant \eta_2(\xi)$ and $\mu_1(\xi) \leqslant \gamma(\xi) \leqslant \mu_2(\xi)$ are satisfied, then*

$$
\left| k(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\Upsilon(\xi)\gamma(\xi)) - {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi) \right| \leqslant \sqrt{\Lambda(\Upsilon, \eta_1, \eta_2)\Lambda(\gamma, \mu_1, \mu_2)}, \tag{3.13}
$$

where

$$
\begin{aligned}
\Lambda(\Upsilon, \eta_1, \eta_2) = {}& ({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi) - {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi) - {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi))\\
&+ k(\xi)({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\Upsilon(\xi)\eta_1(\xi)) - {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi)\\
&+ k(\xi)({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\Upsilon(\xi)\eta_2(\xi)) - {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)\\
&+ {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi) - k(\xi)({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\eta_1(\xi)\eta_2(\xi)),
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
\Lambda(\gamma, \mu_1, \mu_2) = {}& ({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_2(\xi) - {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi))({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi) - {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_1(\xi))\\
&+ k(\xi)({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\gamma(\xi)\eta_1(\xi)) - {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_1(\xi)\\
&+ k(\xi)({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\gamma(\xi)\mu_2(\xi)) - {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_2(\xi)\\
&+ {}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_2(\xi){}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_1(\xi) - k(\xi)({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\mu_1(\xi)\mu_2(\xi)),
\end{aligned} \tag{3.15}
$$

and $k(\xi)$ is given by (3.7).

Proof. Let $\xi \geqslant 0$, $r, s \in [0, \xi]$, and Υ and γ be two positive functions on $[0, \infty)$ such that $\eta_1(\xi) \leqslant \Upsilon(\xi) \leqslant \eta_2(\xi)$ and $\mu_1(\xi) \leqslant \gamma(\xi) \leqslant \mu_2(\xi)$ are satisfied, and let $\Lambda(r, s)$ be defined by

$$
\Lambda(r, s) = (\Upsilon(r) - \Upsilon(s))\,(\gamma(r) - \gamma(s))\,. \tag{3.16}
$$

Multiplying (3.16) by $\frac{(\xi-s)^{\frac{\lambda}{\varphi(l,m)}-1}}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}$ and $\frac{(\xi-r)^{\frac{\lambda}{\varphi(l,m)}-1}}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}$ on both sides and integrating from c to ξ with respect to s and r we obtain

$$\begin{aligned}&\frac{1}{2}\int_c^\xi\int_c^\xi \frac{(\xi-s)^{\frac{\lambda}{\varphi(l,m)}-1}}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}\frac{(\xi-r)^{\frac{\lambda}{\varphi(l,m)}-1}}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}\Lambda(r,s)ds\,dr\\&\quad=k(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\gamma(\xi)-_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi).\end{aligned}\tag{3.17}$$

Applying the Cauchy-Schwarz inequality we get

$$\begin{aligned}&\left(\frac{1}{2}\left[\int_c^\xi\int_c^\xi \frac{(\xi-s)^{\frac{\lambda}{\varphi(l,m)}-1}}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}\frac{(\xi-r)^{\frac{\lambda}{\varphi(l,m)}-1}}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\Upsilon(r)-\Upsilon(s))(\gamma(r)-\gamma(s))dsdr\right]\right)^2\\&\leqslant\int_c^\xi\int_c^\xi \frac{(\xi-s)^{\frac{\lambda}{\varphi(l,m)}-1}}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}\frac{(\xi-r)^{\frac{\lambda}{\varphi(l,m)}-1}}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\Upsilon(r)-\Upsilon(s))^2ds\,dr\\&\quad\times\int_c^\xi\int_c^\xi \frac{(\xi-s)^{\frac{\lambda}{\varphi(l,m)}-1}}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}\frac{(\xi-r)^{\frac{s}{\varphi(l,m)}-1}}{(l\Gamma_{(l,m)})\left(\frac{ls}{\varphi(l,m)}\right)}(\gamma(r)-\gamma(s))^2dsdr.\end{aligned}\tag{3.18}$$

From (3.17) and (3.18), we get

$$\begin{aligned}&(k(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\gamma(\xi).\Upsilon(\xi))-\gamma(\xi).\Upsilon(\xi)))^2\\&\quad\leqslant(k(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^2(\xi))-(_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\Upsilon(\xi))^2(k(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^2(\xi)-(_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\gamma(\xi)))^2.\end{aligned}$$

Since $(\eta_2(\xi)-\Upsilon(\xi))(\Upsilon(\xi)-\eta_1(\xi))\geqslant 0$ and $(\mu_2(\xi)-\gamma(\xi))(\gamma(\xi)-\mu_1(\xi))\geqslant 0$, we have

$$k(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}((\eta_2(\xi)-\Upsilon(\xi))(\Upsilon(\xi)-\eta_1(\xi)))\geqslant 0,$$

and

$$k(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}((\mu_2(\xi)-\gamma(\xi))(\gamma(\xi)-\mu_1(\xi)))\geqslant 0.$$

Thus from Lemma 3.13 we have

$$\begin{aligned}&k(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^2(\xi)-(_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))^2\\&\leqslant(_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)-_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))(_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)-_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi))\\&\quad+k(\xi)(_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\Upsilon(\xi)\eta_1(\xi))-_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi)+k(\xi)(_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\Upsilon(\xi)\eta_2(\xi))\\&\quad-_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)+_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_2(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\eta_1(\xi)\\&\quad-k(\xi)(_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\eta_1(\xi)\eta_2(\xi))=\Lambda(\Upsilon,\eta_1,\eta_2),\end{aligned}\tag{3.19}$$

and

$$\begin{aligned}&k(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^2(\xi)-(_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi))^2\\&=(_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_2(\xi)-_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi))(_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi)-_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_1(\xi))\\&\quad+k(\xi)(_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\gamma(\xi)\eta_1(\xi))-_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_1(\xi)+k(\xi)(_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\gamma(\xi)\mu_2(\xi))\\&\quad-_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_2(\xi)+_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_2(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\mu_1(\xi)\\&\quad-k(\xi)(_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\mu_1(\xi)\mu_2(\xi))=\Lambda(\gamma,\mu_1,\mu_2).\end{aligned}\tag{3.20}$$

Therefore the inequality (3.13) follows from the inequalities (3.19) and (3.20). □

Corollary 3.20. *Result for the geometric* (l, m)*-Riemann-Liouville fractional integral can be obtained if we choose* $\varphi(l, m) = \sqrt{l.m}$ *in Theorem 3.19 and the inequality becomes*

$$\left|k(\xi)_{c^+}G^{\lambda}_{\varphi(l,m)}(\Upsilon(\xi)\gamma(\xi)) -_{c^+} G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma(\xi)\right| \leqslant \sqrt{\Lambda(\Upsilon, \eta_1, \eta_2)\Lambda(\gamma, \mu_1, \mu_2)},$$

where $k(\xi)$*,* $\Lambda(\Upsilon, \eta_1, \eta_2)$*, and* $\Lambda(\gamma, \mu_1, \mu_2)$ *are defined by* (3.7)*,* (3.14)*, and* (3.15)*, respectively.*

Corollary 3.21. *Now the inequality for the the arithmetic* (l, m)*-Riemann-Liouville will now be determined. For this we put* $\varphi(l, m) = \frac{l+m}{2}$ *in Theorem 3.19 and we get*

$$|k(\xi)_{c^+}A^{\lambda}_{\varphi(l,m)}(\Upsilon(\xi)\gamma(\xi)) -_{c^+} A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma(\xi)| \leqslant \sqrt{(\Lambda(\Upsilon, \eta_1, \eta_2)(\Lambda(\gamma, \mu_1, \mu_2))}$$

where $k(\xi)$*,* $\Lambda(\Upsilon, \eta_1, \eta_2)$*, and* $\Lambda(\gamma, \mu_1, \mu_2)$ *are defined by* (3.7)*,* (3.14)*, and* (3.15)*, respectively.*

Corollary 3.22. *Now we shall establish the result for harmonic* (l, m)*-Riemann-Liouville fraction integral, for this we choose* $\varphi(l, m) = \frac{l^2}{m}$ *in Theorem 3.19 and get*

$$|k(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}(\Upsilon(\xi)\gamma(\xi)) -_{c^+} H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma(\xi)| \leqslant \sqrt{(\Lambda(\Upsilon, \eta_1, \eta_2)(\Lambda(\gamma, \mu_1, \mu_2))}$$

where $k(\xi)$*,* $\Lambda(\Upsilon, \eta_1, \eta_2)$*, and* $\Lambda(\gamma, \mu_1, \mu_2)$ *are defined by* (3.7)*,* (3.14)*, and* (3.15)*, respectively.*

Example 3.23. Let $t, T, f, F \in \mathfrak{R}$ and $\Lambda(\gamma, \mu_1, \mu_2) = \Lambda(\gamma, f, F)$ and $\Lambda(\Upsilon, \eta_1, \eta_2) = \Lambda(\Upsilon, t, T)$ in (3.13), we get

$$|k(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}(\Upsilon(\xi)\gamma(\xi)) -_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi)| \leqslant (k(\xi))^2(T-t)(F-f).$$

Theorem 3.24. *Let* $p, q \geqslant 0$*, and* $a, b > 1$*,* $\frac{1}{a} + \frac{1}{b} = 1$*. Let* Υ *and* γ *be positive functions defined on* $[0, \infty)$*, then the following inequalities holds:*

$$b_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^a(\xi) + a_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^b(\xi) \geqslant ab\frac{1}{k(\xi)}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi), \tag{1}$$

$$\begin{aligned} &b_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^a(\xi)_{s^-}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^b(\xi) + a_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^b(\xi)_{s^-}\mathfrak{I}\Upsilon^b(\xi) \\ &\quad \geqslant ab({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{s^-}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi))_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma(\xi)_{s^-}\mathfrak{I}\Upsilon(\xi), \end{aligned} \tag{2}$$

$$\begin{aligned} &b_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^a(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^b(\xi) + a_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^b(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^a(\xi) \\ &\quad \geqslant_{c^+} \mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^{a-1}(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^{b-1}(\xi), \end{aligned} \tag{3}$$

$$\begin{aligned} &{}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^a(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^b(\xi) + a_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^a(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^b(\xi) \\ &\quad \geqslant ab({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^{a-1}(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^{b-1}(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\gamma(\xi)), \end{aligned} \tag{4}$$

$$\begin{aligned} &b_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^a(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^2(\xi) + a_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^2(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^b(\xi) \\ &\quad \geqslant ab({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^{\frac{2}{b}}(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^{\frac{2}{a}}(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\gamma(\xi)), \end{aligned} \tag{5}$$

$$\begin{aligned} &b_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^2(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^b(\xi) + a_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^a(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^2(\xi) \\ &\quad \geqslant ab({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^{\frac{2}{a}}(\xi)\gamma^{\frac{2}{b}}(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^{a-1}(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^{b-1}(\xi)), \end{aligned} \tag{6}$$

$$\begin{aligned} &b_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^2(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^b(\xi) + ak(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^2(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^b(\xi) \\ &\quad \geqslant ab({}_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^{\frac{2}{a}}(\xi)\gamma^{b-1}(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\Upsilon^{\frac{2}{b}}(\xi)_{c^+}\mathfrak{I}^{\lambda}_{\varphi(l,m)}\gamma^{a-1}(\xi)), \end{aligned} \tag{7}$$

where $k(\xi)$ *is given by* (3.7)*.*

Proof. By Young's inequality, we have

$$\frac{p^a}{a} + \frac{q^b}{b} \geqslant pq, \quad (p, q \geqslant 0, a, b > 1, \frac{1}{a} + \frac{1}{b} = 1).$$

Taking $p = \Upsilon(r)$ and $q = \gamma(s)$ in above Young's inequality,

$$\frac{\Upsilon(r)^a}{a} + \frac{\gamma(s)^b}{b} \geqslant \Upsilon(r).\gamma(s), \text{ for all } \Upsilon(r), \gamma(s) \geqslant 0. \tag{3.21}$$

Multiplying (3.21) by $\frac{1}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - r)^{\frac{\lambda}{\varphi(l,m)}-1}$ on both sides and integrating from c to ξ with respect to r,

$$\begin{aligned}&\int_c^\xi \frac{1}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - r)^{\frac{\lambda}{\varphi(l,m)}-1}\frac{\Upsilon(r)^a}{a}dr + \int_c^\xi \frac{1}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - r)^{\frac{\lambda}{\varphi(l,m)}-1}\frac{\gamma(s)^b}{b}dr\\ &\quad\leqslant \int_c^\xi \frac{1}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - r)^{\frac{\lambda}{\varphi(l,m)}-1}\Upsilon(r)\gamma(s)dr,\end{aligned}$$

which becomes

$$\frac{1}{a}{}_{c^+}\mathfrak{I}^\lambda_{\varphi(l,m)}\Upsilon^a(\xi) + \frac{1}{b}k(\xi)\gamma^b(s) \geqslant {}_{c^+}\mathfrak{I}^\lambda_{\varphi(l,m)}\Upsilon(\xi)\gamma(s). \tag{3.22}$$

Now multiplying (3.22) by $\frac{1}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - s)^{\frac{\lambda}{\varphi(l,m)}-1}$ on both sides and integrating from c to ξ with respect to s,

$$\begin{aligned}&\int_c^\xi \frac{1}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - s)^{\frac{\lambda}{\varphi(l,m)}-1}\frac{1}{a}{}_{c^+}\mathfrak{I}^\lambda_{\varphi(l,m)}\Upsilon^a(\xi)ds + \int_c^\xi \frac{1}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - s)^{\frac{\lambda}{\varphi(l,m)}-1}\frac{1}{b}k(\xi)\gamma^b(s)ds\\ &\quad\leqslant \int_c^\xi \frac{1}{(l\Gamma_{(l,m)})\left(\frac{l\lambda}{\varphi(l,m)}\right)}(\xi - s)^{\frac{\lambda}{\varphi(l,m)}-1}_{c^+}\mathfrak{I}^\lambda_{\varphi(l,m)}\Upsilon(\xi)\gamma(s)ds.\end{aligned}$$

After some necessary settings we can write

$$k(\xi)\frac{1}{a}{}_{c^+}\mathfrak{I}^\lambda_{\varphi(l,m)}\Upsilon^a(\xi) + \frac{1}{b}k(\xi)_{c^+}\mathfrak{I}^\lambda_{\varphi(l,m)}\gamma^b(\xi) \geqslant {}_{c^+}\mathfrak{I}^\lambda_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^\lambda_{\varphi(l,m)}\gamma(\xi),$$

which becomes as

$$b_{c^+}\mathfrak{I}^\lambda_{\varphi(l,m)}\Upsilon^a(\xi) + a_{c^+}\mathfrak{I}^\lambda_{\varphi(l,m)}\gamma^b(\xi) \geqslant ab\frac{1}{k(\xi)}{}_{c^+}\mathfrak{I}^\lambda_{\varphi(l,m)}\Upsilon(\xi)_{c^+}\mathfrak{I}^\lambda_{\varphi(l,m)}\gamma(\xi).$$

This complete proof of (1). The remaining parts can be proved using Young's inequality in a same pattern as follows.

(2) $p = \Upsilon(r)\gamma(s)$ and $q = \Upsilon(s)\gamma(r)$.
(3) $p = \frac{\Upsilon(r)}{\gamma(r)}$ and $q = \frac{\Upsilon(s)}{\gamma(s)}$, provided that $\gamma(r), \gamma(s) \neq 0$.
(4) $p = \frac{\Upsilon(s)}{\Upsilon(r)}$ and $q = \frac{\gamma(s)}{\gamma(r)}$, provided that $\Upsilon(r).\gamma(s) \neq 0$.
(5) $p = \Upsilon(r)\gamma^{\frac{2}{a}}(s)$ and $q = \Upsilon^{\frac{2}{b}}(s)\gamma(r)$.
(6) $p = \frac{\Upsilon^{\frac{2}{a}}(r)}{\Upsilon(s)}$ and $q = \frac{\gamma^{\frac{2}{b}}(r)}{\gamma(s)}$, provided that $\Upsilon(s), \gamma(s) \neq 0$.
(7) $p = \frac{\Upsilon^{\frac{2}{a}}(r)}{\gamma(s)}$ and $q = \frac{\Upsilon^{\frac{2}{b}}(s)}{\gamma(r)}$.

□

Corollary 3.25. *Now we shall establish the result for geometric* (l, m)*-Riemann-Liouville fractional integral, for this we choose* $\varphi(l, m) = \sqrt{l.m}$ *and the Theorem 3.24 reduces to*

$$b\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi) + a\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi) \geqslant ab\frac{1}{k(\xi)}\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma(\xi), \tag{1}$$

$$\begin{aligned} &b\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)\, {}_{s^-}G^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi) + a\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi)\, {}_{s^-}G^{\lambda}_{\varphi(l,m)}\Upsilon^{b}(\xi)\\ &\quad \geqslant ab({}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\, {}_{s^-}G^{\lambda}_{\varphi(l,m)}\gamma(\xi))\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma(\xi)\, {}_{s^-}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi), \end{aligned} \tag{2}$$

$$\begin{aligned} &b\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi) + a\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{b}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{a}(\xi)\\ &\quad \geqslant {}_{c^+}\, G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{a-1}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{b-1}(\xi), \end{aligned} \tag{3}$$

$$\begin{aligned} &{}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi) + a\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi)\\ &\quad \geqslant ab({}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{a-1}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{b-1}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\gamma(\xi)), \end{aligned} \tag{4}$$

$$\begin{aligned} &b\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{2}(\xi) + a\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{2}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi)\\ &\quad \geqslant ab({}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{\frac{2}{b}}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{\frac{2}{a}}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\gamma(\xi)), \end{aligned} \tag{5}$$

$$\begin{aligned} &b\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{2}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi) + a\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{2}(\xi)\\ &\quad \geqslant ab({}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{\frac{2}{a}}(\xi)\gamma^{\frac{2}{b}}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{a-1}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{b-1}(\xi)), \end{aligned} \tag{6}$$

$$\begin{aligned} &b\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{2}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi) + ak(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{2}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi)\\ &\quad \geqslant ab({}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{\frac{2}{a}}(\xi)\gamma^{b-1}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\Upsilon^{\frac{2}{b}}(\xi)\, {}_{c^+}G^{\lambda}_{\varphi(l,m)}\gamma^{a-1}(\xi)). \end{aligned} \tag{7}$$

Corollary 3.26. *Now we shall establish the result for arithmetic* (l, m)*-Riemann-Liouville fractional integral, for this we choose* $\varphi(l, m) = \frac{l+m}{2}$ *in Theorem 3.24 and we get*

$$b\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi) + a\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi) \geqslant ab\frac{1}{k(\xi)}\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma(\xi), \tag{1}$$

$$\begin{aligned} &b\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)\, {}_{s^-}A\gamma^{b}(\xi) + a\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi)\, {}_{s^-}A\Upsilon^{b}(\xi)\\ &\quad \geqslant ab({}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\, {}_{s^-}A\gamma(\xi))\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma(\xi)\, {}_{s^-}A\Upsilon(\xi), \end{aligned} \tag{2}$$

$$\begin{aligned} &b\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi) + a\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{b}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{a}(\xi)\\ &\quad \geqslant {}_{c^+}\, A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{a-1}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{b-1}(\xi), \end{aligned} \tag{3}$$

$$\begin{aligned} &{}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi) + a\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi)\\ &\quad \geqslant ab({}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{a-1}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{b-1}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\gamma(\xi)), \end{aligned} \tag{4}$$

$$\begin{aligned} &b\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{2}(\xi) + a\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{2}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi)\\ &\quad \geqslant ab({}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{\frac{2}{b}}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{\frac{2}{a}}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\gamma(\xi)), \end{aligned} \tag{5}$$

$$\begin{aligned} &b\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{2}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi) + a\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{2}(\xi)\\ &\quad \geqslant ab({}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{\frac{2}{a}}(\xi)\gamma^{\frac{2}{b}}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{a-1}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{b-1}(\xi)), \end{aligned} \tag{6}$$

$$\begin{aligned} &b\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{2}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi) + ak(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{2}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi)\\ &\quad \geqslant ab({}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{\frac{2}{a}}(\xi)\gamma^{b-1}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\Upsilon^{\frac{2}{b}}(\xi)\, {}_{c^+}A^{\lambda}_{\varphi(l,m)}\gamma^{a-1}(\xi)), \end{aligned} \tag{7}$$

Corollary 3.27. *For the result for harmonic* (l, m)*-Riemann-Liouville fractional integral, we replace* $\varphi(l, m) = \frac{l^2}{m}$ *in Theorem 3.24 and we obtain*

$$b\, {}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi) + a\, {}_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi) \geqslant ab\frac{1}{k(\xi)}\, {}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\, {}_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma(\xi), \tag{1}$$

$$\begin{aligned} &b_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)_{s^-}H\gamma^{b}(\xi)+a_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi)_{s^-}H\Upsilon^{b}(\xi)\\ &\quad\geqslant ab({}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{s^-}H\gamma(\xi))_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma(\xi)_{s^-}H\Upsilon(\xi), \end{aligned} \tag{2}$$

$$\begin{aligned} &b_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi)+a_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{b}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{a}(\xi)\\ &\quad\geqslant_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{a-1}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{b-1}(\xi), \end{aligned} \tag{3}$$

$$\begin{aligned} &{}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi)+a_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi)\\ &\quad\geqslant ab({}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{a-1}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{b-1}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\gamma(\xi)), \end{aligned} \tag{4}$$

$$\begin{aligned} &b_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{2}(\xi)+a_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{2}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi)\\ &\quad\geqslant ab({}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{\frac{2}{b}}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{\frac{2}{a}}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon(\xi)\gamma(\xi)), \end{aligned} \tag{5}$$

$$\begin{aligned} &b_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{2}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi)+a_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{a}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{2}(\xi)\\ &\quad\geqslant ab({}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{\frac{2}{a}}(\xi)\gamma^{\frac{2}{b}}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{a-1}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{b-1}(\xi)), \end{aligned} \tag{6}$$

$$\begin{aligned} &b_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{2}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi)+ak(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{2}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{b}(\xi)\\ &\quad\geqslant ab({}_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{\frac{2}{a}}(\xi)\gamma^{b-1}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\Upsilon^{\frac{2}{b}}(\xi)_{c^+}H^{\lambda}_{\varphi(l,m)}\gamma^{a-1}(\xi)). \end{aligned} \tag{7}$$

4. Conclusion

In this paper we proved the wide range of Grüs type integral inequalities by using (l, m)-Riemann-Liouville fractional integrals. We deduce the inequalities for the geometric, arithmetic and harmonic (l, m)-Riemann-Liouville fractional integrals as special cases of our general result. These estimates can be used for further developments in applied and pure mathematics. Moreover such results can be extended and develop for different spaces and more generalized fractional integrals as well.

Acknowledgment

All authors are deeply grateful to the referees for their valuable suggestions, which significantly improved the final version of this paper. Professor Miguel Vivas-Cortez acknowledge the funding provided by Pontificia Universidad Católica del Ecuador, Project UIO-077-2024: La derivada fraccionaria generalizada, nuevos resultados y aplicaciones a desigualdades integrales.

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