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Some New Smooth Fuzzy Relational Compositions

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Abstract
Fuzzy relational compositions (FRCs) are the core of the fuzzy relational models (FRMs) which play a significant role in the fuzzy linguistic modeling. In this paper, we introduce some new fuzzy relational compositions. These FRCs are composed of some new t-norms and t-conorms and have some good properties such as differentiability. In this regard, the properties of the proposed t-norms and t-conorms are studied and compared with the other ones. Finally, as the most important applications of the FRCs suggest that the new FRCs are to be used in fuzzy relational modeling of some benchmark problems to justify their usage. We show by simulations that the proposed FRCs yield fuzzy relational dynamic systems with very good modeling capabilities.

Keywords: Differential/Smooth fuzzy relational composition, FRC, list of fuzzy t-norms and t-conorms, n-D-to-one mapping, fuzzy relational modeling.

1. Introduction

Fuzzy relational modeling is a favorite tool for modeling both static functions and dynamic systems [1]. Using smooth or differentiable t-norms and t-conorms in fuzzy relational composition improves the modeling capability of the model [2]. Already there are some smooth or differentiable t-norms and t-conorms introduced in the literature, and in this paper we propose some new ones. To introduce some new t-norm or t-conorm let us remind the definitions and properties of them. In this regard, primary, secondary, and tertiary properties of t-norms and t-conorms are studied in this text.

A t-norm $t: [0,1]^2 \to [0,1]$ (and respectively t-conorm $s: [0,1]^2 \to [0,1]$) is defined by its primary properties as follows.

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Boundary condition: $t(a,1)=a$ $\quad (s(a,0)=a)$ (1a)
Commutativity: $t(a,b)=t(b,a)$ $\quad (s(a,b)=s(b,a))$ (1b)
Associativity: $t(a,t(b,c))=t(t(a,b),c)$ $\quad (s(a,s(b,c))=s(s(a,b),c))$ (1c)
Monotonicity: $b\le c\Rightarrow t(a,b)\le t(a,c)$ $\quad (b\le c\Rightarrow s(a,b)\le s(a,c))$ (1d)

Also the secondary properties of a t-norm (and respectively t-conorm) are as follows. A t-norm (s-norm) may or may not have each of these properties.

Continuity: t: continuous $\quad$ (s: continuous) (2a)
Being Archimedian: $t(a,a)<a, \forall a\in(0,1)$ $\quad (s(a,a)>a, \forall a\in(0,1))$ (2b)
Strict monotonicity: $t(a,b)<t(a,c), \forall a\in(0,1)\ and\ b<c$ $\quad (t(a,b)<t(a,c), \forall a\in(0,1)\ and\ b<c)$ (2c)

Another additional property is considered in this paper as a tertiary property and that is the differentiability of a t-norm and t-conorm which is discussed in the next section after introducing the new fuzzy compositions.

This paper consists of four sections . In section 2, the new t-norms and t-conorms are introduced and their properties are investigated. Then in section 3, the proposed fuzzy relational compositions are used in some modeling problems which approve the use of them. Finally a brief conclusion is presented in section 4.

2. Introducing New Fuzzy Relational Compositions

The new fuzzy relational compositions are of the form s-t where s and t are respectively some new t-conorms and t-norms which are presented in Table 1.

Table 1. The proposed t-norms and t-conorms, five t-norms and five t-conorms

#	t-norm $t(a,b)$	t-conorm $s(a,b)$	Parameter Boundary
1	$\left(\beta^{\log_\beta((\beta-1)a+1)\,\log_\beta((\beta-1)b+1)}-1\right)/(\beta-1)$	$\left(\beta-\beta^{\log_\beta(a+\beta-a\beta)\,\log_\beta(b+\beta-b\beta)}\right)/(\beta-1)$	$\beta\in(1,\infty)$
2	$1-\cos\left(\frac{2}{\pi}\cos^{-1}(1-a)\cos^{-1}(1-b)\right)$	$\frac{((\beta-1)a+1)((\beta-1)b+1)\beta^{-\log_\beta((\beta-1)a+1)\,\log_\beta((\beta-1)b+1)}-1}{(\beta-1)}$	$\beta\in(1,\infty)$
3	$\frac{4}{\pi}\tan^{-1}\left(\tan\left(\frac{\pi}{4}a\right)\tan\left(\frac{\pi}{4}b\right)\right)$	$1-\frac{4}{\pi}\tan^{-1}\left(\tan\left(\frac{\pi}{4}(1-a)\right)\tan\left(\frac{\pi}{4}(1-b)\right)\right)$	---
4	$1-\frac{2}{\pi}\cos^{-1}\left(\sin\left(\frac{\pi}{2}a\right)\sin\left(\frac{\pi}{2}b\right)\right)$	$\frac{2}{\pi}\cos^{-1}\left(\cos\left(\frac{\pi}{2}a\right)\cos\left(\frac{\pi}{2}b\right)\right)$	---
5	$\cos\left(\cos^{-1}a+\cos^{-1}b-\frac{2}{\pi}\cos^{-1}a\cos^{-1}b\right)$	$\cos\left(\frac{2}{\pi}\cos^{-1}a\cos^{-1}b\right)$	---

The differentiability of the components of the fuzzy relational composition is important for us to have the ability of using the gradient-based methods to tune the parameters of the model in fuzzy relational modeling. In this regard we accept a fuzzy relational composition for the smooth fuzzy relational modeling if the contributing t-norm and t-conorm are differentiable everywhere in $[0,1]\times[0,1]$ or at least differentiable almost everywhere in $[0,1]\times[0,1]$.

In Table 1, t-norms 1, 3, and t-conorms 1-3 are differentiable everywhere and the other ones are differentiable almost everywhere. In fact t-norm 2 is nondifferentiable in $a=0$, t-norm and t-conorm 4 are nondifferentiable in $(0,0)$, and t-norm and t-conorm 5 are nondifferentiable in $a=1$.

Also all the proposed t-norms and t-conorms of Table 1 have all of the secondary conditions (2) in addition to the necessary conditions (1). Hence they are well-behavior.

Definition 1. Let $\{L_i\}_{i\in[0,1]}$ be the level sets of a function $f:[0,1]^2\to[0,1]$, $\dim(L_i)=l_i$ for $i\in[0,1]$, and $n=\max_i l_i$. Then f is called n-D-to-one.

The notion of n-D-to-one can be defined for any multi-variable function. Here, for t-norm and t-conorm, $n\le 2$. If a t-norm or t-conorm is 2-D-to-one then it shows a kind of saturation behavior.

Hence being 1-D-to-one is favorite from the viewpoint of effective computational models. Also note that being n-D-to-one and specifically 1-D-to-one for a multi-variable function is somehow an extension of the notion of being one-to-one for a single-variable function. It is worth mentioning that when a t-norm or t-conorm has the secondary properties (2) then it is 1-D-to-one. Therefore all the proposed t-norms and t-conorms of Table 1 are 1-D-to-one. Checking the primary and secondary properties of the proposed t-norms and t-conorms using (1) and (2) is straightforward.

3. Application and Simulation Results

In this section the FRCs made up of the t-norms and t-conorms of Table 1 are applied to fuzzy relational modeling of some dynamic systems. The mean squared error is calculated to assess the identification process. In view of the differentiability, since the proposed FRCs are at least differentiable almost everywhere in their domain, so smooth fuzzy relational modeling is possible through gradient-based methods for parameter tuning.

3.1. Mackey-Glass time series [3]

This time series is in fact the output of a dynamic system represented by the following differential equation.

$$\ddot{x} = \frac{0.2\,x(t-\tau)}{1+x^{10}(t-\tau)} - 0.1\,x(t) \tag{3}$$

For $\tau \geq 17$ the system becomes chaotic. The more the value of τ is, the more chaotic is the system. Here, $\tau = 30$, which represents a very chaotic behavior. One thousand data is produced, 500 for identification, and 500 for verification, and $x(t+6)$ is predicted from $x(t)$, $x(t-6)$, $x(t-12)$, and $x(t-18)$. The initial condition is $x(0) = 1.2$.

Figure 1 shows the result for the verification data where the mean squared error $J = 0.0083$ is achieved. The result of some other methods for this benchmark problem can be found at [4].

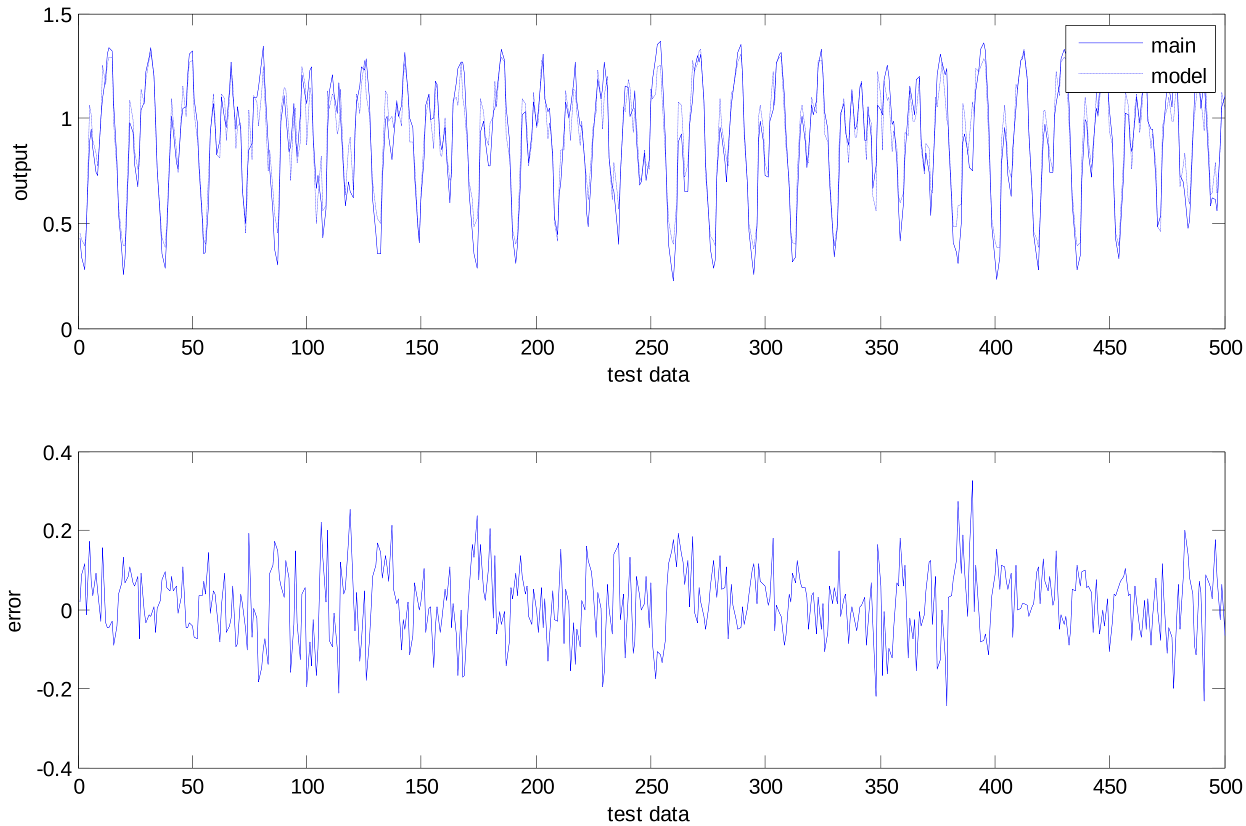

Figure 1. Verification of the FRM identified by Mackey-Glass times series with $\tau = 30$, three linguistic terms, and FRC #1
(Actual output, Model output, and the error curve)

3.2. Nonlinear dynamical system

The system is described by the following equations.

$$\begin{cases} x_1 = y(k) \\ x_2 = y(k-1) \\ x_3 = y(k-2) \\ x_4 = u(k) \\ x_5 = u(k-1) \end{cases}, \quad y(k+1) = \frac{x_1 x_2 x_3 x_5 (x_3 - 1) + x_4}{1 + {x_2}^2 + {x_3}^2} \tag{4a}$$

where the input is as follows.

$$u(k) = \begin{cases} \sin(\pi k/25) & 0 \le k \le 250 \\ 1 & 250 \le k \le 500 \\ -1 & 500 \le k \le 750 \\ 0.3\sin(\pi k/25) + 0.1\sin(\pi k/32) + 0.6\sin(\pi k/10) & 750 \le k \le 1000 \end{cases} \tag{4b}$$

To model the system, 1000 data pairs were generated by Simulink. The identification process yields $J_{learn} = 0.0017$. The result can be seen in Fig. 7. Only two linguistic terms are used in this FRM.

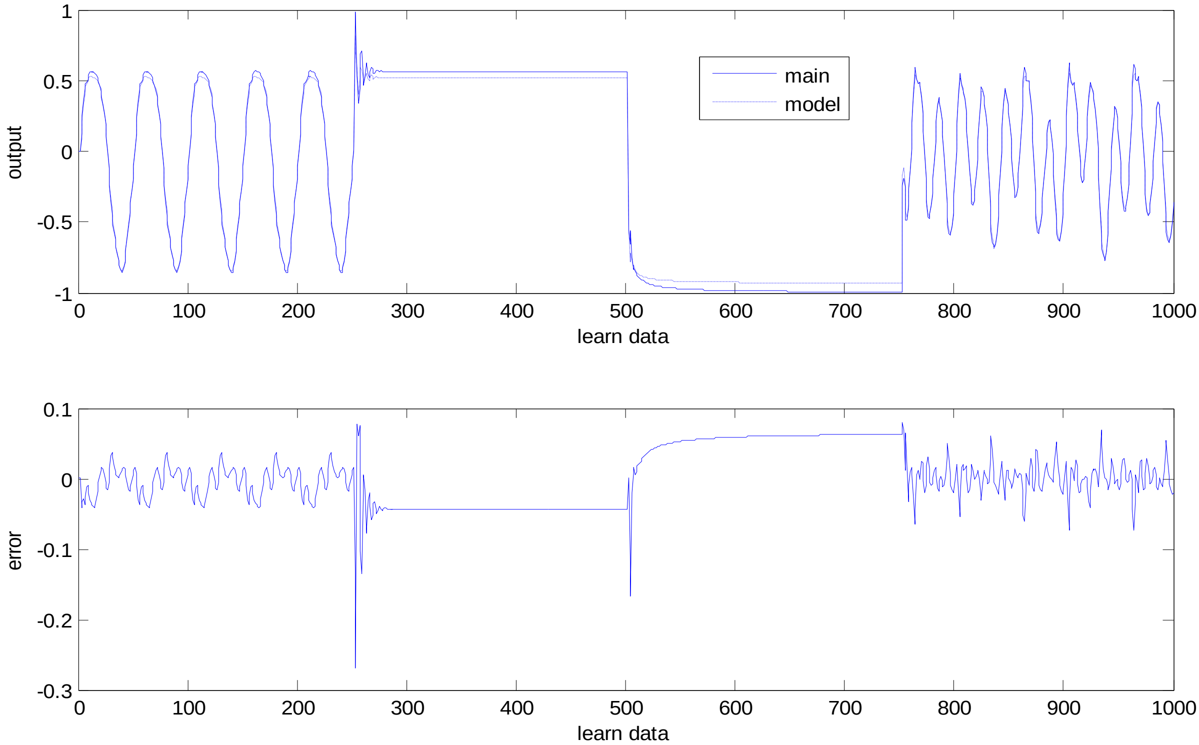

Figure 2. Actual output, model output, and error curve for fuzzy relational modeling of the dynamic system of Section 3.2

3.3. Synchronous generator

In this section the FRM of a synchronous generator is obtained using the proposed FRCs. The input and output of the generator system are respectively the field voltage and the electrical power. Hence the inputs of the FRM is considered as $y(k-1)$ and $u(k-m)$. The identification is done for several values of m as seen in Table 2. This leads to selecting $m = 0$ which means no delay for the input. There exist one thousand data for this problem, where we used the first 300 data for tuning the FRM and the last 700 data for verification of the FRM. The results of identification and verification are shown respectively in Fig. 3 and Fig. 4. The mean squared errors $J_{identification} = 74 \times 10^{-6}$ and $J_{verification} = 82 \times 10^{-6}$ are achieved in this run.

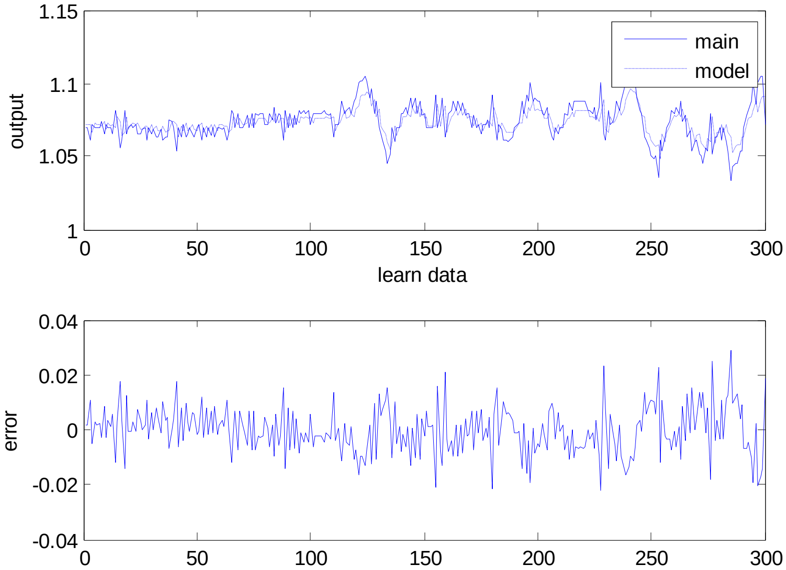

Figure 3. Identification of the FRM for the synchronous generator with only two linguistic terms and FRC #3

Table 2. The mean squared error versus several input delays, for FRM with FRC #4

Input delay m	Mean squared error J
0	0.00011210
1	0.00011221
2	0.00011235
3	0.00011287
4	0.00017631

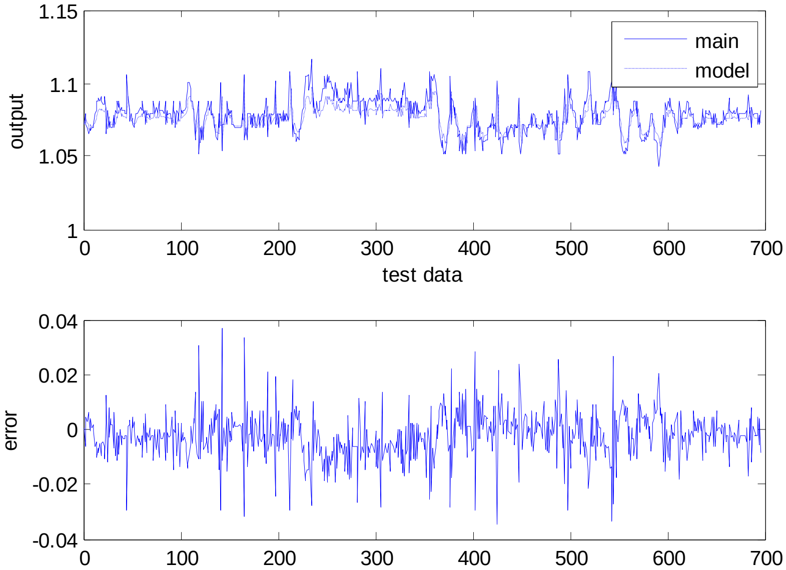

Figure 4. Verification of the FRM for the synchronous generator with only two linguistic terms and FRC #3

4. Conclusion

In this paper, some new t-norms and t-conorms are proposed to establish new FRCs. These t-norms and t-conorms have the secondary properties in addition to the necessary primary ones. They are also differentiable almost everywhere in their domains. This makes them popular to be used in smooth fuzzy relational modeling leading to more accurate models of arbitrary functions and dynamic systems. Simulation results justified the use of the proposed FRCs in FRMs.

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