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Analytical and numerical study on a variable scalar equation

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Abstract

Basically, scalar equations have potential applications in various fields such as the transmission of nerve impulses between neurons through myelin substance and other disciplines. A particular model is well-known as the pantograph equation. The standard version of this scalar equation has been extensively investigated via different analytical and numerical techniques. This paper considers a variable version of the pantograph equation. Usually, constructing an exact or a closed form solution for a variable scalar equation is a challenge. However, this work proposes a developed hybrid approach to overcome such a difficulty. The solution of the current variable version is analytically obtained in different closed forms with addressing the convergence criteria. Under some conditions, such closed forms are successfully converted to different exact ones. Additionally, accurate approximations are provided and examined. Several comparisons with the available exact solutions are conducted as a validation of our approximations. Besides, the accuracy of our approximations is checked for some classes which have no exact solutions. Probably, the results demonstrate the elegance of the proposed approach to deal with a variable version of the Pantograph model.

Keywords: Pantograph, delay, scalar equation, variable, exact solution, series solution.

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1. Introduction

This paper analyzes the extended pantograph delay differential equation (EPDDE):

$$y'(t) = ay(t) + be^{\sigma t}y(ct), \quad y(0) = \lambda, \quad |c| < 1, \quad t \geqslant 0, \tag{1.1}$$

where a, b, σ, and λ are real constants. If $\sigma = 0$, i.e., in the absence of the exponential function, the EPDDE (1.1) reduces to the standard pantograph delay differential equation (SPDDE):

$$y'(t) = ay(t) + by(ct), \quad y(0) = \lambda, \quad t \geqslant 0. \tag{1.2}$$

The cases $b = 0$, $c = 0$, and $c = 1$ transform Eq. (1.1) to three initial value problems (IVPs) of linear ordinary differential equations (ODEs), which can be easily solved via standard methods. Accordingly,

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these cases are trivial and are not of interest. Hence, the cases just mentioned will be excluded when solving the EPDDE (1.1). In the Refs. [18, 19, 34–37], several authors solved the SPDDE (1.2) by means of various numerical methods. On the other side, some authors focused on deducing analytical solutions for the SPDDE (1.2) using different approaches such as the Adomian decomposition method (ADM) [3], the Homotopy Perturbation Method (HPM) [2], a direct ansatz method [15], and the Laplace Transform (LT) [6, 7].

Multi/generalized forms for the SPDDE (1.2) were discussed in Refs. [16, 17]. When $a = -1$, $\sigma = 0$, $c = b = 1/q$ $(q > 1)$, the EPDDE (1.1) reduces to the famous Ambartsumian delay equation (ADE) [32]. The exact solutions of the classical and fractional ADE have been obtained in Ref. [10] and Ref. [14], respectively. Additionally, the ADE with a variable coefficient was solved recently in Ref. [8].

The main purpose of this work is to obtain the exact solution of the EPDDE (1.1) via a simple approach. Indeed, the LT approach was implemented as an effective tool to solve linear models [1, 4, 5, 9, 12, 13, 20–22, 33]. However, the LT requires comprehensive work to calculate the inverse LT as a final step. Moreover, the solution obtained by such an approach leads, sometimes, to closed form series solution which may not be summed to an exact form, see Ref. [6] for example. In the field of partial differential equations, a number of methods have been developed to extract exact and numerical solutions, see for example Refs. [24–31]. Thus, it is our objective to propose an efficient procedure to deal with the EPDDE (1.1). The suggested procedure is based on developing a specific transformation to reduce the EPDDE (1.1) to another equivalent model that takes the same form as the SPDDE (1.2) but with different constant coefficients. Then, the already available solutions in the literature for the SPDDE are to be invested in to construct the exact solution of the current model. In addition, various exact solutions will be provided for the EPDDE (1.1) under certain constraints of the involved parameters. Limitations of the present approach depend on the type/form of the variable coefficient. This means that changing the coefficient from the exponential form to other forms requires separate analysis.

2. Analysis

Theorem 2.1. *For $c \in \mathbb{R} - \{1\}$, the transformation $y(t) = e^{\mu t}\psi(t)$, reduces the EPDDE (1.1) to*

$$\psi'(t) = (a-\mu)\psi(t) + b\psi(ct), \quad \psi(0) = \lambda,$$

where

$$\mu = \frac{\sigma}{1-c}.$$

Proof. Suppose a transformation in the form:

$$y(t) = e^{\mu t}\psi(t), \tag{2.1}$$

where μ is an auxiliary parameter and to be determined. Inserting Eq. (2.1) into Eq. (1.1), then

$$e^{\mu t}\psi'(t) + \mu e^{\mu t}\psi(t) = a e^{\mu t}\psi(t) + b e^{(\sigma+\mu c)t}\psi(ct),$$

i.e.,

$$\psi'(t) + \mu\psi(t) = a\psi(t) + b e^{[\sigma+\mu(c-1)]t}\psi(ct),$$

or

$$\psi'(t) = (a-\mu)\psi(t) + b e^{[\sigma+\mu(c-1)]t}\psi(ct). \tag{2.2}$$

Setting $\sigma + \mu(c-1) = 0$, gives $\mu = \frac{\sigma}{1-c}$. Accordingly, Eq. (2.2) takes the form:

$$\psi'(t) = (a-\mu)\psi(t) + b\psi(ct), \quad \psi(0) = \lambda, \tag{2.3}$$

which completes the proof. □

Remark 2.2. According to the above theorem, the solution of the EPDDE (1.1) is given as

$$y(t) = e^{\frac{\sigma t}{1-c}}\psi(t), \quad c \neq 1, \tag{2.4}$$

where $\psi(t)$ is any solution of the SPDDE:

$$\psi'(t) = \left(a - \frac{\sigma}{1-c}\right)\psi(t) + b\psi(ct), \quad \psi(0) = \lambda. \tag{2.5}$$

The last two equations are to be implemented to generate the subsequent solutions. The specific transformation (2.4) was developed to successfully convert the EPDDE (1.1) to a new form which follows the SPDDE (1.2) with different parameters as appeared in Eq. (2.3).

3. Solutions of the SPDDE (1.2): $y'(t) = ay(t) + by(ct)$

This section is devoted to listing some well-known solutions in the literature for the SPDDE (1.2): $y'(t) = ay(t) + by(ct)$ under the initial condition $y(0) = \lambda$. Such solutions are to be invested in a subsequent section to establish the solution of the present EPDDE (1.1) in different analytical forms.

3.1. *Power series solution (PSS)*

In Ref. [23], the author obtained the following PSS for the SPDDE (1.2):

$$y(t) = \lambda\left[1 + \sum_{i=1}^{\infty}\left(\prod_{k=1}^{i}\left(a + bc^{k-1}\right)\right)\frac{t^i}{i!}\right]. \tag{3.1}$$

The PSS (3.1) can be expressed as

$$y(t) = \lambda\sum_{i=0}^{\infty}\left(\prod_{k=1}^{i}\left(a + bc^{k-1}\right)\right)\frac{t^i}{i!}, \tag{3.2}$$

where $\prod_{k=1}^{i}\left(a + bc^{k-1}\right) = 1$ at $i = 0$.

3.2. *Exponential function solution (EFS)*

Very recently, the authors [2] derived the solution of the SPDDE (1.2) in terms of the exponential functions in the form:

$$y(t) = \lambda\sum_{i=0}^{\infty}\left(\frac{b}{a}\right)^i\sum_{j=0}^{i}\frac{(-1)^j c^{\frac{1}{2}(i-j)(i-j-1)} e^{ac^j t}}{(c:c)_{i-j}(c:c)_j}, \tag{3.3}$$

where $(c:c)_j$ is defined as

$$(c:c)_j = \prod_{k=0}^{j-1}\left(1 - c^{k+1}\right) = \prod_{k=1}^{j}\left(1 - c^k\right).$$

Generally, $(\alpha:\beta)_j$ is known as the Pochhammer symbol and defined by the product:

$$(\alpha:\beta)_j = \prod_{k=0}^{j-1}\left(1 - \alpha\beta^k\right) = \prod_{k=1}^{j}\left(1 - \alpha\beta^{k-1}\right).$$

In addition, the authors [15] applied a direct ansatz method to obtain the solution of the SPDDE (1.2) in terms of the exponential functions in a simpler form, given by

$$y(t) = \lambda\,(-b/a:c)_\infty\sum_{i=0}^{\infty}\frac{(-b/a)^i e^{ac^i t}}{(c:c)_i}, \quad \text{provided that} \quad |b/a| < 1,\ |c| < 1,$$

where $(-b/a:c)_\infty$ is given by

$$(-b/a:c)_\infty = \prod_{k=0}^{\infty}\left(1 + (b/a)c^k\right) = \prod_{k=1}^{\infty}\left(1 + (b/a)c^{k-1}\right).$$

4. Solutions of the EPDDE (1.1): $y'(t) = ay(t) + be^{\sigma t}y(ct)$

4.1. *Power series solution (PSS)*

Based on the previous section, the PSS of Eq. (2.5) can be obtained via replacing a in Eq. (3.2) with $a-\mu$ (i.e., $a-\dfrac{\sigma}{1-c}$), then

$$\psi(t) = \lambda \sum_{i=0}^{\infty}\left(\prod_{k=1}^{i}\left(a-\frac{\sigma}{1-c}+bc^{k-1}\right)\right)\frac{t^i}{i!}, \tag{4.1}$$

and accordingly, the PSS of the EPDDE (1.1) can be obtained by substituting (4.1) into (2.4) to give

$$y(t) = \lambda e^{\dfrac{\sigma t}{1-c}} \sum_{i=0}^{\infty}\left(\prod_{k=1}^{i}\left(a-\frac{\sigma}{1-c}+bc^{k-1}\right)\right)\frac{t^i}{i!}, \quad c \neq 1. \tag{4.2}$$

It will be shown later that the solution (4.2) implies several exact solutions at special relations between the involved parameters a, b, σ, and c.

4.2. *Exponential function solution (EFS)*

The EFS of Eq. (2.5) has two forms. The first form can be obtained via replacing a in Eq. (3.3) with $a-\mu$ (i.e., $a-\frac{\sigma}{1-c}$), then

$$\psi(t) = \lambda \sum_{i=0}^{\infty}\left(\frac{b}{\left(a-\frac{\sigma}{1-c}\right)}\right)^i \sum_{j=0}^{i}\frac{(-1)^j c^{\frac{1}{2}(i-j)(i-j-1)} e^{\left(a-\frac{\sigma}{1-c}\right)c^j t}}{(c:c)_{i-j}(c:c)_j}, \tag{4.3}$$

provided that $\sigma \neq a(1-c)$ and $c \neq 1$. Thus, the first form of the EFS for the EPDDE (1.1) can be obtained by inserting (4.3) into (2.4), which yields

$$y(t) = \lambda e^{\frac{\sigma t}{1-c}} \sum_{i=0}^{\infty}\left(\frac{b}{\left(a-\frac{\sigma}{1-c}\right)}\right)^i \sum_{j=0}^{i}\frac{(-1)^j c^{\frac{1}{2}(i-j)(i-j-1)} e^{\left(a-\frac{\sigma}{1-c}\right)c^j t}}{(c:c)_{i-j}(c:c)_j}, \quad \sigma \neq a(1-c), \quad c \neq 1. \tag{4.4}$$

Proceeding as above, the second form of $\psi(t)$ reads

$$\psi(t) = \lambda \left(-b/\left(a-\frac{\sigma}{1-c}\right):c\right)_{\infty} \sum_{i=0}^{\infty}\frac{\left(-b/\left(a-\frac{\sigma}{1-c}\right)\right)^i e^{\left(a-\frac{\sigma}{1-c}\right)c^i t}}{(c:c)_i}. \tag{4.5}$$

Therefore, the EFS for the EPDDE (1.1) is derived by substituting Eq. (4.5) into Eq. (2.4), hence

$$y(t) = \lambda \left(-b/\left(a-\frac{\sigma}{1-c}\right):c\right)_{\infty} e^{\frac{\sigma t}{1-c}} \sum_{i=0}^{\infty}\frac{\left(-b/\left(a-\frac{\sigma}{1-c}\right)\right)^i e^{\left(a-\frac{\sigma}{1-c}\right)c^i t}}{(c:c)_i}, \tag{4.6}$$

where the conditions $\left|b/\left(a-\frac{\sigma}{1-c}\right)\right| < 1$ and $|c| < 1$ ensure the convergence of the solution (4.5) in addition to the solution in the form (4.6).

5. Exact solution at special cases

In order to facilitate the derivation of the results of this section, we rewrite the solution given in Eq. (4.2) as

$$y(t) = \lambda e^{\frac{\sigma t}{1-c}} \sum_{i=0}^{\infty} w_i(c) \frac{t^i}{i!}, \tag{5.1}$$

where $w_i(c)$ is defined by

$$w_i(c) = \prod_{k=1}^{i} \left(a - \frac{\sigma}{1-c} + bc^{k-1} \right). \tag{5.2}$$

Existence of exact solutions under different relations of the parameters a, b, σ, and c is revealed in this section.

5.1. $a - \frac{\sigma}{1-c} + b = 0$

The next theorem provides the exact solution for the EPDDE (1.1) under the condition $a - \frac{\sigma}{1-c} + b = 0$. In this case, it will be shown that the first term of the infinite series (5.1) survives while the other terms vanish. Hence, explicit form for the exact solution is expected.

Theorem 5.1. *For $c \neq 1$ and $a, b \in \mathbb{R}$ such that $a - \frac{\sigma}{1-c} + b = 0$, then the EPDDE (1.1) becomes*

$$y'(t) = ay(t) + be^{(a+b)(1-c)t} y(ct), \quad y(0) = \lambda, \quad t \geqslant 0,$$

which has the exact solution:

$$y(t) = \lambda e^{(a+b)t}.$$

Proof. From (5.1), we have

$$y(t) = \lambda e^{\frac{\sigma t}{1-c}} \left[w_0(c) + w_1(c)t + w_2(c)\frac{t^2}{2!} + \cdots \right].$$

We deduce form Eq. (5.2) that

$$\begin{aligned}
w_0(c) &= 1, \\
w_1(c) &= a - \frac{\sigma}{1-c} + b, \\
w_2(c) &= \left(a - \frac{\sigma}{1-c} + b\right)\left(a - \frac{\sigma}{1-c} + bc\right), \\
w_3(c) &= \left(a - \frac{\sigma}{1-c} + b\right)\left(a - \frac{\sigma}{1-c} + bc\right)\left(a - \frac{\sigma}{1-c} + bc^2\right), \\
&\vdots \\
w_i(c) &= \left(a - \frac{\sigma}{1-c} + b\right)\left(a - \frac{\sigma}{1-c} + bc\right)\left(a - \frac{\sigma}{1-c} + bc^2\right)\cdots\left(a - \frac{\sigma}{1-c} + bc^{i-1}\right).
\end{aligned} \tag{5.3}$$

If $a - \frac{\sigma}{1-c} + b = 0$, then $w_1(c) = w_2(c) = w_3(c) = \cdots = 0$, i.e., $w_i(c) = 0$, $\forall i \geqslant 1$. Thus

$$y(t) = \lambda e^{\frac{\sigma t}{1-c}} w_0(c) = \lambda e^{\frac{\sigma t}{1-c}}. \tag{5.4}$$

Solving $a - \frac{\sigma}{1-c} + b = 0$ for σ gives $\sigma = (a+b)(1-c)$. Substituting this value of σ into (5.4) yields

$$y(t) = \lambda e^{(a+b)t}.$$

In this case, the EPDDE (1.1) takes the form:

$$y'(t) = ay(t) + be^{(a+b)(1-c)t} y(ct), \quad y(0) = \lambda, \quad t \geqslant 0,$$

which completes the proof. □

Remark 5.2. It may be important to refer to that the exact solution $y(t) = \lambda e^{(a+b)t}$ is independent of c and remains valid for the EPDDE $y'(t) = ay(t) + be^{(a+b)(1-c)t}y(ct)$, $y(0) = \lambda$ for any $c \in \mathbb{R} - \{1\}$.

5.2. $a - \frac{\sigma}{1-c} + bc = 0$

The next lemma shows that the exact solution of the EPDDE (1.1) when $a - \frac{\sigma}{1-c} + bc = 0$ is expressed as an exponential function multiplied by a polynomial of first degree in t.

Lemma 5.3. *For $c \neq 1$ and $a, b \in \mathbb{R}$ such that $a - \frac{\sigma}{1-c} + bc = 0$, then the EPDDE (1.1) becomes*

$$y'(t) = ay(t) + be^{(a+bc)(1-c)t}y(ct), \quad y(0) = \lambda, \quad t \geqslant 0,$$

and its exact solution reads

$$y(t) = \lambda e^{(a+bc)t}\left[1 + b(1-c)t\right].$$

Proof. Assume that $a - \frac{\sigma}{1-c} + bc = 0$, then $\sigma = (a+bc)(1-c)$. From Eqs. (5.3), one can deduce that

$$w_0(c) = 1, \quad w_1(c) = b(1-c),$$

while $w_i(c) = 0$, $\forall i \geqslant 2$. Therefore, the series solution in Eq. (5.1) reduces to the exact solution:

$$y(t) = \lambda e^{\frac{\sigma t}{1-c}}\left[1 + b(1-c)t\right].$$

Inserting the value of $\sigma = (a+bc)(1-c)$ into the last equation implies

$$y(t) = \lambda e^{(a+bc)t}\left[1 + b(1-c)t\right],$$

which is the exact solution of the corresponding EPDDE (1.1):

$$y'(t) = ay(t) + be^{(a+bc)(1-c)t}y(ct), \quad y(0) = \lambda, \quad t \geqslant 0,$$

and this completes the proof. □

5.3. $a - \frac{\sigma}{1-c} + bc^2 = 0$

In this case, the exact solution of the EPDDE (1.1) is obtained as an exponential function multiplied by a polynomial of second degree in t as shown in the lemma below.

Lemma 5.4. *For $c \neq 1$ and $a, b \in \mathbb{R}$ such that $a - \frac{\sigma}{1-c} + bc^2 = 0$, then the EPDDE (1.1) becomes*

$$y'(t) = ay(t) + be^{(a+bc^2)(1-c)t}y(ct), \quad y(0) = \lambda, \quad t \geqslant 0,$$

with the exact solution:

$$y(t) = \lambda e^{(a+bc^2)t}\left[1 + b(1-c^2)t + b^2c(1-c)(1-c^2)\frac{t^2}{2!}\right].$$

Proof. Suppose that $a - \frac{\sigma}{1-c} + bc^2 = 0$, then $\sigma = (a+bc^2)(1-c)$. One can find from Eqs. (5.3) that

$$w_0(c) = 1, \quad w_1(c) = b(1-c^2), \quad w_2(c) = b^2c(1-c)(1-c^2),$$

while $w_i(c) = 0$, $\forall i \geqslant 3$. Hence, the series solution (5.1) transforms to the exact one, given by

$$y(t) = \lambda e^{\frac{\sigma t}{1-c}}\left[1 + b(1-c^2)t + b^2c(1-c)(1-c^2)\frac{t^2}{2}\right]. \tag{5.5}$$

Employing the value $\sigma = (a+bc^2)(1-c)$ into Eq. (5.5) leads to

$$y(t) = \lambda e^{(a+bc^2)t}\left[1 + b(1-c^2)t + b^2c(1-c)(1-c^2)\frac{t^2}{2!}\right],$$

and the EPDDE (1.1) becomes

$$y'(t) = ay(t) + be^{(a+bc^2)(1-c)t}y(ct), \quad y(0) = \lambda, \quad t \geqslant 0,$$

which finalizes the proof. □

5.4. $a - \frac{\sigma}{1-c} + bc^3 = 0$

This case shows that the exact solution of the EPDDE (1.1) is a product of exponential function and a polynomial of third degree in t, discussed in the next lemma.

Lemma 5.5. *For $c \neq 1$ and $a, b \in \mathbb{R}$ such that $a - \frac{\sigma}{1-c} + bc^3 = 0$, then the EPDDE (1.1) becomes*

$$y'(t) = ay(t) + be^{(a+bc^3)(1-c)t}y(ct), \quad y(0) = \lambda, \quad t \geqslant 0,$$

which follows the exact solution:

$$y(t) = \lambda e^{(a+bc^3)t}\left[1 + b(1-c^3)t + b^2c(1-c^2)(1-c^3)\frac{t^2}{2!} + b^3c^3(1-c)(1-c^2)(1-c^3)\frac{t^3}{3!}\right]. \tag{5.6}$$

Proof. The proof follows immediately by repeating the same analysis of the previous lemmas through considering $a - \frac{\sigma}{1-c} + bc^3 = 0$. In such case, one can find from Eqs. (5.3) that

$$w_0(c) = 1, \quad w_1(c) = b(1-c^3), \quad w_2(c) = b^2c(1-c^2)(1-c^3), \quad w_3(c) = b^3c^3(1-c)(1-c^2)(1-c^3),$$

while $w_i(c) = 0$, $\forall i \geqslant 4$. Thus, the series solution in Eq. (5.1) becomes the exact one in Eq. (5.6), which is the desired result. □

5.5. $a - \frac{\sigma}{1-c} + bc^n = 0$, $n \in \mathbb{N}^+$

This section generalizes the results of the previous sections. Here, we show that the solution is given as a product of an exponential function and a polynomial of degree n in t.

Theorem 5.6. *For $c \neq 1$ and $a, b \in \mathbb{R}$ such that $a - \frac{\sigma}{1-c} + bc^n = 0$ ($n \in \mathbb{N}^+$), then the EPDDE (1.1) becomes*

$$y'(t) = ay(t) + be^{(a+bc^n)(1-c)t}y(ct), \quad y(0) = \lambda, \quad t \geqslant 0,$$

with the exact solution:

$$y(t) = \lambda e^{(a+bc^n)t}\sum_{i=0}^{n} w_i \frac{t^i}{i!}, \tag{5.7}$$

where w_i is given by

$$w_i = b^i c^{\frac{i(i-1)}{2}} \prod_{k=0}^{i-1}\left(1 - c^n\left(\frac{1}{c}\right)^k\right) = b^i c^{\frac{i(i-1)}{2}}\left(c^n : \frac{1}{c}\right)_i, \quad c \neq 0. \tag{5.8}$$

Proof. In view of the above analysis, we have from Eqs. (5.3) at $a - \frac{\sigma}{1-c} + bc^n = 0$ that

$$\begin{aligned}
w_0(c) &= 1,\\
w_1(c) &= b\left(1-c^n\right),\\
w_2(c) &= b^2c\left(1-c^n\right)\left(1-c^{n-1}\right),\\
w_3(c) &= b^3c^3\left(1-c^n\right)\left(1-c^{n-1}\right)\left(1-c^{n-2}\right),\\
w_4(c) &= b^4c^6\left(1-c^n\right)\left(1-c^{n-1}\right)\left(1-c^{n-2}\right)\left(1-c^{n-3}\right),\\
&\vdots
\end{aligned} \tag{5.9}$$

which follows the pattern:

$$w_i = b^i c^{\frac{i(i-1)}{2}} \prod_{k=0}^{i-1}\left(1 - c^{n-k}\right), \quad i \geqslant 0.$$

It should be noted that at $i = n+1$ we have from (5.9) that $w_{n+1} = 0$, which implies $w_i = 0$, $\forall i \geqslant n+1$. This means that the infinite series includes only n terms, therefore

$$y(t) = \lambda e^{\frac{\sigma t}{1-c}} \sum_{i=0}^{n} w_i \frac{t^i}{i!},$$

or equivalently

$$y(t) = \lambda e^{(a+bc^n)t} \sum_{i=0}^{n} w_i \frac{t^i}{i!}.$$

The general term w_i can be rewritten as

$$w_i = b^i c^{\frac{i(i-1)}{2}} \prod_{k=0}^{i-1} \left(1 - c^n \left(\frac{1}{c}\right)^k\right) = b^i c^{\frac{i(i-1)}{2}} \left(c^n : \frac{1}{c}\right)_i, \quad c \neq 0,$$

and this completes the proof. □

6. Numerical results

In the previous sections, the exact solutions of the EPDDE (1.1) were obtained under specific relations of the included parameters via the PSS and the Theorem 5.6 given in Eqs. (5.7)-(5.8). In addition, two forms of the EFS were constructed as approximate solutions given by Eq. (4.4) and (4.6). Since the general exact solution (5.7)-(5.8) of the EPDDE (1.1) was obtained under the constraint $\sigma = (a + bc^n)(1-c)$, then the behavior of the exact solution depends mainly on n. In this regard, Figures 1 and 2 show the behavior of the exact solution (5.7)-(5.8) when $n = 1, 2, 3$ (Figure 1) and $n = 4, 5, 6$ (Figure 1), respectively.

The rest of this section is devoted to estimate the accuracy of the EFS (4.4) in view of the available exact solution. The N-term approximate EFS (4.4) corresponding to the PSS under the condition $\sigma = (a + bc^n)(1-c)$ reads

$$\Theta_N(t) = \lambda e^{(a+bc^n)t} \sum_{i=0}^{N-1} \left(-c^{-n}\right)^i \sum_{j=0}^{i} \frac{(-1)^j c^{\frac{1}{2}(i-j)(i-j-1)} e^{-bc^{n+j}t}}{(c:c)_{i-j}(c:c)_j}, \quad c \neq 1.$$

Figure 3 displays the comparison between the approximate solutions $\Theta_N(t)$, $N = 7, 8, 9$ and the exact solution (5.7)-(5.8) when $n = 1$. This figure shows that the coincidence between the approximations $\Theta_N(t)$ and the exact solution increases as the number of term N increases. This conclusion is confirmed in Figures 4-6 at different values of n such as $n = 2$ in Figure 4, $n = 3$ in Figure 5, and $n = 4$ in Figure 6.

The numerical results in Figures 7 and 8 confirm the convergence of the approximations $\Theta_N(t)$ derived from the EFS solution (4.4) at selected values of λ, a, b, c, and σ. A final step is to estimate the accuracy of the approximations $\Theta_N(t)$ through evaluating the residuals $RE_N(t)$:

$$RE_N(t) = \left|\Theta_N'(t) - a\Theta_N(t) - be^{\sigma t}\Theta_N(ct)\right|, \quad N \geqslant 1, \quad c \neq 1.$$

In this context, the obtained residuals in Figures 9 and 10 confirm that all $RE_N(t)$ tend to zero even in a large domain.

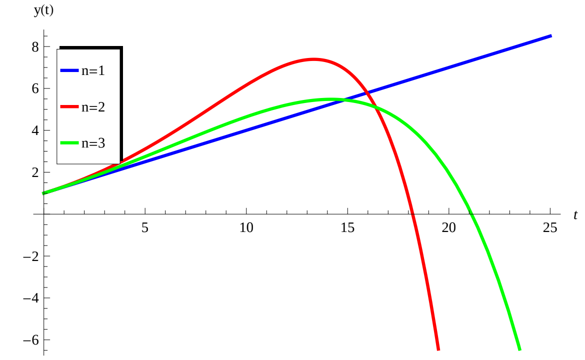

Figure 1: Plots of the exact solution (5.7)-(5.8) for the EPDDE $y'(t) = ay(t) + be^{(a+bc^n)(1-c)t}y(ct)$, $y(0) = \lambda$ when $\lambda = 1$, $a = 0.1$, $b = 0.2$, and $c = -0.5$ at different values of n, $n = 1, 2, 3$.

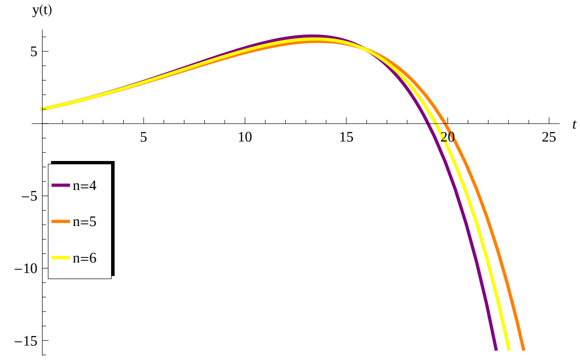

Figure 2: Plots of the exact solution (5.7)-(5.8) for the EPDDE $y'(t) = ay(t) + be^{(a+bc^n)(1-c)t}y(ct)$, $y(0) = \lambda$ when $\lambda = 1$, $a = 0.1$, $b = 0.2$, and $c = -0.5$ at different values of n, $n = 4, 5, 6$.

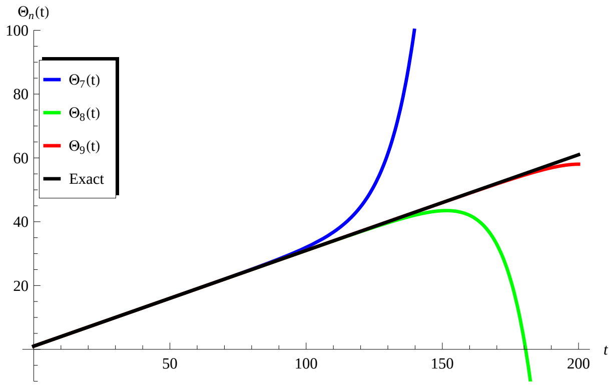

Figure 3: Comparison between the approximate solutions $\Theta_N(t)$, $N = 7, 8, 9$ and the exact solution (5.7)-(5.8) for the EPDDE $y'(t) = ay(t) + be^{(a+bc^n)(1-c)t}y(ct)$, $y(0) = \lambda$ at $\lambda = 1$, $a = 0.1$, $b = 0.2$, and $c = -0.5$ when $n = 1$.

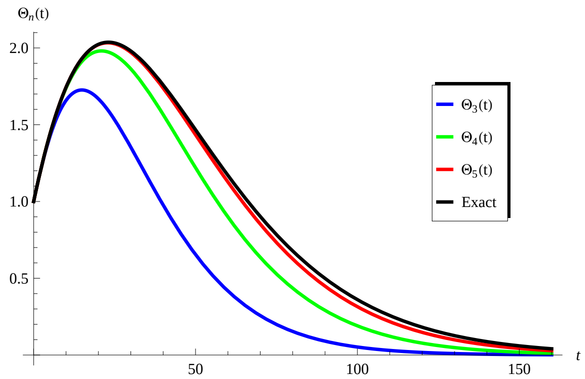

Figure 4: Comparison between the approximate solutions $\Theta_N(t)$, $N = 3, 4, 5$ and the exact solution (5.7)-(5.8) for the EPDDE $y'(t) = ay(t) + be^{(a+bc^n)(1-c)t}y(ct)$, $y(0) = \lambda$ at $\lambda = 1$, $a = 0.1$, $b = 0.2$, and $c = -0.5$ when $n = 2$.

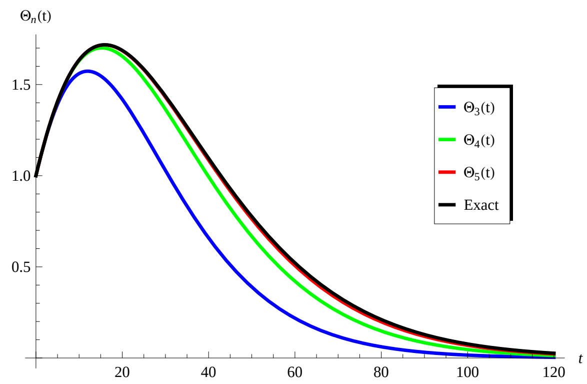

Figure 5: Comparison between the approximate solutions $\Theta_N(t)$, $N = 3, 4, 5$ and the exact solution (5.7)-(5.8) for the EPDDE $y'(t) = ay(t) + be^{(a+bc^n)(1-c)t}y(ct)$, $y(0) = \lambda$ at $\lambda = 1$, $a = 0.1$, $b = 0.2$, and $c = -0.5$ when $n = 3$.

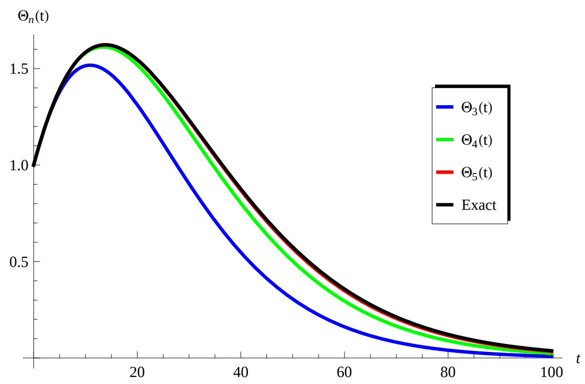

Figure 6: Comparison between the approximate solutions $\Theta_N(t)$, $N = 3, 4, 5$ and the exact solution (5.7)-(5.8) for the EPDDE $y'(t) = ay(t) + be^{(a+bc^n)(1-c)t}y(ct)$, $y(0) = \lambda$ at $\lambda = 1$, $a = 0.1$, $b = 0.2$, and $c = -0.5$ when $n = 4$.

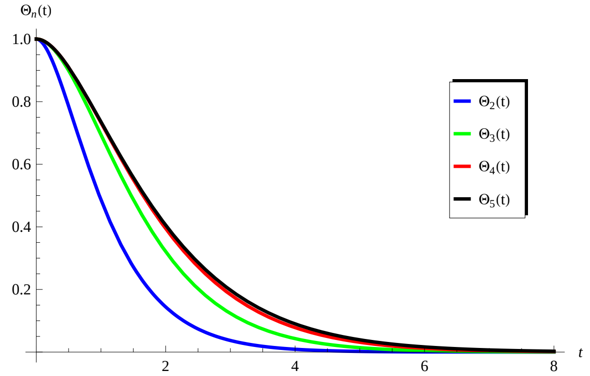

Figure 7: Plots of the approximate solutions $\Theta_N(t)$, $N = 2, 3, 4, 5$ for the EPDDE $y'(t) = ay(t) + be^{\sigma t}y(ct)$, $y(0) = \lambda$ at $\lambda = 1$, $a = -2$, $b = 2$, $c = -0.5$, and $\sigma = -0.5$.

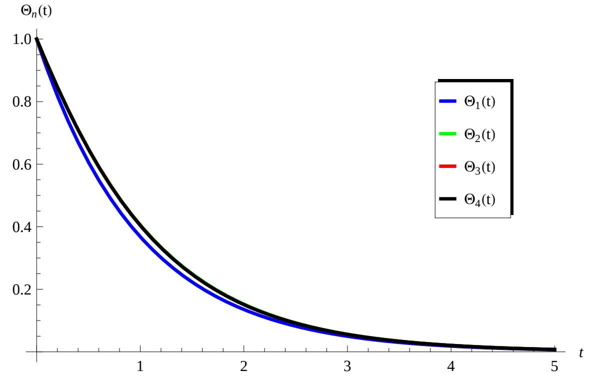

Figure 8: Plots of the approximate solutions $\Theta_N(t)$, $N = 1, 2, 3, 4$ for the EPDDE $y'(t) = ay(t) + be^{\sigma t}y(ct)$, $y(0) = \lambda$ at $\lambda = 1$, $a = -1$, $b = 0.2$, $c = -0.5$, and $\sigma = -3$.

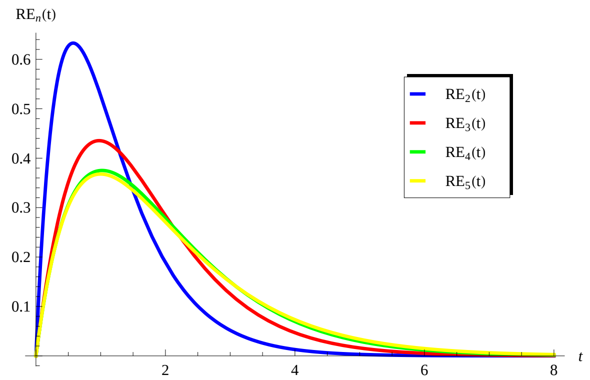

Figure 9: Plots of the residuals $RE_N(t)$, $N = 2, 3, 4, 5$ for the EPDDE $y'(t) = ay(t) + be^{\sigma t}y(ct)$, $y(0) = \lambda$ at $\lambda = 1$, $a = -2$, $b = 2$, $c = -0.5$, and $\sigma = -0.5$.

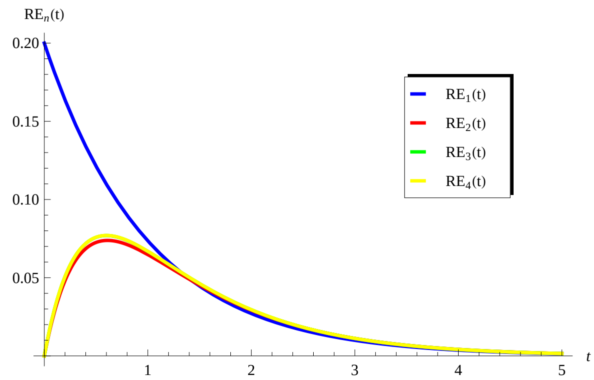

Figure 10: Plots of the residuals $RE_N(t)$, $N = 1, 2, 3, 4$ for the EPDDE $y'(t) = ay(t) + be^{\sigma t}y(ct)$, $y(0) = \lambda$ at $\lambda = 1$, $a = -1$, $b = 0.2$, $c = -0.5$, and $\sigma = -3$.

7. Conclusions

The extended pantograph delay differential equation (EPDDE) was investigated in this paper by incorporating a term of exponential function into the standard pantograph delay differential equation (SPDDE). An effective transformation was developed to reduce the EPDDE to the SPDDE in which no exponential term exists. With the aid of the available solutions for the SPDDE, the solution of the EPDDE was established in different analytical forms. In addition, the exact solution of the current extended model was determined under a certain relation which governs the involved parameters and the exponent of the exponential term. Besides, the approximate solution of the investigated model was also evaluated and checked for convergence in the absence of such a relation. The accuracy of the obtained approximate solution was examined by performing some comparisons with the exact solution. Finally, the effectiveness of the obtained approximations is assessed by utilizing the residuals. It was observed that these residuals approach zero, even in enlarged domains, which confirms the efficiency of the current proposed method. The present results may be extended to other types of scalar equations, such as the transmission model of nerve impulses between neurons through myelin substance, which covers all the nerves in the brain and nervous system in humans [11].

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