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Euler polynomials coefficient estimates for bi-univalent functions defined by subordinations

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Abstract

Our research endeavors focus on the introduction of novel subclasses of analytic functions, which are defined in terms of Euler polynomials. The primary objective of our investigation is to estimate the Fekete-Szego functional problem and determine the Maclaurin coefficients, specifically $|c_2|$ and $|c_3|$, for this particular subfamily. Furthermore, we will present a number of innovative findings that arise when we specialize the parameters employed in our core discoveries.

Keywords: Analytic functions, Euler polynomials, convolution, bi-univalent functions, Fekete-Szegö problem.

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1. preliminaries

Originating from Leonhard Euler's research in the eighteenth century, Euler polynomials (EP) are crucial in comprehending complex functions and their geometric properties. EP play a pivotal role in the characterization of conformal mappings that locally preserve angles within the context of geometric function theory (GFT). Moreover, these polynomials find extensive application across various domains of GFT, this includes the study of Schwarz-Christoffel mappings, univalent functions, and the theory of Riemann surfaces.

As a result of the widespread use of EP in pure mathematics, many researchers have begun exploring different areas. Currently, The field of GFT primarily focuses on exploring the geometric properties of

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specialized functions and their corresponding counterparts. To delve deeper into the geometric characteristics of these functions, we recommend consulting the sources listed in references [12, 25], along with other relevant scholarly literature on the subject.

Let $\mathcal{F}$ represent the set of analytic functions d that are defined in the open unit disk $\Lambda = \{\nu \in \mathbb{C} : |\nu| < 1\}$ such that $d(0) = 0$ and $d'(0) - 1 = 0$. Therefore, each $d \in \mathcal{F}$ has a Maclaurin series in the following form:

$$d(\nu) = \nu + \sum_{\iota=2}^{\infty} c_\iota \nu^\iota, \quad (\nu \in \Lambda). \tag{1.1}$$

We also define Ψ as the set of functions that are univalent in Λ.

Every mathematical function $d \in \Psi$ has an inverse, denoted as d^{-1}, which is defined by:

$$d^{-1}(d(\nu)) = \nu \text{ and } w = d(d^{-1}(w)) \qquad (\nu \in \Lambda,\ |w| < r_0(d);\ r_0(d) \geqslant \frac{1}{4}),$$

where

$$d^{-1}(w) = q(w) = w - c_2 w^2 + (2c_2^2 - c_3)w^3 - (c_4 + 5, c_2^3 - 5c_3c_2)w^4 + \cdots. \tag{1.2}$$

A function d is considered bi-univalent in Λ if both d and d^{-1} are univalent in Λ. Let Π represent the class of all bi-univalent functions in Λ as defined in (1.1).

In the class Π, the example is $d(\nu) = \frac{\nu}{1-\nu}$. However, it should be noted that $d(\nu) = \frac{\nu}{1-\nu^2}$ is not a member of Π. For more information on interesting function classes within Π, please refer to [2].

Miller and Mocanu [18] who were the first to introduce the concept of differential subordination. For more information, refer to [19] and [20]. In this context, we say that the function d is subordinate to q, denoted as $d \prec q$, if both d and q are analytic in Λ and there exists a function $w \in \mathcal{F}$ in Λ such that

$$w(0) = 0 \text{ and } |w(\nu)| < 1, \qquad (\nu \in \Omega)$$

such that

$$d(\nu) = q(w(\nu)).$$

Also, if q is univalent in Λ, then

$$d(\nu) \prec q(\nu) \quad \text{if and only if} \quad d(0) = q(0) \text{ and } d(\Lambda) \subset q(\Lambda).$$

GFT makes effective use of EP, which are essential tools in the field of mathematical analysis. These polynomials hold particular significance in the study of conformal mappings and complex analysis.

The EP are commonly defined using the generating function $\Theta_\iota(\kappa)$ (see, [17, 24]):

$$\mathcal{B}(\kappa, h) = \frac{2e^{h\kappa}}{e^h + 1} = \sum_{\iota=0}^{\infty} \Theta_\iota(\kappa) \frac{h^\iota}{\iota!}, \qquad \left(\frac{1}{2} < \kappa \leqslant 1,\ |h| < \pi\right).$$

An explicit formula for $\Theta_j(\kappa)$ is given by

$$\Theta_j(\kappa) = \sum_{\iota=0}^{j} \frac{1}{2^\iota} \sum_{u=0}^{\iota} (-1)^u \binom{\iota}{u} (\kappa + u)^j.$$

Now, $\Theta_\iota(\kappa)$ can be expressed in terms of Θ_u using the equation above:

$$\Theta_\iota(\kappa) = \sum_{u=0}^{\iota} \frac{\Theta_u}{2^u} \binom{\iota}{u} (\kappa - \frac{1}{2})^{\iota - u}.$$

The initial values of EP are:

$$\begin{aligned} \Theta_0(\kappa) &= 1; \\ \Theta_1(\kappa) &= \frac{2\kappa-1}{2}; \\ \Theta_2(\kappa) &= \kappa^2-\kappa; \\ \Theta_3(\kappa) &= \frac{4\kappa^3-6\kappa^2+1}{4}; \\ \Theta_4(\kappa) &= \kappa^4-2\kappa^3+\kappa. \end{aligned} \tag{1.3}$$

In recent years, numerous studies have been conducted on GFT, specifically focusing on coefficient estimates. Within this field, various subclasses of the class Π have been introduced. These studies have also obtained non-sharp estimates on the coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1), as discussed in references ([1, 3–11, 14–16, 21, 22, 26–28]).

The primary objective of this study is to explore the properties of bi-univalent functions related to the open unit disk. The subsequent section will provide definitions of the class under investigation, along with examples to support this exploration. In Section 3, we will derive coefficient estimates for the new class, while Section 3 will focus on evaluating the Fekete-Szego functional. Additionally, Section 4 will present corollaries that correspond to the examples provided, derived from the theorems established in the preceding sections.

2. Definition and Examples

In this section, we start by introducing a definition for the new subclasses $\mathcal{F}_{\Pi}^{\mu}(\alpha,\varphi,\kappa)$ that is connected to Euler polynomials.

Definition 2.1. Provided that the following subordination conditions are satisfied for a function $d \in \Lambda$ given by Equation (1.1), then $d \in \mathcal{F}_{\Pi}^{\mu}(\alpha,\varphi,\kappa)$:

$$\mu\left[\begin{array}{c}\left(\frac{d(\nu)}{\nu}\right)^{\alpha} \\ +\frac{1+e^{i\varphi}}{2}\left(\frac{\nu d''(\nu)}{d'(\nu)}\right)\end{array}\right]+(1-\mu)\left[\begin{array}{c}(d'(\nu))^{\alpha} \\ +\frac{1+e^{i\varphi}}{2}(\nu d''(\nu))\end{array}\right] \prec \mathcal{B}(\kappa,\nu)=\sum_{\iota=0}^{\infty}\Theta_{\iota}(\kappa)\frac{\nu^{\iota}}{\iota!} \tag{2.1}$$

and

$$\mu\left[\begin{array}{c}\left(\frac{q(w)}{w}\right)^{\alpha}+ \\ \frac{1+e^{i\varphi}}{2}\left(\frac{wq''(w)}{q(w)}\right)\end{array}\right]+(1-\mu)\left[\begin{array}{c}(q'(w))^{\alpha} \\ +\frac{1+e^{i\varphi}}{2}(wq''(w))\end{array}\right] \prec \mathcal{B}(\kappa,w)=\sum_{\iota=0}^{\infty}\Theta_{\iota}(\kappa)\frac{w^{\iota}}{\iota!}, \tag{2.2}$$

where $0 \leqslant \mu \leqslant 1$, $\frac{1}{2} < \kappa \leqslant 1$, $-\pi < \varphi \leqslant \pi$, $\alpha \geqslant 1$, $\nu, w \in \Lambda$ and $q = b^{-1}$.

Remark 2.2. Many subclasses can be found by taking special values for the parameters μ, α and κ in Definition 2.1.

This example corresponds to the special case where $\mu = 0$. Setting $\mu = 0$ simplifies the general form of Definition 2.1. This reduction allows us to study the behavior of the definition in this specific case.

Example 2.3. *Provided that the following subordination conditions are satisfied for a function* $d \in \Lambda$ *given by Equation* (1.1), *then* d *belongs to the set* $\mathcal{F}_{\Pi}^{0}(\alpha,\varphi,\kappa)$:

$$(d'(\nu))^{\alpha}+\frac{1+e^{i\varphi}}{2}(\nu d''(\nu)) \prec \mathcal{B}(\kappa,\nu)=\sum_{\iota=0}^{\infty}\Theta_{\iota}(\kappa)\frac{\nu^{\iota}}{\iota!}$$

and

$$(q'(w))^{\alpha}+\frac{1+e^{i\varphi}}{2}(wq''(w)) \prec \mathcal{B}(\kappa,w)=\sum_{\iota=0}^{\infty}\Theta_{\iota}(\kappa)\frac{w^{\iota}}{\iota!},$$

where $\frac{1}{2} < \kappa \leqslant 1$ $\nu, w \in \Lambda$ *and* $d = b^{-1}$.

Here, $\mu = 1$ is considered. By substituting $\mu = 1$ into the general form of Definition 2.1. This showcases how the definition adapts when μ takes on a different value.

Example 2.4. *Provided that the specified subordination conditions are satisfied for a function* $d \in \Lambda$ *given by Equation* (1.1)*, then* d *belongs to the set* $\mathcal{F}^1_\Pi(\alpha, \varphi, \kappa)$*:*

$$\left(\frac{d(\nu)}{\nu}\right)^\alpha + \frac{1+e^{i\varphi}}{2}\left(\frac{\nu d''(\nu)}{d'(\nu)}\right) \prec \mathcal{B}(\kappa, \nu) = \sum_{\iota=0}^{\infty} \Theta_\iota(\kappa)\frac{\nu^\iota}{\iota!}$$

and

$$\left(\frac{q(w)}{w}\right)^\alpha + \frac{1+e^{i\varphi}}{2}\left(\frac{wq''(w)}{q(w)}\right) \prec \mathcal{B}(\kappa, w) = \sum_{\iota=0}^{\infty} \Theta_\iota(\kappa)\frac{w^\iota}{\iota!},$$

where$\frac{1}{2} < \kappa \leqslant 1$ $\nu, w \in \Lambda$*and* $d = b^{-1}$.

In this following example , we set $\mu = 1$, $\alpha = 1$, and $\varphi = 0$. This combination of values provides insight into the definition when both parameters are constrained in a specific way, further simplifying the structure of Definition 2.1.

Example 2.5. *Provided that the specified subordination conditions are satisfied for a function* $d \in \Lambda$ *given by Equation* (1.1)*, then* b *belongs to the set* $\mathcal{F}^1_\Pi(1, 0, \kappa)$*:*

$$\frac{d(\nu)}{\nu} + \frac{\nu d''(\nu)}{d'(\nu)} \prec \mathcal{B}(\kappa, \nu) = \sum_{\iota=0}^{\infty} \Theta_\iota(\kappa)\frac{\nu^\iota}{\iota!}$$

and

$$\frac{q(w)}{w} + \frac{wq''(w)}{q(w)} \prec \mathcal{B}(\kappa, w) = \sum_{\iota=0}^{\infty} \Theta_\iota(\kappa)\frac{w^\iota}{\iota!},$$

where$\frac{1}{2} < \kappa \leqslant 1$ $\nu, w \in \Lambda$*and* $d = b^{-1}$.

Finally, $\mu = 0$, $\alpha = 1$, and $\varphi = 0$ are substituted. This represents a highly specific scenario where both parameters are minimized or neutralized. The resulting form allows us to analyze the simplest or baseline case derived from Definition 2.1, which might serve as a foundation for comparison with more complex cases.

Example 2.6. *Provided that the specified subordination conditions are satisfied for a function* $d \in \Lambda$ *given by Equation* (1.1)*, then* d *belongs to the set* $\mathcal{F}^0_\Pi(1, 0, \kappa)$*:*

$$d'(\nu) + \nu d''(\nu) \prec \mathcal{B}(\kappa, \nu) = \sum_{\iota=0}^{\infty} \Theta_\iota(\kappa)\frac{\nu^\iota}{\iota!}$$

and

$$q'(w) + wq''(w) \prec \mathcal{B}(\kappa, w) = \sum_{\iota=0}^{\infty} \Theta_\iota(\kappa)\frac{w^\iota}{\iota!},$$

where $\frac{1}{2} < \kappa \leqslant 1$ $\nu, w \in \Lambda$*and* $d = b^{-1}$.

lemma 2.7 ([23])**.** *If* p *is an element of* $\mathcal{D}$*, then for every* n*,* $|m_n|$ *is less than or equal to 2. Here,* $\mathcal{D}$ *represents the collection of all analytic functions in* Λ *that satisfy Re* $(p(\nu)) > 0$*, where* $p(\nu) = 1 + m_1\nu + m_2^2\nu + \cdots \quad (\nu \in \Lambda)$*.*

3. Bounds of the class $\mathcal{F}_{\Pi}^{\mu}(\alpha,\varphi,\kappa)$

For a function $d \in \Lambda$, we solve the Fekete-Szegö problem (see [13]) for the class $\mathcal{F}_{\Pi}^{\mu}(\alpha,\varphi,\kappa)$, giving the coefficient estimates.

Theorem 3.1. *Let* $d \in \Pi$ *given by* (1.1) *belongs to the class* $\mathcal{F}_{\Pi}^{\mu}(\alpha,\varphi,\kappa)$ *where* $0 \leqslant \mu \leqslant 1$, $\frac{1}{2} < \kappa \leqslant 1$, $-\pi < \varphi \leqslant \pi$, $\alpha \geqslant 1$, $\nu, w \in \Lambda$ *and* $d = b^{-1}$. *Then*

$$|c_2| \leqslant \sqrt{\Upsilon(\mu,\kappa)},$$

$$|c_3| \leqslant \frac{\left|2\alpha\left(\alpha\left(2-\mu\right)+1\right)+\mu\alpha\left(3-\alpha\right)+2\left(e^{i\varphi}+1\right)\left(3-2\mu\right)\right|\left(2\kappa-1\right)^2}{2\left|3e^{i\varphi}+\alpha\left(3-2\mu\right)+3\right|\left(e^{i\varphi}+\alpha\left(2-\mu\right)+1\right)^2}+\left|\frac{\left(2\kappa-1\right)}{2(3e^{i\varphi}+\alpha\left(3-2\mu\right)+3)}\right|.$$

and

$$\left|c_3-\varkappa c_2^2\right| \leqslant \begin{cases} \frac{2\kappa-1}{(3e^{i\varphi}+\alpha(3-2\mu)+3)} & 0 \leqslant \mathcal{F}^{\kappa}(\alpha,\mu) < \frac{2\kappa-1}{2(3e^{i\varphi}+\alpha(3-2\mu)+3)}, \\ 2\mathcal{F}^{\kappa}(\alpha,\mu) & \mathcal{F}^{\kappa}(\alpha,\mu) \geqslant \frac{2\kappa-1}{2(3e^{i\varphi}+\alpha(3-2\mu)+3)}. \end{cases}$$

where

$$\Upsilon(\mu,\kappa) = \frac{\left(2\kappa-1\right)^3}{\left|\left[\begin{array}{c} 2\left(3\left(e^{i\varphi}+1\right)+\alpha\left(3-2\mu\right)\right)\left(\frac{2\kappa-1}{2}\right)^2 \\ -\left(e^{i\varphi}+\alpha\left(2-\mu\right)+1\right)^2\left(\kappa^2-3\kappa+1\right)\end{array}\right]\right|},$$

and

$$\mathcal{F}^{\kappa}(\alpha,\mu) = \left|\frac{\left[2\alpha\left(\alpha\left(2-\mu\right)+1\right)+\mu\alpha\left(3-\alpha\right)+2\left(e^{i\varphi}+1\right)\left(3-2\mu\right)\right]}{2(3e^{i\varphi}+\alpha\left(3-2\mu\right)+3)}-\varkappa\right|\Upsilon(\mu,\kappa).$$

Proof. Since $d(\nu) = \nu + \sum_{i=2}^{\infty} c_i \nu^i \in \mathcal{F}_{\Pi}^{\mu}(\alpha,\varphi,\kappa)$, So from Definition 2.1, we can write

$$\mu\left[\begin{array}{c}\left(\frac{d(\nu)}{\nu}\right)^{\alpha} \\ +\frac{1+e^{i\varphi}}{2}\left(\frac{\nu d''(\nu)}{d'(\nu)}\right)\end{array}\right]+(1-\mu)\left[\begin{array}{c}(d'(\nu))^{\alpha} \\ +\frac{1+e^{i\varphi}}{2}\left(\nu d''(\nu)\right)\end{array}\right] \prec \mathcal{B}(\kappa,\nu) \tag{3.1}$$

and

$$\mu\left[\begin{array}{c}\left(\frac{q(w)}{w}\right)^{\alpha}+ \\ \frac{1+e^{i\varphi}}{2}\left(\frac{wq''(w)}{q(w)}\right)\end{array}\right]+(1-\mu)\left[\begin{array}{c}(q'(w))^{\alpha} \\ +\frac{1+e^{i\varphi}}{2}\left(wq''(w)\right)\end{array}\right] \prec \mathcal{B}(\kappa,w). \tag{3.2}$$

We have two functions, r and s, both mapping from Λ to Λ. These functions satisfy the conditions $r(0) = s(0) = 0$ and $|r(\nu)| < 1$, $|s(w)| < 1$ for all $\nu, w \in \Lambda$. Given this, we can define γ and λ as follows:

$$\gamma(\nu) = \frac{r(\nu)+1}{1-r(\nu)} = 1+\gamma_1\nu+\gamma_2\nu^2+\gamma_3\nu^3+\cdots, \ |\gamma_\iota| \leqslant 2 \text{ for all } \iota \in \mathbb{N}.$$

$$\Rightarrow r(\nu) = \frac{\gamma(\nu)-1}{\gamma(\nu)+1} = \frac{\gamma_1}{2}\nu+\left(\frac{\gamma_2}{2}-\frac{\gamma_1^2}{4}\right)\nu^2+\frac{1}{2}\left(\gamma_3-\gamma_1\gamma_2+\frac{\gamma_1^3}{4}\right)\nu^3+\cdots \tag{3.3}$$

and

$$\lambda(w) = \frac{s(w)+1}{1-s(w)} = 1+\lambda_1 w+\lambda_2 w^2+\lambda_3 w^3+\cdots, \ |\lambda_\iota| \leqslant 2 \text{ for all } \iota \in \mathbb{N}.$$

$$\Rightarrow s(w) = \frac{\lambda(w)-1}{\lambda(w)+1} = \frac{\lambda_1}{2}w+\left(\frac{\lambda_2}{2}-\frac{\lambda_1^2}{4}\right)w^2+\frac{1}{2}\left(\lambda_3-\lambda_1\lambda_2+\frac{\lambda_1^3}{4}\right)w^3+\cdots. \tag{3.4}$$

Using (3.3) and (3.4), we get

$$\begin{aligned}\mathcal{B}(\kappa, r(\nu)) &= \Theta_0(\kappa) + \frac{\Theta_1(\kappa)}{2}\gamma_1 \nu + \left(\frac{\Theta_1(\kappa)}{2}\left(\gamma_2 - \frac{\gamma_1^2}{2}\right) + \frac{\Theta_2(\kappa)}{8}\gamma_1^2\right)\nu^2 \\ &+ \left(\frac{\Theta_1(\kappa)}{2}\left(\gamma_3 - \gamma_1\gamma_2 + \frac{\gamma_1^3}{4}\right) + \frac{\Theta_2(\kappa)}{4}\left(\gamma_1\gamma_2 - \frac{\gamma_1^3}{2}\right) + \frac{\Theta_3(\kappa)}{48}\gamma_1^3\right)\nu^3 + \cdots\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\mathcal{B}(\kappa, s(w)) &= \Theta_0(\kappa) + \frac{\Theta_1(\kappa)}{2}\lambda_1 w + \left(\frac{\Theta_1(\kappa)}{2}\left(\lambda_2 - \frac{\lambda_1^2}{2}\right) + \frac{\Theta_2(\kappa)}{8}\lambda_1^2\right)w^2 \\ &+ \left(\frac{\Theta_1(\kappa)}{2}\left(\lambda_3 - \lambda_1\lambda_2 + \frac{\lambda_1^3}{4}\right) + \frac{\Theta_2(\kappa)}{4}\left(\lambda_1\lambda_2 - \frac{\lambda_1^3}{2}\right) + \frac{\Theta_3(\kappa)}{48}\lambda_1^3\right)w^3 + \cdots\end{aligned} \tag{3.6}$$

Based on (3.1), (3.2), and the previous two equations, we can conclude

$$(e^{i\varphi} + \alpha(2-\mu) + 1)c_2 = \frac{\Theta_1(\kappa)}{2}\gamma_1, \tag{3.7}$$

$$(3e^{i\varphi} + \alpha(3-2\mu) + 3)c_3 - \left[\begin{array}{c} 2\alpha(\alpha-1) \\ -\mu\left(\frac{3\alpha(\alpha-1)}{2} + 2\left(e^{i\varphi}+1\right)\right)\end{array}\right]c_2^2 = \frac{\Theta_1(\kappa)}{2}\left(\gamma_2 - \frac{\gamma_1^2}{2}\right) + \frac{\Theta_2(\kappa)}{8}\gamma_1^2, \tag{3.8}$$

$$-(e^{i\varphi} + \alpha(2-\mu) + 1)c_2 = \frac{\Theta_1(\kappa)}{2}\lambda_1, \tag{3.9}$$

and

$$\left[\begin{array}{c} 2(3\left(e^{i\varphi}+1\right) + \alpha(\alpha+2)) \\ -\mu\left(2\left(e^{i\varphi}+1\right) - \frac{\alpha(\alpha+3)}{2} + 2\alpha(\alpha+2)\right)\end{array}\right]c_2^2 - \left[3e^{i\varphi} + \alpha(3-2\mu) + 3\right]c_3 = \frac{\Theta_1(\kappa)}{2}\left(\lambda_2 - \frac{\lambda_1^2}{2}\right) + \frac{\Theta_2(\kappa)}{8}\lambda_1^2. \tag{3.10}$$

Adding equations (3.7) and (3.9) and some simplification, we get

$$\gamma_1 = -\lambda_1 \text{ and } \gamma_1^2 = \lambda_1^2 \tag{3.11}$$

and

$$2(e^{i\varphi} + \alpha(2-\mu) + 1)^2 c_2^2 = \Theta_1^2(\kappa)(\gamma_1^2 + \lambda_1^2). \tag{3.12}$$

$$\Rightarrow c_2^2 = \frac{\Theta_1^2(\kappa)(\gamma_1^2 + \lambda_1^2)}{2(e^{i\varphi} + \alpha(2-\mu) + 1)^2} \tag{3.13}$$

Adding (3.8) to (3.10) gives

$$2\left(3\left(e^{i\varphi}+1\right) + \alpha(3-2\mu)\right)c_2^2 = 2\Theta_1(\kappa)(\gamma_2 + \lambda_2) + (\gamma_1^2 + \lambda_1^2)\left(\frac{1}{2}\Theta_2(\kappa) - \Theta_1(\kappa)\right).$$

By (3.11), we have

$$2\left(\begin{array}{c} 3\left(e^{i\varphi}+1\right) \\ +\alpha(3-2\mu)\end{array}\right)c_2^2 = 2\Theta_1(\kappa)(\gamma_2 + \lambda_2) + \gamma_1^2\left(\Theta_2(\kappa) - 2\Theta_1(\kappa)\right). \tag{3.14}$$

Also, appling (3.11) in (3.12)

$$\gamma_1^2 = \frac{(e^{i\varphi} + \alpha(2-\mu) + 1)^2 c_2^2}{\Theta_1^2(\kappa)}. \tag{3.15}$$

Replacing γ_1^2 in (3.14)

$$c_2^2 = \frac{2\Theta_1^3(\kappa)(\gamma_2 + \lambda_2)}{[\Omega(\alpha, \mu)]}, \tag{3.16}$$

where

$$\Omega(\alpha,\mu)=\begin{array}{c}2\left(3\left(e^{i\varphi}+1\right)+\alpha\left(3-2\mu\right)\right)\Theta_1^2(\kappa)\\-(e^{i\varphi}+\alpha\left(2-\mu\right)+1)^2\left(\Theta_2(\kappa)-2\Theta_1(\kappa)\right)\end{array}$$

$$\Rightarrow |c_2|^2=\frac{2\Theta_1^3(\kappa)\left(|\gamma_2|+|\lambda_2|\right)}{|\Omega(\alpha,\mu)|}.$$

Applying Lemma 2.7 and (1.3), we have:

$$\begin{aligned}|c_2|&\leqslant\sqrt{\frac{(2\kappa-1)^3}{\left|\left[\begin{array}{c}2\left(3\left(e^{i\varphi}+1\right)+\alpha\left(3-2\mu\right)\right)\left(\frac{2\kappa-1}{2}\right)^2\\-(e^{i\varphi}+\alpha\left(2-\mu\right)+1)^2\left(\kappa^2-3\kappa+1\right)\end{array}\right]\right|}}\\&=\sqrt{\Upsilon(\mu,\kappa)}.\end{aligned}$$

By subtracting (3.10) from (3.8) and then considering (3.11) along with some calculations, we can obtain

$$2(3e^{i\varphi}+\alpha\left(3-2\mu\right)+3)c_3-\left[2\alpha\left(\alpha\left(2-\mu\right)+1\right)+\mu\alpha\left(3-\alpha\right)+2\left(e^{i\varphi}+1\right)\left(3-2\mu\right)\right]c_2^2=\frac{\Theta_1(\kappa)\left(\gamma_2-\lambda_2\right)}{2}.$$

By (3.13) we obtain

$$c_3=\frac{\left[F\left(\alpha,\mu\right)\right]\Theta_1^2(\kappa)(\gamma_1^2+\lambda_1^2)}{4(3e^{i\varphi}+\alpha\left(3-2\mu\right)+3)(e^{i\varphi}+\alpha\left(2-\mu\right)+1)^2}+\frac{\Theta_1(\kappa)\left(\gamma_2-\lambda_2\right)}{4(3e^{i\varphi}+\alpha\left(3-2\mu\right)+3)},\tag{3.17}$$

where

$$F\left(\alpha,\mu\right)=\ 2\alpha\left(\alpha\left(2-\mu\right)+1\right)+\mu\alpha\left(3-\alpha\right)+2\left(e^{i\varphi}+1\right)\left(3-2\mu\right).$$

By (1.3) and (3.11)

$$c_3=\frac{\left[F\left(\alpha,\mu\right)\right]\left(2\kappa-1\right)^2}{2(3e^{i\varphi}+\alpha\left(3-2\mu\right)+3)(e^{i\varphi}+\alpha\left(2-\mu\right)+1)^2}+\frac{\left(2\kappa-1\right)}{2(3e^{i\varphi}+\alpha\left(3-2\mu\right)+3)}.\tag{3.18}$$

By applying Lemma 2.7 and equation (1.3), we can deduce the following:

$$\begin{aligned}|c_3|\leqslant&\frac{|F\left(\alpha,\mu\right)|\left(2\kappa-1\right)^2}{2\left|3e^{i\varphi}+\alpha\left(3-2\mu\right)+3\right|(e^{i\varphi}+\alpha\left(2-\mu\right)+1)^2}\\&+\left|\frac{\left(2\kappa-1\right)}{2(3e^{i\varphi}+\alpha\left(3-2\mu\right)+3)}\right|.\end{aligned}$$

From (3.17), we obtain

$$\begin{aligned}c_3-\varkappa c_2^2=&\frac{\Theta_1(\kappa)\left(\gamma_2-\lambda_2\right)}{4(3e^{i\varphi}+\alpha\left(3-2\mu\right)+3)}\\&+\left[\begin{array}{c}\frac{\left[2\alpha\left(\alpha\left(2-\mu\right)+1\right)+\mu\alpha\left(3-\alpha\right)+2\left(e^{i\varphi}+1\right)\left(3-2\mu\right)\right]}{2(3e^{i\varphi}+\alpha\left(3-2\mu\right)+3)}\\-\varkappa.\end{array}\right]c_2^2.\end{aligned}$$

Applying the triangular inequality with assist (1.3), we obtain:

$$\left|c_3-\varkappa c_2^2\right|\leqslant\frac{2\kappa-1}{2(3e^{i\varphi}+\alpha\left(3-2\mu\right)+3)}+\mathcal{F}^{\kappa}(\alpha,\mu).$$

If

$$\mathcal{F}^{\kappa}(\alpha,\mu)\leqslant\frac{2\kappa-1}{2(3e^{i\varphi}+\alpha\left(3-2\mu\right)+3)},$$

we obtain

$$\left|c_3 - \varkappa c_2^2\right| \leqslant \frac{2\kappa - 1}{(3e^{i\varphi} + \alpha(3-2\mu)+3)};$$

and if:

$$\mathcal{F}^{\kappa}(\alpha,\mu) \geqslant \frac{2\kappa - 1}{2(3e^{i\varphi} + \alpha(3-2\mu)+3)},$$

we obtain

$$\left|c_3 - \varkappa c_2^2\right| \leqslant 2\mathcal{F}^{\kappa}(\alpha,\mu).$$

The assertions made by Theorem 3.1 are as follows. □

4. Some Corollaries

By substituting $\mu = 1$ in Theorems 3.1, we obtain the following corollary.

Corollary 4.1. *Let* $d \in \Pi$ *given by* (1.1) *belongs to the class* $\mathcal{F}_{\Pi}^{1}(\alpha,\varphi,\kappa)$ *where* $\frac{1}{2} < \kappa \leqslant 1$, $-\pi < \varphi \leqslant \pi$, $\alpha \geqslant 1$, $\nu, w \in \Lambda$ *and* $d = b^{-1}$*. Then*

$$|c_2| \leqslant \sqrt{\Upsilon(1,\kappa)},$$

$$|c_3| \leqslant \frac{\left|2\alpha(\alpha+1) + \alpha(3-\alpha) + 2\left(e^{i\varphi}+1\right)\right|(2\kappa-1)^2}{2\left|3e^{i\varphi}+\alpha+3\right|\left(e^{i\varphi}+\alpha+1\right)^2} + \left|\frac{(2\kappa-1)}{2(3e^{i\varphi}+\alpha+3)}\right|,$$

and

$$\left|c_3 - \varkappa c_2^2\right| \leqslant \begin{cases} \frac{2\kappa-1}{(3e^{i\varphi}+\alpha+3)} & 0 \leqslant \mathcal{F}^{\kappa}(\alpha,1) < \frac{2\kappa-1}{2(3e^{i\varphi}+\alpha+3)}, \\ 2\mathcal{F}^{\kappa}(\alpha,1) & \mathcal{F}^{\kappa}(\alpha,1) \geqslant \frac{2\kappa-1}{2(3e^{i\varphi}+\alpha+3)}, \end{cases}$$

where

$$\Upsilon(1,\kappa) = \frac{(2\kappa-1)^3}{\left|\left[\begin{array}{c} 2\left(3\left(e^{i\varphi}+1\right)+\alpha\right)\left(\frac{2\kappa-1}{2}\right)^2 \\ -(e^{i\varphi}+\alpha+1)^2\left(\kappa^2-3\kappa+1\right)\end{array}\right]\right|},$$

and

$$\mathcal{F}^{\kappa}(\alpha,1) = \left|\frac{\left[2\alpha(\alpha+1)+\alpha(3-\alpha)+2\left(e^{i\varphi}+1\right)\right]}{2(3e^{i\varphi}+\alpha+3)} - \varkappa\right|\Upsilon(1,\kappa).$$

The next corollary follows from setting $\mu = 0$ in Theorems 3.1.

Corollary 4.2. *Let* $d \in \Pi$ *given by* (1.1) *belongs to the class* $\mathcal{F}_{\Pi}^{0}(\alpha,\varphi,\kappa)$ *where* $\frac{1}{2} < \kappa \leqslant 1$, $-\pi < \varphi \leqslant \pi$, $\alpha \geqslant 1$, $\nu, w \in \Lambda$ *and* $d = b^{-1}$*. Then*

$$|c_2| \leqslant \sqrt{\Upsilon(0,\kappa)},$$

$$|c_3| \leqslant \frac{\left|2\alpha(2\alpha+1) + 6\left(e^{i\varphi}+1\right)\right|(2\kappa-1)^2}{2\left|3e^{i\varphi}+3\alpha+3\right|\left(e^{i\varphi}+2\alpha+1\right)^2} + \left|\frac{(2\kappa-1)}{2(3e^{i\varphi}+3\alpha+3)}\right|,$$

and

$$\left|c_3 - \varkappa c_2^2\right| \leqslant \begin{cases} \frac{2\kappa-1}{(3e^{i\varphi}+3\alpha+3)} & 0 \leqslant \mathcal{F}^{\kappa}(\alpha,0) < \frac{2\kappa-1}{2(3e^{i\varphi}+3\alpha+3)}, \\ 2\mathcal{F}^{\kappa}(\alpha,\mu) & \mathcal{F}^{\kappa}(\alpha,0) \geqslant \frac{2\kappa-1}{2(3e^{i\varphi}+3\alpha+3)}, \end{cases}$$

where

$$\Upsilon(0,\kappa)=\frac{(2\kappa-1)^3}{\left|\left[\begin{array}{c}2\left(3\left(e^{i\varphi}+1\right)+3\alpha\right)\left(\frac{2\kappa-1}{2}\right)^2\\-(e^{i\varphi}+2\alpha+1)^2\left(\kappa^2-3\kappa+1\right)\end{array}\right]\right|},$$

and

$$\mathcal{F}^{\kappa}(\alpha,0)=\left|\frac{\left[2\alpha\left(2\alpha+1\right)+6\left(e^{i\varphi}+1\right)\right]}{2(3e^{i\varphi}+3\alpha+3)}-\varkappa\right|\Upsilon(0,\kappa).$$

Corollary 4.3. *Let* $d\in\Pi$ *given by* (1.1) *belongs to the class* $\mathcal{F}^{1}_{\Pi}(1,0,\kappa)$ *where* $\frac{1}{2}<\kappa\leqslant 1$, $\nu,w\in\Lambda$ *and* $q=b^{-1}$. *Then*

$$|c_2|\leqslant\sqrt{\Upsilon(1,\kappa)},$$

$$|c_3|\leqslant\frac{10\,(2\kappa-1)^2}{126}+\frac{|(2\kappa-1)|}{14},$$

and

$$\left|c_3-\varkappa c_2^2\right|\leqslant\begin{cases}\frac{2\kappa-1}{7} & 0\leqslant\mathcal{F}^{\kappa}(1,1)<\frac{2\kappa-1}{14},\\ 2\mathcal{F}^{\kappa}(1,1) & \mathcal{F}^{\kappa}(1,1)\geqslant\frac{2\kappa-1}{14},\end{cases}$$

where

$$\Upsilon(1,\kappa)=\frac{(2\kappa-1)^3}{\left|\left[14\left(\frac{2\kappa-1}{2}\right)^2-9\left(\kappa^2-3\kappa+1\right)\right]\right|},$$

and

$$\mathcal{F}^{\kappa}(1,1)=\left|\frac{5}{14}-\varkappa\right|\Upsilon(1,\kappa).$$

Corollary 4.4. *Let* $d\in\Pi$ *given by* (1.1) *belongs to the class* $\mathcal{F}^{0}_{\Pi}(1,0,\kappa)$ *where* $\frac{1}{2}<\kappa\leqslant 1$, $\nu,w\in\Lambda$ *and* $q=b^{-1}$. *Then*

$$|c_2|\leqslant\sqrt{\Upsilon(0,\kappa)},$$

$$|c_3|\leqslant\frac{(2\kappa-1)^2}{16}+\left|\frac{(2\kappa-1)}{18}\right|,$$

and

$$\left|c_3-\varkappa c_2^2\right|\leqslant\begin{cases}\frac{2\kappa-1}{9} & 0\leqslant\mathcal{F}^{\kappa}(1,0)<\frac{2\kappa-1}{18},\\ 2\mathcal{F}^{\kappa}(1,0) & \mathcal{F}^{\kappa}(1,0)\geqslant\frac{2\kappa-1}{18},\end{cases}$$

where

$$\Upsilon(0,\kappa)=\frac{(2\kappa-1)^3}{\left|\left[18\left(\frac{2\kappa-1}{2}\right)^2-16\left(\kappa^2-3\kappa+1\right)\right]\right|},$$

and

$$\mathcal{F}^{\kappa}(1,0)=|1-\varkappa|\,\Upsilon(0,\kappa).$$

5. Conclusions

Recently, there has been a surge of interest among prominent mathematicians in studying polynomials and special functions due to their applications in various mathematical and scientific fields. The goal of this paper is to present new subclasses of analytical and univalent functions using EP. We have established an upper bound estimate for the coefficients of functions belonging to the classes $\mathcal{F}_{\Pi}^{\mu}(\alpha,\varphi,\kappa)$, $\mathcal{F}_{\Pi}^{1}(\alpha,\varphi,\kappa)$, and $\mathcal{F}_{\Pi}^{0}(\alpha,\varphi,\kappa)$, and have successfully solved the Fekete-Szego problem. Future research may explore the Hankel determinant associated with these classes. The Caputo derivative operator is anticipated to play a significant role in various fields of mathematics, science, and technology.

References

[1] T. Al-Hawary, A. Amourah, A. Alsoboh, O. Alsalhi, *A new comprehensive subclass of analytic bi-univalent functions related to gegenbauer polynomials*, Symmetry, **15** (2023), 1–11. 1
[2] O. Alnajar, A. Amourah, J. Salah, M. Darus, *Fekete–Szegö Functional Problem for Analytic and Bi-Univalent Functions Subordinate to Gegenbauer Polynomials*, Contemp. Math., **5** (2024), 5731–5742. 1
[3] A. Alsoboh, A. Amourah, J. Salah, *Bi-univalent functions using Bell distribution associated with Meixner-Pollaczek polynomials*, Int. J. Math. Comput. Sci., **19** (2024), 1077–1092. 1
[4] A. Amourah, T. Al-Hawary, B. A. Frasin, *Application of Chebyshev polynomials to certain class of bi-Bazilevič functions of order* $\alpha+i\beta$, Afr. Mat., **32** (2021), 1059–1066.
[5] A. Amourah, M. Alomari, F. Yousef, A. Alsoboh, *Consolidation of a Certain Discrete Probability Distribution with a Subclass of Bi-Univalent Functions Involving Gegenbauer Polynomials*, Math. Probl. Eng. **2022** (2022), 6 pages.
[6] A. Amourah, A. Alsoboh, D. Breaz, S. M. El-Deeb, *A Bi-Starlike Class in a Leaf-like Domain Defined through Subordination via q-Calculus*, Mathematics, **12** (2024), 13 pages.
[7] A. Amourah, N. Anakira, M. J. Mohammed, M. Jasim, *Jacobi polynomials and bi-univalent functions*, Int. J. Math. Comput. Sci., **19** (2024), 957–968.
[8] A. Amourah, B. A. Frasin, M. Ahmad, F. Yousef, *Exploiting the pascal distribution series and gegenbauer polynomials to construct and study a new subclass of analytic bi-univalent functions*, Symmetry **14** (2022), 8 pages.
[9] A. Amourah, B. A. Frasin, G. Murugusundaramoorthy, T. Al-Hawary, *Bi-Bazilevič functions of order* $\vartheta+i\delta$ *associated with* (p,q)*-Lucas polynomials*, AIMS Math., **6** (2021), 4296–4305.
[10] N. Anakira, M. J. Mohammed, I. Irianto, A. Amourah, O. N. Oqilat, *Exact solution of system of multi-photograph type delay differential equations via new algorithm based on homotopy perturbation method*, Results Nonlinear Anal. **7** (2024), 187–197.
[11] S. Bulut, N. Magesh, C. Abirami, *A comprehensive class of analytic bi-univalent functions by means of Chebyshev polynomials*, J. Fract. Calc. Appl., **8** (2017), 32–39. 1
[12] J. Dziok, H. M. Srivastava, *Certain subclasses of analytic functions associated with the generalized hypergeometric function*, Integral Transforms Spec. Funct., **14** (2003), 7–18. 1
[13] M. Fekete, G. Szegö,*Eine Bemerkung Uber Ungerade Schlichte Funktionen*, J. Lond. Math. Soc., **8** (1933), 85–89. 3
[14] A. Hussen, *An application of the Mittag-Leffler-type Borel distribution and Gegenbauer polynomials on a certain subclass of bi-univalent functions*, Heliyon, **10** (2024), 8 pages. 1
[15] A. Hussen, M. Illafe, *Coefficient bounds for a certain subclass of bi-univalent functions associated with Lucas-balancing polynomials*, Mathematics, **11** (2023), 8 pages.
[16] M. Illafe, F. Yousef, M. Haji Mohd, S. Supramaniam, *Initial coefficients estimates and Fekete?Szego inequality problem for a general subclass of bi-univalent functions defined by subordination*, Axioms, **12** (2023), 10 pages. 1
[17] V. Kac, P. Cheung, *Quantum calculus*, Springer-Verlag, New York, (2002). 1
[18] S. S. Miller, P. T. Mocanu, *Second-order differential inequalities in the complex plane*, J. Math. Anal. Appl., **65** (1978), 289–305. 1
[19] S. S. Miller, P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J., **28** (1981), 157–172. 1
[20] S. S. Miller, P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker, New York, (2000). 1
[21] G. Murugusundaramoorthy, N. Magesh, V. Prameela, *Coefficient bounds for certain subclasses of bi-univalent function*, Abstr. Appl. Anal., **2013** (2013), 3 pages. 1
[22] Z. Peng, G. Murugusundaramoorthy, T. Janani, *Coefficient estimate of biunivalent functions of complex order associated with the Hohlov operator*, J. Complex Anal., **2014** (2014), 6 pages. 1
[23] C. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, (1975). 2.7
[24] H. M. Srivastava, *Some formulas for the Bernoulli and Euler polynomials at rational arguments*, Math. Proc. Cambridge Philos. Soc., **129** (2000), 77–84. 1
[25] H. M. Srivastava, *Some families of Mittag-Leffler type functions and associated operators of fractional calculus (survey)*, TWMS J. Pure Appl. Math., **7** (2016), 123–145. 1

[26] H. M. Srivastava, A. K. Mishra, P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett., **23** (2010), 1188–1192. 1

[27] F. Yousef, T. Al-Hawary, G. Murugusundaramoorthy, *Fekete-Szegö functional problems for some subclasses of bi-univalent functions defined by Frasin differential operator*, Afr. Mat., **30** (2019), 495–503.

[28] F. Yousef, S. Alroud, M. Illafe, *New subclasses of analytic and bi-univalent functions endowed with coefficient estimate problems*, Anal. Math. Phys., **11** (2021), 12 pages. 1