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Cubic Fermatean fuzzy decision model for project team evaluation using Sugeno–Weber operators and maximizing deviation technique

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Abstract

This paper introduces a novel cubic Fermatean fuzzy (CFF) Sugeno–Weber (SuW) aggregation-based method for multi-criteria group decision-making (MCGDM) under uncertainty. First, we define new SuW operational laws for CFF sets and investigate their key properties. Based on these, we develop several power aggregation operators (AOs) that effectively combine evaluation information in fuzzy environments. These AOs are then integrated into a novel MCGDM framework employing two types of maximizing deviation models to determine unknown criteria weights. The proposed method is validated through a real-world case study assessing project team member performance across multiple evaluation criteria. Furthermore, sensitivity analysis and comparative evaluation demonstrate the robustness and superiority of the proposed method over existing aggregation techniques, confirming its practical significance.

Keywords: Sugeno-Weber t-norm, power operators, cubic fermatean fuzzy set, maximizing deviation, MCGDM.

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1. Introduction

In everyday decision-making, conflict resolution often depends on data analysis, machine learning, and the systematic gathering of knowledge [13, 32, 36, 51]. One of the main obstacles in strategic planning is the absence of solid, verifiable data. This challenge can be mitigated through the application of mathematical models that support the development of effective decision-making (DM) strategies [1, 28, 37]. These methodologies enable organizations to assess and rank diverse viewpoints, identifying the most favorable choices while discarding the least suitable. Such structured evaluation aids in thoroughly examining, classifying, and selecting among various alternatives. Multi-criteria group decision-making

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(MCGDM) is widely recognized as a powerful approach for identifying optimal solutions by considering multiple influencing factors. Traditionally, it was presumed that all information regarding alternatives, attributes, and weights was available in precise, numerical form. However, the core objective in addressing evaluation and DM challenges lies in gathering and organizing information for systematic analysis. Given the inherent uncertainty in complex, real-world scenarios, many MCGDM problems feature imprecise or ambiguous input data. To address this uncertainty, Zadeh [62] introduced the fuzzy set (FS) theory, which allows for the modeling of vagueness in decision contexts by assigning each element a membership degree (MD) ranging from 0 to 1. Despite its usefulness, FS has limitations—particularly in cases where decision experts (DEs) also need to consider the degree of non-membership (or disaffiliation). To resolve this shortcoming, Atanassov [8] developed the intuitionistic fuzzy set (IFS) framework, which expands FS by incorporating both membership and non-membership degrees, offering a richer means of representing uncertainty. IFS has since been applied across various fields, including decision analysis [58], pattern recognition [16], and control systems [35]. To further refine uncertainty modeling in DM processes, Yager [60] introduced the concept of Pythagorean fuzzy sets (PyFS), which extend IFS by allowing a more flexible relationship between affiliation and disaffiliation measures. This advancement has significantly improved the ability to evaluate uncertain scenarios, leading to its adoption in numerous applications [2, 5, 61].

As DM environments grow increasingly complex, DEs face greater difficulty in delivering precise and reliable evaluations. Traditional models like IFS and PyFS were developed to tackle the inherent uncertainty and cognitive ambiguity in human reasoning, which is often both nuanced and subjective. To accommodate the varying backgrounds and timing constraints of DEs, it is essential to provide a broader informational framework that reflects their context and expertise, enabling more meaningful and tailored assessments. To address this requirement, Senapati and Yager [47, 48] introduced Fermatean fuzzy sets (FFS), a novel model that permits the sum of the cubes of the membership and non-membership values to remain within the unit interval [0,1]. This characteristic enhances FFS's capability to handle uncertainty more effectively than IFS and PyFS. Since their introduction, FFS have spurred considerable academic interest, leading to significant developments in fuzzy theory [9, 17, 22, 52]. Building on this foundation, various FFS-based extensions have been proposed to enrich information representation in uncertain environments. Notable examples include the Fermatean fuzzy linguistic set [30], interval-valued Fermatean fuzzy sets [21], interval-valued hesitant Fermatean fuzzy sets [34], and cubic Fermatean fuzzy sets (CFFS) [42]. Figure 1 presents an overview of different extensions of FSs. These extensions have demonstrated strong applicability in a wide range of DM models. In particular, the CFFS framework has led to the development of advanced algorithms specifically designed to manage uncertainty in real-world decision scenarios [14, 18, 40, 42, 53]. In many real-life evaluations, experts express their opinions in terms of both crisp and interval-valued beliefs with varying levels of agreement and hesitation. Existing models like IFS, PyFS, and even FFS are often limited when handling such multi-layered and asymmetric uncertainty. For instance, consider a case where a panel of medical experts evaluates the success probability of a newly introduced treatment for a complex disease. Based on initial clinical trials, the treatment success is estimated between 50–70%, but due to limited data and individual biases, there's still a hesitation degree of around 0.25. Conversely, the experts collectively agree that the failure probability does not exceed 30–45%, with only minor disagreement (hesitation ≈ 0.10). This evaluation may be represented as: $(\langle [0.50, 0.70];\ 0.25\rangle,\ \langle [0.30, 0.45];\ 0.10\rangle)$.

In the above context, traditional FSs (which only handle single membership values) or intuitionistic fuzzy sets (which cannot accommodate interval-valued or hesitation data) fall short. PyFSs and FFSs, while extending the allowable ranges, still assume crisp (point-based) membership and non-membership components. Moreover, they do not adequately reflect the co-existence of interval information and degrees of hesitancy. In contrast, the CFFS not only supports interval-valued membership/non-membership components but also integrates hesitation degrees, making it an ideal model for complex real-world problems where data is both imprecise and hesitant. Hence, the choice of CFFS over other fuzzy extensions is justified by its ability to simultaneously capture layered uncertainty, expert hesitation, and information spread.

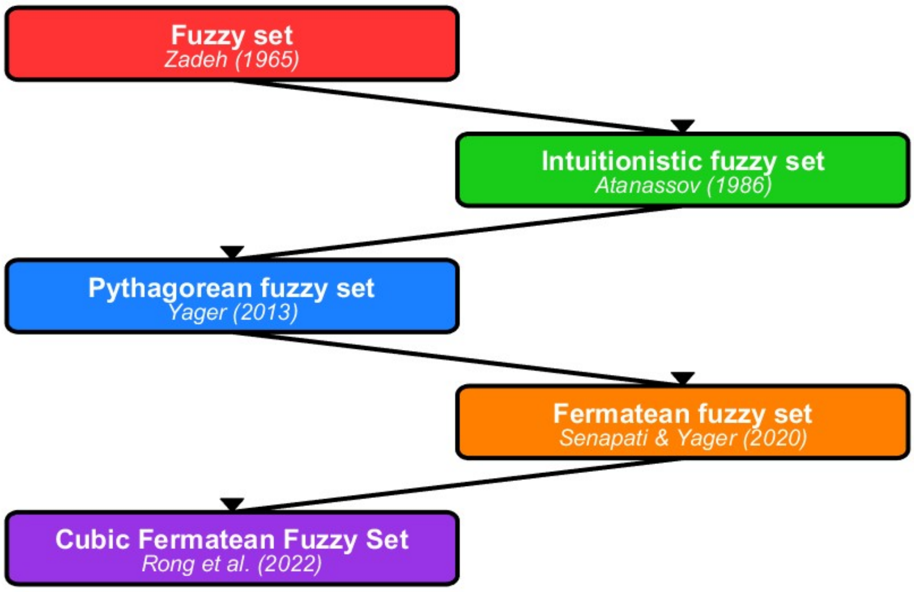

Figure 1: Hierarchy of fuzzy set extensions

Sugeno [50] developed a family of nilpotent T-conorm (TCO), with asymptotic members including the drastic sum and probabilistic sum. Meanwhile, Weber [55] initiated a family of nilpotent T-norm (TNO), featuring the product and drastic product as asymptotic members. These TNO and TCO are dual, with the Sugeno TCO parameter $\zeta \in]-1,\infty[$ corresponding to the Weber TNO parameter $\mathfrak{g} = \frac{-\zeta}{1+\zeta} \in]-1,\infty[$. This duality has led to the combined naming of Sugeno-Weber (SuW) TNO and TCO. For more details, refer to subsection 4.7 of Klement et al. [25]. The inclusion of the SuW $\mathfrak{g}$ parameter enhances DEs' ability to formulate optimistic or pessimistic conclusions, thereby improving risk management effectiveness. Recently, this topic has garnered significant attention in the DM field from numerous researchers. Sarkar et al. [45] utilized SuW aggregation tools to explore suitable methods within the t-spherical fuzzy hypersoft framework. Ashraf et al. [7] investigated circular spherical fuzzy SuW aggregation operators (AOs) and their basic properties. Wang et al. [54] conducted an in-depth analysis of q-rung orthopair fuzzy mathematical approaches, specifically addressing solar panel selection. Another study [6] introduced the concept of complex spherical FS, presenting operational rules and creating AOs based on SuW TNO and TCO. Recently, Senapati et al. [46] extended SuW TNO and TCO to the dual hesitant q-rung orthopair fuzzy setting, applying this to healthcare supply chain management. While these strategies offer several advantages for handling complex human opinions, there is a noticeable gap in research on the CFF environment utilizing dominant SuW TNO and TCO operations. To better meet the demands of certain DM processes, various AOs need to be adapted and updated according to SuW TNO and TCO, ensuring more authentic and effective DM solutions.

AOs are vital in addressing DM dilemmas [3, 4, 19, 20, 38]. They have become a significant research topic, yielding extensive and ongoing investigation results. Yager [59] originated the power average (PA) operator to mitigate the influence of biased experts and ensure fair outcomes. Xu and Yager [57] further developed power geometric (PG) operators, which consider the common support interrelatedness of aggregated values to reduce the impact of irrelevant information provided by experts. These operators are utilized in various fields. Xiong et al. [56] explored the use of power geometric operators for proportional hesitant fuzzy linguistic scenarios. Zindani et al. [63] explored a DM technique based on Schweizer–Sklar power AOs. Kou et al. [27] expanded on this by integrating the extended PA operator into the linguistic Pythagorean fuzzy context, introducing several new AOs. Recently, Liu et al. [31] proposed a Dombi extended power AOs utilizing single-valued neutrosophic credibility numbers and examined its application in selecting data collection schemes for intelligent transportation systems. However, no research has yet addressed the application of SuW operations within the context of CFFS or their use in DM techniques under CFFS. Therefore, it is crucial to integrate the strengths of both SuW and PA operators to develop effective AOs in this setting. This necessitates combining SuW operations and PA operators to create AOs such as CFF SuW power averaging (CFFSuWPA), CFF SuW power weighted averaging (CFFSuWPWA), CFF SuW power geometric (CFFSuWPG), and CFF SuW power weighted geometric (CFFSuWPWG). These newly derived operators can then be used to develop novel DM techniques for addressing DM challenges within the CFF information framework.

Based on the preceding discussion, the research motivations for this study are outlined as follows:

i). The CFFS empowers DEs by providing an extended representational space that accommodates both interval and crisp values along with degrees of hesitation. This enhanced expressiveness enables the capture of complex, vague, or conflicting information commonly encountered in real-world DM. By allowing for a more complex modeling of agreement and disagreement, the CFF framework supports flexible, precise, and context-sensitive evaluations—capabilities that are essential for navigating the layered uncertainties of modern group decision environments.

ii). In the application of CFFs, although algebraic TNOs and TCNOs are commonly used, researchers have highlighted that other advanced triangular operations—such as the SuW t-norms and co-norms—possess desirable properties for flexible information fusion [25, 26]. These operators offer general parameter-based structures that allow fine-grained control over the aggregation process. Integrating SuW TNO and TCO with the CFF framework enhances its modeling power, enabling more nuanced and adaptable aggregation under complex uncertainty.

iii). SuW-based operators also demonstrate strong capability in managing ambiguity and conflict in expert assessments, as emphasized by recent studies in fuzzy reasoning systems [45]. However, despite their proven value in traditional fuzzy contexts, they remain underutilized within the CFF domain. This research addresses this gap by embedding SuW-based operations directly into the CFF structure, enhancing its theoretical richness and real-world applicability across a wider range of uncertain decision environments.

iv). In many DM scenarios, expert assessments may be biased due to intentional exaggeration, uncertainty, or perceptual errors—especially in subjective domains such as service quality evaluation. Such distortions can mislead the outcome if not properly managed. To mitigate this, prior works have suggested the use of PA and PG operators [57, 59], which reduce the influence of extreme values. By incorporating these power-based strategies into a CFF framework, our model ensures balanced, objective, and distortion-resilient decision outcomes.

v). Accurately assigning weights to evaluation criteria is crucial in MCGDM, but in practical settings, such weight information is often partially known or entirely unavailable. The measurement of alternatives and ranking according to compromise solution method [42] effectively handles fully unknown weights; however, existing CFF-based models have not adequately addressed partial-weight uncertainty. This study contributes by incorporating a maximizing deviation-based model tailored for CFFSs, thereby closing this gap and enhancing the decision framework's realism and adaptability.

Building on the motivations outlined above, the primary contributions of this research are as follows:

i). Foundational operational laws and essential properties of CFF information are systematically established using SuW norms. These operations form the mathematical backbone for further analytical developments in the domain of FS-based DM under uncertainty.

ii). To address scenarios involving extremely large or small-scale data, we introduce a set of novel power-based AOs (CFFSuWPWA and CFFSuWPWG) grounded in SuW TNO and TCO. These operators offer adjustable flexibility and are rigorously analyzed to ensure they retain desirable algebraic and decision-theoretic properties.

iii). A maximizing deviation-based weight determination model is proposed, enabling robust computation of criteria weights when such information is entirely missing or only partially known. This approach improves realism in uncertain environments and avoids biases introduced by assumed or equal-weight strategies.

iv). A hybrid MCGDM algorithm is developed that seamlessly combines the proposed SuW-based operators with the maximizing deviation model. This integration enhances the model's capacity to handle group decision-making settings characterized by diverse expert inputs and uncertain judgments.

v). The proposed methodology is validated through a real-world application involving the performance evaluation of project team members. This practical example demonstrates the method's ability to produce reliable and discriminative rankings, even under conditions of imprecise and conflicting information.

The originality of this study lies in the synergy between CFFS and SuW-based power AOs—an integration not previously explored in the literature. While prior models have utilized basic t-norms or fixed operators within the CFF environment, the introduction of SuW-based operators provides adjustable flexibility, enabling more precise modeling of decision-maker attitudes and uncertainty levels. This is particularly useful in scenarios involving extreme values or highly imbalanced data, where traditional operators often fail to preserve discrimination between alternatives.

Moreover, the development of a maximizing deviation-based weighting model tailored for the CFF environment represents a significant innovation. Unlike existing techniques that often assume fully known or equal weights, our model handles partial and complete uncertainty in a mathematically consistent manner. This allows for more realistic application in real-world settings where weight information is vague or unavailable.

In combining these components into a unified hybrid DM framework, this research advances both theoretical foundations and practical tools for MCGDM. The proposed approach not only improves the interpretability and robustness of group decisions but also broadens the scope of CFF applications to complex, data-sensitive domains.

Based on the identified gaps and motivations discussed above, the following research questions are formulated to guide the current study:

i). How can SuW t-norms and t-conorms be effectively integrated into the CFFS framework to enhance modeling of uncertainty in MCGDM?

ii). What new power AOs can be developed under the CFFS environment, and how do they address biased or extreme evaluations from DEs?

iii). How can a maximizing deviation model be formulated within the CFFS context to determine criteria weights when such weights are partially or entirely unknown?

iv). What is the practical effectiveness and comparative advantage of the proposed CFF SuW-based DM technique in real-world scenarios involving incomplete, uncertain, or biased information?

The flow diagram representing the developed work is shown in Fig. 2. The subsequent sections of this paper are structured as follows: Section 2 briefly reviews key background concepts relevant to this research. Section 3 introduces core operations for CFF contexts utilizing SuW norms. In Section 4, several CFF SuW power operators are presented and their properties are examined. Section 5 presents a step-by-step procedure of the MCGDM technique under the formulated operators where the criteria weights are determined using the proposed maximizing deviation model. Section 6 validates the efficacy of the proposed methodology by applying it to a practical scenario. Furthermore, it employs a sensitivity analysis to investigate the susceptibility of the results to variations within the objects of analysis. Section 7 conducts a detailed comparative analysis with other prevailing studies. In Section 8, we conclude the paper.

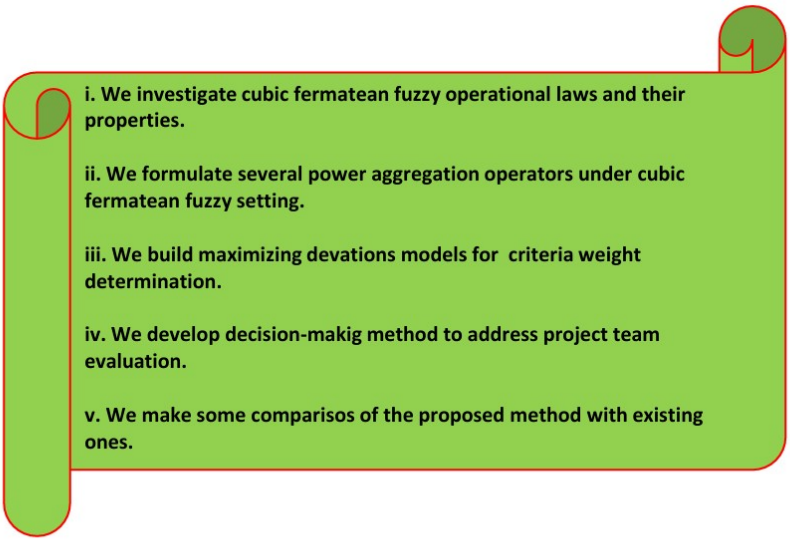

Figure 2: Graphical representation of the proposed work

2. Preliminaries

To facilitate a comprehensive understanding of the study, the following section presents the essential background information that will be subsequently employed.

Definition 2.1 ([33])**.** A mapping $\top : [0,1] \times [0,1] \longrightarrow [0,1]$ is considered TNO if it meets the following four conditions:

i. $\top(\mathsf{H}_1, \mathsf{H}_2) = \top(\mathsf{H}_2, \mathsf{H}_1)$;

ii. $\top(\mathsf{H}_1, \mathsf{H}_2) \leqslant \top(\mathsf{H}_3, \mathsf{H}_4)$ if $\mathsf{H}_1 \leqslant \mathsf{H}_3, \mathsf{H}_2 \leqslant \mathsf{H}_4$;

iii. $\top(\mathsf{H}, 1) = \mathsf{H}$;

iv. $\top(\mathsf{H}_1, \top(\mathsf{H}_2, \mathsf{H}_3)) = \top(\top(\mathsf{H}_1, \mathsf{H}_2), \mathsf{H}_3)$;

$\forall\, \mathsf{H}, \mathsf{H}_1, \mathsf{H}_2, \mathsf{H}_3, \mathsf{H}_4 \in [0,1]$.

Some examples of TNOs

1). $\top_P(\mathsf{H}_1, \mathsf{H}_2) = \mathsf{H}_1\mathsf{H}_2$ (product TNO),

2). $\top_M(\mathsf{H}_1, \mathsf{H}_2) = \min(\mathsf{H}_1, \mathsf{H}_2)$ (minimum TNO),

3). $\top_L(\mathsf{H}_1, \mathsf{H}_2) = \max(\mathsf{H}_1 + \mathsf{H}_2 - 1, 0)$ (Lukasiewicz TNO),

4). $\top_D(\mathsf{H}_1, \mathsf{H}_2) = \begin{cases} \mathsf{H}_1, & \text{if } \mathsf{H}_2 = 1 \\ \mathsf{H}_2, & \text{if } \mathsf{H}_1 = 1 \\ 0, & \text{otherwise.} \end{cases}$ (drastic TNO)

$\forall\, H_1, H_2 \in [0,1]$.

Definition 2.2 ([25]). A mapping $\perp : [0,1] \times [0,1] \longrightarrow [0,1]$ is considered TCO if it meets the following four conditions:

i. $\perp(H_1, H_2) = \perp(H_2, H_1)$;

ii. $\perp(H_1, H_2) \leqslant \perp(H_3, H_4)$ if $H_1 \leqslant H_3, H_2 \leqslant H_4$;

iii. $\perp(H, 0) = H$;

iv. $\perp(H_1, \perp(H_2, H_3)) = \perp(\perp(H_1, H_2), H_3)$;

$\forall\, H, H_1, H_2, H_3, H_4 \in [0,1]$.

Some examples of TCOs

1). $\perp_P(H_1, H_2) = H_1 + H_2 - H_1H_2$ (probabilistic sum),

2). $\perp_M(H_1, H_2) = \max(H_1, H_2)$ (maximum TCO),

3). $\perp_L(H_1, H_2) = \min(H_1 + H_2, 1)$ (Lukasiewicz TCO),

4). $\perp_D(H_1, H_2) = \begin{cases} H_1, & \text{if } H_2 = 0 \\ H_2, & \text{if } H_1 = 0 \text{ (drastic TCO)} \\ 1, & \text{otherwise.} \end{cases}$

$\forall\, H_1, H_2 \in [0,1]$.

Definition 2.3 ([23]). The SuW TNO and TCN are expressed, respectively, as follows:

$$\top^{\pounds}(H_1, H_2) = \begin{cases} \top_D(H_1, H_2), & \text{if } \pounds = -1 \\ \max\left\{0, \frac{H_1 + H_2 - 1 + \pounds H_1 H_2}{1 + \pounds}\right\}, & \text{if } -1 < \pounds < \infty\ \forall H_1, H_2 \in [0,1] \\ \top_P(H_1, H_2), & \text{if } \pounds = \infty \end{cases} \tag{2.1}$$

and

$$\perp^{\pounds}(H_1, H_2) = \begin{cases} \perp_D(H_1, H_2), & \text{if } \pounds = -1 \\ \min\left\{1, H_1 + H_2 - \frac{\pounds}{1+\pounds} H_1 H_2\right\}, & \text{if } -1 < \pounds < \infty\ \forall H_1, H_2 \in [0,1] \\ \perp_P(H_1, H_2), & \text{if } \pounds = \infty \end{cases} \tag{2.2}$$

Definition 2.4 ([48]). The Fermatean fuzzy set (FFS) A under the universal set $\mathcal{U}$ is given by

$$A = \{(x, \overline{m}(x), \overline{n}(x)) \,|\, x \in \mathcal{U}\}, \tag{2.3}$$

wherein $\overline{m}(x)$ and $\overline{n}(x)$ respectively denote the grades of affiliation and disaffiliation of the element $x \in \mathcal{U}$, with the constraint that $0 \leqslant (\overline{m}(x))^3 + (\overline{n}(x))^3 \leqslant 1$. The pair $A = (\overline{m}(x), \overline{n}(x))$ is used to represent a Fermatean fuzzy number (FFN), and is simplified as $A = (\overline{m}, \overline{n})$ with $0 \leqslant (\overline{m})^3 + (\overline{n})^3 \leqslant 1$. The hesitancy degree of $x \in \mathcal{U}$ is given by $\pi(x) = \sqrt[3]{1 - (\overline{m}(x))^3 - (\overline{n}(x))^3}$.

Subsequently, Sergi et al. [49] extended the FFS to develop the concept of interval-valued FFS. This was achieved by using interval numbers to represent the membership and non-membership values, thereby enhancing experts' ability to express uncertainty more effectively.

Definition 2.5 ([49]). The interval-valued FFS (IVFFS) B under the universal set $\mathcal{U}$ is given by

$$\mathrm{B} = \left\{ \left(x, \left[\overline{m}^{L}(x), \overline{m}^{U}(x) \right], \left[\overline{n}^{L}(x), \overline{n}^{U}(x) \right] \right) | x \in \mathcal{U} \right\}, \tag{2.4}$$

wherein $\left[\overline{m}^{L}(x), \overline{m}^{U}(x)\right]$ and $\left[\overline{n}^{L}(x), \overline{n}^{U}(x)\right]$ respectively denote the grades of affiliation and disaffiliation of the element $x \in \mathcal{U}$, such that $0 \leqslant \overline{m}^{L}(x) \leqslant \overline{m}^{U}(x) \leqslant 1, 0 \leqslant \overline{n}^{L}(x) \leqslant \overline{n}^{U}(x) \leqslant 1$ and $0 \leqslant \left(\overline{m}^{U}(x)\right)^{3} + \left(\overline{n}^{U}(x)\right)^{3} \leqslant 1$. The pair $\mathrm{B} = \left(\left[\overline{m}^{L}(x), \overline{m}^{U}(x)\right], \left[\overline{n}^{L}(x), \overline{n}^{U}(x)\right]\right)$ is used to represent an interval-valued Fermatean fuzzy number (IVFFN), and is simplified as $\mathrm{B} = \left(\left[\overline{m}^{L}, \overline{m}^{U}\right], \left[\overline{n}^{L}, \overline{n}^{U}\right]\right)$ with $0 \leqslant \left(\overline{m}^{U}\right)^{3} + \left(\overline{n}^{U}\right)^{3} \leqslant 1$. The hesitancy degree of $x \in \mathcal{U}$ is given by $\pi(x) = \left[\pi^{L}(x), \pi^{U}(x)\right]$, where $\pi^{L}(x) = \sqrt[3]{1 - \left(\overline{m}^{L}(x)\right)^{3} - \left(\overline{n}^{L}(x)\right)^{3}}$ and $\pi^{U}(x) = \sqrt[3]{1 - \left(\overline{m}^{U}(x)\right)^{3} - \left(\overline{n}^{U}(x)\right)^{3}}$.

Definition 2.6 ([42]). The cubic FFS (CFFS) F under the universal set $\mathcal{U}$ is given by

$$\mathrm{F} = \{(x, \curlyvee(x), \curlywedge(x)) | x \in \mathcal{U}\}, \tag{2.5}$$

where $\curlyvee(x) = ([\underline{m}(x), \overline{m}(x)], [\underline{n}(x), \overline{n}(x)])$ denotes IVFFN while $\curlywedge(x) = (\underset{\cdot}{t}(x), \beta(x))$ denotes FFN for all $x \in X$ such that $0 \leqslant \underline{m}(x) \leqslant \overline{m}(x) \leqslant 1$, $0 \leqslant \underline{n}(x) \leqslant \overline{n}(x) \leqslant 1$ and $0 \leqslant (\overline{m}(x))^{3} + (\overline{n}(x))^{3} \leqslant 1$. Also, $0 \leqslant \underset{\cdot}{t}(x), \beta(x) \leqslant 1$ and $0 \leqslant (\underset{\cdot}{t}(x))^{3} + (\beta(x))^{3} \leqslant 1$. To keep it simple, the pair $\mathrm{F} = (([\underline{m}, \overline{m}], [\underline{n}, \overline{n}]), (\underset{\cdot}{t}, \beta))$ is known as cubic FFN (CFFN).

Definition 2.7 ([15]). Assume three CFFNs F, F_1, F_2 and $\wr > 0$. Then we have

1. $\mathrm{F}_1 \oplus \mathrm{F}_2 = \left(\begin{array}{c} \left(\left[\left(\underline{m}_1^3 + \underline{m}_2^3 - \underline{m}_1^3\underline{m}_2^3\right)^{1/3}, \left(\overline{m}_1^3 + \overline{m}_2^3 - \overline{m}_1^3\overline{m}_2^3\right)^{1/3}\right], [\underline{n}_1\underline{n}_2, \overline{n}_1\overline{n}_2]\right), \\ \left(\left(\underset{\cdot}{t}_1^3 + \underset{\cdot}{t}_2^3 - \underset{\cdot}{t}_1^3\underset{\cdot}{t}_2^3\right)^{1/3}, \beta_1\beta_2\right) \end{array} \right)$;
2. $\mathrm{F}_1 \otimes \mathrm{F}_2 = \left(\begin{array}{c} \left([\underline{m}_1\underline{m}_2, \overline{m}_1\overline{m}_2], \left[\left(\underline{n}_1^3 + \underline{n}_2^3 - \underline{n}_1^3\underline{n}_2^3\right)^{1/3}, \left(\overline{n}_1^3 + \overline{n}_2^3 - \overline{n}_1^3\overline{n}_2^3\right)^{1/3}\right]\right), \\ \left(\underset{\cdot}{t}_1\underset{\cdot}{t}_2, \left(\beta_1^3 + \beta_2^3 - \beta_1^3\beta_2^3\right)^{1/3}\right) \end{array} \right)$;
3. $\mathrm{F}^{\wr} = \left(\begin{array}{c} \left(\left[\underline{m}^{\wr}, \overline{m}^{\wr}\right], \left[\left(1 - \left(1 - \underline{n}^3\right)^{\wr}\right)^{1/3}, \left(1 - \left(1 - \overline{n}^3\right)^{\wr}\right)^{1/3}\right]\right), \\ \left(\underset{\cdot}{t}_{\mathrm{F}}^{\wr}, \left(1 - \left(1 - \beta^3\right)^{\wr}\right)^{1/3}\right) \end{array} \right)$;
4. $\wr\mathrm{F} = \left(\begin{array}{c} \left(\left[\left(1 - \left(1 - \underline{m}^3\right)^{\wr}\right)^{1/3}, \left(1 - \left(1 - \overline{m}^3\right)^{\wr}\right)^{1/3}\right], \left[\underline{n}^{\wr}, \overline{n}^{\wr}\right]\right), \\ \left(\left(1 - \left(1 - \underset{\cdot}{t}^3\right)^{\wr}\right)^{1/3}, \beta^{\wr}\right) \end{array} \right)$;
5. $\mathrm{F}^{c} = (([\underline{n}, \overline{n}], [\underline{m}, \overline{m}]), (\beta, \underset{\cdot}{t}))$.

Definition 2.8 ([42]). Let F be a CFFN. Then the score S and A are given as follows:

$$\mathrm{S}(\mathrm{F}) = \frac{1}{2}\left[\frac{1}{4}\left[\underline{m}^3 - \underline{n}^3 + \overline{m}^3 - \overline{n}^3\right] + \frac{1}{2}\left(\underset{\cdot}{t}^3 - \beta^3 + 1\right)\right], \mathrm{S}(\mathrm{F}) \in [0,1], \tag{2.6}$$

$$\mathrm{A}(\mathrm{F}) = \frac{1}{2}\left[\frac{1}{2}\left[\underline{m}^3 + \underline{n}^3 + \overline{m}^3 + \overline{n}^3\right] + \left(\underset{\cdot}{t}^3 + \beta^3\right)\right], \mathrm{A}(\mathrm{F}) \in [0,1]. \tag{2.7}$$

For two CFFNs F_1 and F_2, if $\mathrm{S}(\mathrm{F}_1) > \mathrm{S}(\mathrm{F}_2)$, then F_1 is considered superior to F_2. If $\mathrm{S}(\mathrm{F}_1) = \mathrm{S}(\mathrm{F}_2)$, we then compute their accuracy function: if $\mathrm{A}(\mathrm{F}_1) > \mathrm{A}(\mathrm{F}_2)$, then F_1 is superior to F_2, if $\mathrm{A}(\mathrm{F}_1) = \mathrm{A}(\mathrm{F}_2)$, then F_1 is equivalent to F_2.

Definition 2.9 ([42]). Let F_1, F_2 be two CFFNs, then the CFF Euclidean distance measure between them is given by

$$d(\mathrm{F}_1, \mathrm{F}_2) = \left(\frac{1}{6}\left(\left|\underline{m}_1^3 - \underline{m}_2^3\right|^2 + \left|\overline{m}_1^3 - \overline{m}_2^3\right|^2 + \left|\underline{n}_1^3 - \underline{n}_2^3\right|^2 + \left|\overline{n}_1^3 - \overline{n}_2^3\right|^2 + \left|\underset{\cdot}{t}_1^3 - \underset{\cdot}{t}_2^3\right|^2 + \left|\beta_1^3 - \beta_2^3\right|^2\right)\right)^{\frac{1}{2}}. \tag{2.8}$$

Definition 2.10 ([59]). Let $R_j (j = 1, 2, .., n)$ be a collection of real numbers $(R_j \geqslant 0)$. Then PA operator is given by

$$PA(R_1, R_2, ..., R_n) = \sum_{j=1}^{n} \frac{R_j (1 + T(R_j))}{\sum_{j=1}^{n} (1 + T(R_j))}, \tag{2.9}$$

and weighted power average (WPA) operator is expressed as

$$WPA(R_1, R_2, ..., R_n) = \sum_{j=1}^{n} \frac{R_j (1 + T(R_j)) w_j}{\sum_{j=1}^{n} (1 + T(R_j))}. \tag{2.10}$$

Definition 2.11 ([57]). Let $R_j (j = 1, 2, .., n)$ be a collection of real numbers $(R_j \geqslant 0)$. Then PG operator is given by

$$PG(R_1, R_2, ..., R_n) = \prod_{j=1}^{n} R_j^{\frac{(1+T(R_j))}{\sum_{j=1}^{n}(1+T(R_j))}}, \tag{2.11}$$

and weighted power geometric (WPG) operator is expressed as

$$WPG(R_1, R_2, ..., R_n) = \prod_{j=1}^{n} R_j^{\frac{w_j(1+T(R_j))}{\sum_{j=1}^{n}(1+T(R_j))}}. \tag{2.12}$$

The variables used in Definitions 2.10 and 2.11 can be described as follows: $w_j \in [0, 1]$ is the component of a weight vector. $T(R_j) = \sum_{i=1}^{n} Sup(R_j, R_i)_{j \neq i}$, where $Sup(R_j, R_i)$ is the support measure that fulfills the following conditions: i). $Sup(R_j, R_i) \in [0, 1]$, ii). $Sup(R_j, R_i) = Sup(R_i, R_j)$, iii). $Sup(R_j, R_i) \geqslant Sup(R_l, R_k)$ if $|R_j - R_i| < |R_l - R_k|$. $Sup(R_j, R_i) = 1 - d(R_j, R_i)$, where d denotes the distance measure.

3. Sugeno–Weber Triangular Norm Based Operations

This section explores the operations and some properties of SuW operations within the framework of CFFNs. We achieve the extension of these SuW operations by incorporating both SuW-TNO and SuW-TCN.

Definition 3.1. Assume three CFFNs F, F_1, F_2 and $\wr > 0$. Then, the basic operations of SuW norms are defined as follows:

a) $$F_1 \oplus F_2 = \left(\begin{array}{c} \left(\begin{array}{c} \left[\sqrt[3]{\underline{m}_1^3 + \underline{m}_2^3 - \frac{\pounds}{1+\pounds}\underline{m}_1^3\underline{m}_2^3}, \sqrt[3]{\overline{m}_1^3 + \overline{m}_2^3 - \frac{\pounds}{1+\pounds}\overline{m}_1^3\overline{m}_2^3} \right], \\ \left[\sqrt[3]{\frac{\underline{n}_1^3 + \underline{n}_2^3 - 1 + \pounds\underline{n}_1^3\underline{n}_2^3}{1+\pounds}}, \sqrt[3]{\frac{\overline{n}_1^3 + \overline{n}_2^3 - 1 + \pounds\overline{n}_1^3\overline{n}_2^3}{1+\pounds}} \right] \end{array} \right), \\ \left(\sqrt[3]{t_1^3 + t_2^3 - \frac{\pounds}{1+\pounds}t_1^3t_2^3}, \sqrt[3]{\frac{\beta_1^3 + \beta_2^3 - 1 + \pounds\beta_1^3\beta_2^3}{1+\pounds}} \right) \end{array} \right);$$

b) $$F_1 \otimes F_2 = \left(\begin{array}{c} \left(\begin{array}{c} \left[\sqrt[3]{\frac{\underline{m}_1^3 + \underline{m}_2^3 - 1 + \pounds\underline{m}_1^3\underline{m}_2^3}{1+\pounds}}, \sqrt[3]{\frac{\overline{m}_1^3 + \overline{m}_2^3 - 1 + \pounds\overline{m}_1^3\overline{m}_2^3}{1+\pounds}} \right], \\ \left[\sqrt[3]{\underline{n}_1^3 + \underline{n}_2^3 - \frac{\pounds}{1+\pounds}\underline{n}_1^3\underline{n}_2^3}, \sqrt[3]{\overline{n}_1^3 + \overline{n}_2^3 - \frac{\pounds}{1+\pounds}\overline{n}_1^3\overline{n}_2^3} \right] \end{array} \right), \\ \left(\sqrt[3]{\frac{t_1^3 + t_2^3 - 1 + \pounds t_1^3t_2^3}{1+\pounds}}, \sqrt[3]{\beta_1^3 + \beta_2^3 - \frac{\pounds}{1+\pounds}\beta_1^3\beta_2^3} \right) \end{array} \right);$$

c) $$F^{\wr}=\left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\underline{m}^3+1}{1+\pounds}\right)^{\wr}-1\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\overline{m}^3+1}{1+\pounds}\right)^{\wr}-1\right)}\right],\\ \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\underline{n}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\wr}\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\overline{n}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\wr}\right)}\right]\end{array}\right),\\ \left(\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds t^3+1}{1+\pounds}\right)^{\wr}-1\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\beta^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\wr}\right)}\right)\end{array}\right);$$

d) $$\wr F=\left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\underline{m}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\wr}\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\overline{m}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\wr}\right)}\right],\\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\underline{n}^3+1}{1+\pounds}\right)^{\wr}-1\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\overline{n}^3+1}{1+\pounds}\right)^{\wr}-1\right)}\right]\end{array}\right),\\ \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-t^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\wr}\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\beta^3+1}{1+\pounds}\right)^{\wr}-1\right)}\right)\end{array}\right).$$

Theorem 3.2. *Let F_1, F_2, F_3 be three CFFNs, and $\wr>0$ then the following properties hold.*

i) $F_1\oplus F_2=F_2\oplus F_1$;

ii) $F_1\otimes F_2=F_2\otimes F_1$;

iii) $(F_1\oplus F_2)\oplus F_3=F_1\oplus(F_2\oplus F_3)$;

iv) $(F_1\otimes F_2)\otimes F_3=F_1\otimes(F_2\otimes F_3)$;

v) $\wr(F_1\oplus F_2)=\wr F_1\oplus\wr F_2$;

vi) $(F_1\otimes F_2)^{\wr}=F_1^{\wr}\otimes F_2^{\wr}$.

Proof. **i)** Consider $F_1\oplus F_2=$

$$\left(\begin{array}{c}\left(\left[\sqrt[3]{\underline{m}_1^3+\underline{m}_2^3-\frac{\pounds}{1+\pounds}\underline{m}_1^3\underline{m}_2^3},\sqrt[3]{\overline{m}_1^3+\overline{m}_2^3-\frac{\pounds}{1+\pounds}\overline{m}_1^3\overline{m}_2^3}\right],\left[\sqrt[3]{\frac{\underline{n}_1^3+\underline{n}_2^3-1+\pounds\underline{n}_1^3\underline{n}_2^3}{1+\pounds}},\sqrt[3]{\frac{\overline{n}_1^3+\overline{n}_2^3-1+\pounds\overline{n}_1^3\overline{n}_2^3}{1+\pounds}}\right]\right),\\ \left(\sqrt[3]{t_1^3+t_2^3-\frac{\pounds}{1+\pounds}t_1^3t_2^3},\sqrt[3]{\frac{\beta_1^3+\beta_2^3-1+\pounds\beta_1^3\beta_2^3}{1+\pounds}}\right)\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\left[\sqrt[3]{\underline{m}_2^3+\underline{m}_1^3-\frac{\pounds}{1+\pounds}\underline{m}_2^3\underline{m}_1^3},\sqrt[3]{\overline{m}_2^3+\overline{m}_1^3-\frac{\pounds}{1+\pounds}\overline{m}_2^3\overline{m}_1^3}\right],\left[\sqrt[3]{\frac{\underline{n}_2^3+\underline{n}_1^3-1+\pounds\underline{n}_2^3\underline{n}_1^3}{1+\pounds}},\sqrt[3]{\frac{\overline{n}_2^3+\overline{n}_1^3-1+\pounds\overline{n}_2^3\overline{n}_1^3}{1+\pounds}}\right]\right),\\ \left(\sqrt[3]{t_2^3+t_1^3-\frac{\pounds}{1+\pounds}t_2^3t_1^3},\sqrt[3]{\frac{\beta_2^3+\beta_1^3-1+\pounds\beta_2^3\beta_1^3}{1+\pounds}}\right)\end{array}\right)$$

$=F_2\oplus F_1$.

ii) Similar to Part i).

iii) $(F_1\oplus F_2)\oplus F_3=$

$$\left(\left(\begin{array}{c}\left(\left[\sqrt[3]{\underline{m}_1^3+\underline{m}_2^3-\frac{\pounds}{1+\pounds}\underline{m}_1^3\underline{m}_2^3},\sqrt[3]{\overline{m}_1^3+\overline{m}_2^3-\frac{\pounds}{1+\pounds}\overline{m}_1^3\overline{m}_2^3}\right],\left[\sqrt[3]{\frac{\underline{n}_1^3+\underline{n}_2^3-1+\pounds\underline{n}_1^3\underline{n}_2^3}{1+\pounds}},\sqrt[3]{\frac{\overline{n}_1^3+\overline{n}_2^3-1+\pounds\overline{n}_1^3\overline{n}_2^3}{1+\pounds}}\right]\right),\\ \left(\sqrt[3]{t_1^3+t_2^3-\frac{\pounds}{1+\pounds}t_1^3t_2^3},\sqrt[3]{\frac{\beta_1^3+\beta_2^3-1+\pounds\beta_1^3\beta_2^3}{1+\pounds}}\right)\end{array}\right)\right)\oplus F_3$$

$$=\left(\begin{array}{c}\left[\begin{array}{c}\sqrt[3]{\left(\underline{m}_1^3+\underline{m}_2^3-\frac{\pounds}{1+\pounds}\underline{m}_1^3\underline{m}_2^3\right)+\underline{m}_3^3-\frac{\pounds}{1+\pounds}\left(\underline{m}_1^3+\underline{m}_2^3-\frac{\pounds}{1+\pounds}\underline{m}_1^3\underline{m}_2^3\right)\underline{m}_3^3},\\ \sqrt[3]{\left(\overline{m}_1^3+\overline{m}_2^3-\frac{\pounds}{1+\pounds}\overline{m}_1^3\overline{m}_2^3\right)+\overline{m}_3^3-\frac{\pounds}{1+\pounds}\left(\overline{m}_1^3+\overline{m}_2^3-\frac{\pounds}{1+\pounds}\overline{m}_1^3\overline{m}_2^3\right)\overline{m}_3^3}\end{array}\right],\\ \left[\sqrt[3]{\frac{\left(\frac{\underline{n}_1^3+\underline{n}_2^3-1+\pounds\underline{n}_1^3\underline{n}_2^3}{1+\pounds}\right)+\underline{n}_3^3-1+\pounds\left(\frac{\underline{n}_1^3+\underline{n}_2^3-1+\pounds\underline{n}_1^3\underline{n}_2^3}{1+\pounds}\right)\underline{n}_3^3}{1+\pounds}},\sqrt[3]{\frac{\left(\frac{\overline{n}_1^3+\overline{n}_2^3-1+\pounds\overline{n}_1^3\overline{n}_2^3}{1+\pounds}\right)+\overline{n}_3^3-1+\pounds\left(\frac{\overline{n}_1^3+\overline{n}_2^3-1+\pounds\overline{n}_1^3\overline{n}_2^3}{1+\pounds}\right)\overline{n}_3^3}{1+\pounds}}\right],\\ \left(\begin{array}{c}\sqrt[3]{\left(t_1^3+t_2^3-\frac{\pounds}{1+\pounds}t_1^3t_2^3\right)+t_3^3-\frac{\pounds}{1+\pounds}\left(t_1^3+t_2^3-\frac{\pounds}{1+\pounds}t_1^3t_2^3\right)t_3^3},\\ \sqrt[3]{\frac{\left(\frac{\beta_1^3+\beta_2^3-1+\pounds\beta_1^3\beta_2^3}{1+\pounds}\right)+\beta_3^3-1+\pounds\left(\frac{\beta_1^3+\beta_2^3-1+\pounds\beta_1^3\beta_2^3}{1+\pounds}\right)\beta_3^3}{1+\pounds}}\end{array}\right)\end{array}\right)$$

$$= \left(\begin{array}{c} \left[\begin{array}{c} \sqrt[3]{\underline{m}_1^3 + \left(\underline{m}_2^3+\underline{m}_3^3 - \frac{\pounds}{1+\pounds}\underline{m}_2^3\underline{m}_3^3\right) - \frac{\pounds}{1+\pounds}\underline{m}_1^3\left(\underline{m}_2^3+\underline{m}_3^3-\frac{\pounds}{1+\pounds}\underline{m}_2^3\underline{m}_3^3\right)}, \\ \sqrt[3]{\overline{m}_1^3 + \left(\overline{m}_2^3+\overline{m}_3^3 - \frac{\pounds}{1+\pounds}\overline{m}_2^3\overline{m}_3^3\right) - \frac{\pounds}{1+\pounds}\overline{m}_1^3\left(\overline{m}_2^3+\overline{m}_3^3-\frac{\pounds}{1+\pounds}\overline{m}_2^3\overline{m}_3^3\right)} \end{array} \right], \\ \left[\sqrt[3]{\frac{\underline{n}_1^3 + \left(\frac{\underline{n}_2^3+\underline{n}_3^3-1+\pounds\underline{n}_2^3\underline{n}_3^3}{1+\pounds}\right) - 1 + \pounds\underline{n}_1^3\left(\frac{\underline{n}_2^3+\underline{n}_3^3-1+\pounds\underline{n}_2^3\underline{n}_3^3}{1+\pounds}\right)}{1+\pounds}}, \sqrt[3]{\frac{\overline{n}_1^3 + \left(\frac{\overline{n}_2^3+\overline{n}_3^3-1+\pounds\overline{n}_2^3\overline{n}_3^3}{1+\pounds}\right) - 1 + \pounds\overline{n}_1^3\left(\frac{\overline{n}_2^3+\overline{n}_3^3-1+\pounds\overline{n}_2^3\overline{n}_3^3}{1+\pounds}\right)}{1+\pounds}} \right], \\ \left(\begin{array}{c} \sqrt[3]{t_1^3 + \left(t_2^3+t_3^3 - \frac{\pounds}{1+\pounds}t_2^3t_3^3\right) - \frac{\pounds}{1+\pounds}t_1^3\left(t_2^3+t_3^3-\frac{\pounds}{1+\pounds}t_2^3t_3^3\right)}, \\ \sqrt[3]{\frac{\beta_1^3 + \left(\frac{\beta_2^3+\beta_3^3-1+\pounds\beta_2^3\beta_3^3}{1+\pounds}\right) - 1 + \pounds\beta_1^3\left(\frac{\beta_2^3+\beta_3^3-1+\pounds\beta_2^3\beta_3^3}{1+\pounds}\right)}{1+\pounds}} \end{array} \right) \end{array} \right)$$

$$= F_1 \oplus \left(\left(\left(\begin{array}{c} \left[\sqrt[3]{\underline{m}_2^3+\underline{m}_3^3-\frac{\pounds}{1+\pounds}\underline{m}_2^3\underline{m}_3^3}, \sqrt[3]{\overline{m}_2^3+\overline{m}_3^3-\frac{\pounds}{1+\pounds}\overline{m}_2^3\overline{m}_3^3} \right], \\ \left[\sqrt[3]{\frac{\underline{n}_2^3+\underline{n}_3^3-1+\pounds\underline{n}_2^3\underline{n}_3^3}{1+\pounds}}, \sqrt[3]{\frac{\overline{n}_2^3+\overline{n}_3^3-1+\pounds\overline{n}_2^3\overline{n}_3^3}{1+\pounds}} \right] \end{array} \right), \\ \left(\sqrt[3]{t_2^3+t_3^3-\frac{\pounds}{1+\pounds}t_2^3t_3^3}, \sqrt[3]{\frac{\beta_2^3+\beta_3^3-1+\pounds\beta_2^3\beta_3^3}{1+\pounds}} \right) \right) \right)$$

$$= F_1 \oplus (F_2 \oplus F_3).$$

iv) Similar to Part iii).

v) $$\wr(F_1 \oplus F_2) = \wr \left(\left(\begin{array}{c} \left[\sqrt[3]{\underline{m}_1^3+\underline{m}_2^3-\frac{\pounds}{1+\pounds}\underline{m}_1^3\underline{m}_2^3}, \sqrt[3]{\overline{m}_1^3+\overline{m}_2^3-\frac{\pounds}{1+\pounds}\overline{m}_1^3\overline{m}_2^3} \right], \\ \left[\sqrt[3]{\frac{\underline{n}_1^3+\underline{n}_2^3-1+\pounds\underline{n}_1^3\underline{n}_2^3}{1+\pounds}}, \sqrt[3]{\frac{\overline{n}_1^3+\overline{n}_2^3-1+\pounds\overline{n}_1^3\overline{n}_2^3}{1+\pounds}} \right] \end{array} \right), \\ \left(\sqrt[3]{t_1^3+t_2^3-\frac{\pounds}{1+\pounds}t_1^3t_2^3}, \sqrt[3]{\frac{\beta_1^3+\beta_2^3-1+\pounds\beta_1^3\beta_2^3}{1+\pounds}} \right) \right)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left[\begin{array}{c} \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\left(\underline{m}_1^3+\underline{m}_2^3-\frac{\pounds}{1+\pounds}\underline{m}_1^3\underline{m}_2^3\right)\left(\frac{\pounds}{1+\pounds}\right)\right)^{\wr}\right)}, \\ \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\left(\overline{m}_1^3+\overline{m}_2^3-\frac{\pounds}{1+\pounds}\overline{m}_1^3\overline{m}_2^3\right)\left(\frac{\pounds}{1+\pounds}\right)\right)^{\wr}\right)} \end{array} \right], \\ \left[\begin{array}{c} \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\left(\frac{\underline{n}_1^3+\underline{n}_2^3-1+\pounds\underline{n}_1^3\underline{n}_2^3}{1+\pounds}\right)+1}{1+\pounds}\right)^{\wr}-1\right)}, \\ \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\left(\frac{\overline{n}_1^3+\overline{n}_2^3-1+\pounds\overline{n}_1^3\overline{n}_2^3}{1+\pounds}\right)+1}{1+\pounds}\right)^{\wr}-1\right)} \end{array} \right] \end{array} \right), \\ \left(\begin{array}{c} \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\left(t_1^3+t_2^3-\frac{\pounds}{1+\pounds}t_1^3t_2^3\right)\left(\frac{\pounds}{1+\pounds}\right)\right)^{\wr}\right)}, \\ \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\left(\frac{\beta_1^3+\beta_2^3-1+\pounds\beta_1^3\beta_2^3}{1+\pounds}\right)+1}{1+\pounds}\right)^{\wr}-1\right)} \end{array} \right) \end{array} \right)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\underline{m}_1^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\wr}\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\overline{m}_1^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\wr}\right)} \right], \\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\underline{n}_1^3+1}{1+\pounds}\right)^{\wr}-1\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\overline{n}_1^3+1}{1+\pounds}\right)^{\wr}-1\right)} \right] \end{array} \right), \\ \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-t_1^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\wr}\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\beta_1^3+1}{1+\pounds}\right)^{\wr}-1\right)} \right) \end{array} \right)$$

$$\oplus\left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\underline{m}_2^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\wr}\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\overline{m}_2^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\wr}\right)}\right],\\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\underline{n}_2^3+1}{1+\pounds}\right)^{\wr}-1\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\overline{n}_2^3+1}{1+\pounds}\right)^{\wr}-1\right)}\right]\end{array}\right),\\ \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-t_2^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\wr}\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\beta_2^3+1}{1+\pounds}\right)^{\wr}-1\right)}\right)\end{array}\right)=\wr F_1\oplus\wr F_2.$$

vi) Similar to Part v).

□

4. Cubic Fermatean Fuzzy Sugeno–Weber Power aggregation operators

This section offers several averaging and geometric AOs that use the proposed operational principles to effectively assemble m, n-CQROF data.

4.1. *Cubic Fermatean Fuzzy Sugeno–Weber averaging aggregation operators*

Building on the operational laws of SuW t-norms, we develop new mathematical operators for CFF information, specifically the CFF Sugeno-Weber average (CFFSuWPA) and CFF Sugeno-Weber power weighted average (CFFSuWPWA) operators.

Definition 4.1. Let $F_j=\left(\left(\left[\underline{m}_j,\overline{m}_j\right],\left[\underline{n}_j,\overline{n}_j\right]\right),\left(t_j,\beta_j\right)\right)$; $j=1,2,...,n$ be a class of CFFNs. Then the CFFSuWPA operator is defined as follows:

$$\mathrm{CFFSuWPA}\left(F_1,F_2,...,F_n\right)=\overset{n}{\underset{j=1}{\oplus}}\Phi_j F_j, \tag{4.1}$$

where $\Phi_j=\frac{\left(1+T\left(F_j\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(F_j\right)\right)}$, and $T\left(F_j\right)=\sum_{i=1}^{n}\mathrm{Sup}\left(F_j,F_i\right)_{j\neq i}$.

Theorem 4.2. *For a class of CFFNs* $F_j=\left(\left(\left[\underline{m}_j,\overline{m}_j\right],\left[\underline{n}_j,\overline{n}_j\right]\right),\left(t_j,\beta_j\right)\right)$; $j=1,2,...,n$, *the aggregated value of the CFFSuWPA operator remains a CFFN, resulting in the following:*

$$\begin{aligned}&\mathrm{CFFSuWPA}\left(F_1,F_2,...,F_n\right)\\&=\left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\underline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\overline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}\right],\\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\underline{n}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\overline{n}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}\right]\end{array}\right),\\ \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-t_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\beta_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}\right)\end{array}\right).\end{aligned} \tag{4.2}$$

Proof. We can use the induction approach to show the preceding expression. For $n=2$, we have

$$\Phi_1F_1=\left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\underline{m}_1^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_1}\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\overline{m}_1^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_1}\right)}\right],\\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\underline{n}_1^3+1}{1+\pounds}\right)^{\Phi_1}-1\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\overline{n}_1^3+1}{1+\pounds}\right)^{\Phi_1}-1\right)}\right]\end{array}\right),\\ \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-t_1^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_1}\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\beta_1^3+1}{1+\pounds}\right)^{\Phi_1}-1\right)}\right)\end{array}\right),$$

$$\Phi_2F_2=\left(\left(\begin{array}{c}\left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\underline{m}_{2}^{3}\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_2}\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\overline{m}_{2}^{3}\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_2}\right)}\right],\\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\underline{n}_{2}^{3}+1}{1+\pounds}\right)^{\Phi_2}-1\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\overline{n}_{2}^{3}+1}{1+\pounds}\right)^{\Phi_2}-1\right)}\right]\end{array}\right),\begin{array}{c}\\ \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\underset{\cdot}{t}_{2}^{3}\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_2}\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\beta_{2}^{3}+1}{1+\pounds}\right)^{\Phi_2}-1\right)}\right)\end{array}\right),$$

$$CFFSuWPA\left(F_1,F_2\right)=\overset{2}{\underset{j=1}{\oplus}}\Phi_jF_j=\Phi_1F_1\oplus\Phi_2F_2$$

$$\begin{aligned}&=\left(\left(\begin{array}{c}\left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\underline{m}_{1}^{3}\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_1}\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\overline{m}_{1}^{3}\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_1}\right)}\right],\\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\underline{n}_{1}^{3}+1}{1+\pounds}\right)^{\Phi_1}-1\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\overline{n}_{1}^{3}+1}{1+\pounds}\right)^{\Phi_1}-1\right)}\right]\end{array}\right),\begin{array}{c}\\ \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\underset{\cdot}{t}_{1}^{3}\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_1}\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\beta_{1}^{3}+1}{1+\pounds}\right)^{\Phi_1}-1\right)}\right)\end{array}\right)\\ &\oplus\left(\left(\begin{array}{c}\left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\underline{m}_{2}^{3}\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_2}\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\overline{m}_{2}^{3}\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_2}\right)}\right],\\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\underline{n}_{2}^{3}+1}{1+\pounds}\right)^{\Phi_2}-1\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\overline{n}_{2}^{3}+1}{1+\pounds}\right)^{\Phi_2}-1\right)}\right]\end{array}\right),\begin{array}{c}\\ \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\underset{\cdot}{t}_{2}^{3}\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_2}\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\beta_{2}^{3}+1}{1+\pounds}\right)^{\Phi_2}-1\right)}\right)\end{array}\right)\\ &=\left(\left(\begin{array}{c}\left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{2}\left(1-\underline{m}_{j}^{3}\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{2}\left(1-\overline{m}_{j}^{3}\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}\right],\\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{2}\left(\frac{\pounds\underline{n}_{j}^{3}+1}{1+\pounds}\right)^{\Phi_j}-1\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{2}\left(\frac{\pounds\overline{n}_{j}^{3}+1}{1+\pounds}\right)^{\Phi_j}-1\right)}\right]\end{array}\right),\begin{array}{c}\\ \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{2}\left(1-\underset{\cdot}{t}_{j}^{3}\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{2}\left(\frac{\pounds\beta_{j}^{3}+1}{1+\pounds}\right)^{\Phi_j}-1\right)}\right)\end{array}\right).\end{aligned}$$

Assume that Eq. (4.2) is valid for $n=k$.

$$CFFSuWPA\left(F_1,F_2,...,F_k\right)=\overset{k}{\underset{j=1}{\oplus}}\Phi_jF_j=$$

$$\left(\left(\begin{array}{c}\left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k}\left(1-\underline{m}_{j}^{3}\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k}\left(1-\overline{m}_{j}^{3}\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}\right],\\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k}\left(\frac{\pounds\underline{n}_{j}^{3}+1}{1+\pounds}\right)^{\Phi_j}-1\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k}\left(\frac{\pounds\overline{n}_{j}^{3}+1}{1+\pounds}\right)^{\Phi_j}-1\right)}\right]\end{array}\right),\begin{array}{c}\\ \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k}\left(1-\underset{\cdot}{t}_{j}^{3}\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k}\left(\frac{\pounds\beta_{j}^{3}+1}{1+\pounds}\right)^{\Phi_j}-1\right)}\right)\end{array}\right). \quad (4.3)$$

Following this, we can verify for $n=k+1$.

$$\text{CFFSuWPA}(F_1, F_2, ..., F_{k+1}) = \overset{k}{\underset{j=1}{\oplus}} \Phi_j F_j \oplus \Phi_{k+1} F_{k+1} =$$

$$\left(\left(\begin{array}{c} \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k}\left(1-\underline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k}\left(1-\overline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)} \right], \\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k}\left(\frac{\pounds\underline{n}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k}\left(\frac{\pounds\overline{n}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)} \right] \end{array} \right), \atop \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k}\left(1-t_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k}\left(\frac{\pounds\beta_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)} \right) \right)$$

$$\oplus \left(\left(\begin{array}{c} \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\underline{m}_{k+1}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_{k+1}}\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\overline{m}_{k+1}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_{k+1}}\right)} \right], \\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\underline{n}_{k+1}^3+1}{1+\pounds}\right)^{\Phi_{k+1}}-1\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\overline{n}_{k+1}^3+1}{1+\pounds}\right)^{\Phi_{k+1}}-1\right)} \right] \end{array} \right), \atop \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-t_{k+1}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_{k+1}}\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\beta_{k+1}^3+1}{1+\pounds}\right)^{\Phi_{k+1}}-1\right)} \right) \right)$$

$$= \left(\left(\begin{array}{c} \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k+1}\left(1-\underline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k+1}\left(1-\overline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)} \right], \\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k+1}\left(\frac{\pounds\underline{n}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k+1}\left(\frac{\pounds\overline{n}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)} \right] \end{array} \right), \atop \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k+1}\left(1-t_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k+1}\left(\frac{\pounds\beta_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)} \right) \right).$$

Thus, Theorem 4.2 is verified. □

Theorem 4.3. *Let* $F_j = \left(\left(\left[\underline{m}_j, \overline{m}_j\right], \left[\underline{n}_j, \overline{n}_j\right]\right), \left(t_j, \beta_j\right)\right)$; $j = 1, 2, ..., n$, *be a class of CFFNs such that* $F_j = F \ \forall \ j$. *Then*

$$\text{CFFSuWPA}(F_1, F_2, ..., F_n) = F. \tag{4.4}$$

Proof. Since $F_j = F \ \forall \ j$, based on Eq. (4.4), we get

$$\text{CFFSuWPA}(F_1, F_2, ..., F_n) =$$

$$\left(\left(\begin{array}{c} \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\underline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\overline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)} \right], \\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\underline{n}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\overline{n}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)} \right] \end{array} \right), \atop \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-t_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\beta_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)} \right) \right)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\underline{m}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\overline{m}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)} \right], \\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\underline{n}^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\overline{n}^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)} \right] \end{array} \right), \\ \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-t^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\beta^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)} \right) \end{array} \right)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\underline{m}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\sum_{j=1}^{n}\Phi_j}\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\overline{m}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\sum_{j=1}^{n}\Phi_j}\right)} \right], \\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\underline{n}^3+1}{1+\pounds}\right)^{\sum_{j=1}^{n}\Phi_j}-1\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\overline{n}^3+1}{1+\pounds}\right)^{\sum_{j=1}^{n}\Phi_j}-1\right)} \right] \end{array} \right), \\ \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-t^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\sum_{j=1}^{n}\Phi_j}\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\beta^3+1}{1+\pounds}\right)^{\sum_{j=1}^{n}\Phi_j}-1\right)} \right) \end{array} \right)$$

$= (([\underline{m}, \overline{m}], [\underline{n}, \overline{n}]), (t, \beta)) = F$. Thus, Theorem 4.3 is verified. □

Theorem 4.4. *Let* $F_j = (([\underline{m}_j, \overline{m}_j], [\underline{n}_j, \overline{n}_j]), (t_j, \beta_j))$ *and* $\breve{F}_j = (([\breve{\underline{m}}_j, \breve{\overline{m}}_j], [\breve{\underline{n}}_j, \breve{\overline{n}}_j]), (\breve{t}_j, \breve{\beta}_j))$ *;* $j = 1, 2, ..., n$*, be two classes of CFFNs such that* $\underline{m}_j \leqslant \breve{\underline{m}}_j$*,* $\overline{m}_j \leqslant \breve{\overline{m}}_j$*,* $\underline{n}_j \geqslant \breve{\underline{n}}_j$*,* $\overline{n}_j \geqslant \breve{\overline{n}}_j$*,* $t_j \leqslant \breve{t}_j$ *and* $\beta_j \geqslant \breve{\beta}_j$ $\forall$ j*. Then*

$$CFFSuWPA(F_1, F_2, ..., F_n) \leqslant CFFSuWPA(\breve{F}_1, \breve{F}_2, ..., \breve{F}_n). \tag{4.5}$$

Proof. Since $\underline{m}_j \leqslant \breve{\underline{m}}_j$, $\overline{m}_j \leqslant \breve{\overline{m}}_j$, $\underline{n}_j \geqslant \breve{\underline{n}}_j$, $\overline{n}_j \geqslant \breve{\overline{n}}_j$, $t_j \leqslant \breve{t}_j$ and $\beta_j \geqslant \breve{\beta}_j$ $\forall$ j. From this, we get $\underline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right) \leqslant \breve{\underline{m}}_j^3\left(\frac{\pounds}{1+\pounds}\right)$, $\Longrightarrow 1-\underline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right) \geqslant 1-\breve{\underline{m}}_j^3\left(\frac{\pounds}{1+\pounds}\right) \Longrightarrow \prod_{j=1}^{n}\left(1-\underline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j} \geqslant \prod_{j=1}^{n}\left(1-\breve{\underline{m}}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}$ $\Longrightarrow \left(1-\prod_{j=1}^{n}\left(1-\underline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right) \leqslant \left(1-\prod_{j=1}^{n}\left(1-\breve{\underline{m}}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right) \Longrightarrow \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\underline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)} \leqslant \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\breve{\underline{m}}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}$.

Similarly, we have $\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\overline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)} \leqslant \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\breve{\overline{m}}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}$, and

$$\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-t_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)} \leqslant \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\breve{t}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}.$$

Next $\prod_{j=1}^{n}\left(\frac{\pounds\underline{n}_j^3+1}{1+\pounds}\right)^{\Phi_j} \geqslant \prod_{j=1}^{n}\left(\frac{\pounds\breve{\underline{n}}_j^3+1}{1+\pounds}\right)^{\Phi_j}$, $\Longrightarrow \left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\underline{n}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right) \geqslant \left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\breve{\underline{n}}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)$,

$\Longrightarrow \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\underline{n}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)} \geqslant \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\breve{\underline{n}}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}$.

Similarly, we can get $\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\overline{n}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)} \geqslant \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\breve{\overline{n}}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}$, and

$$\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\beta_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)} \geqslant \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\breve{\beta}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}.$$

This implies that

$$\left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\underline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\overline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}\right],\\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\underline{n}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\overline{n}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}\right]\end{array}\right),\\ \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-t_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\beta_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}\right)\end{array}\right)$$

$$\leqslant\left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\underline{\breve{m}}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\breve{\overline{m}}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}\right],\\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\underline{\breve{n}}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\breve{\overline{n}}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}\right]\end{array}\right),\\ \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\breve{t}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds\breve{\beta}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}\right)\end{array}\right).$$

Hence, $\mathrm{CFFSuWPA}\left(F_1,F_2,...,F_n\right)\leqslant \mathrm{CFFSuWPA}\left(\breve{F}_1,\breve{F}_2,...,\breve{F}_n\right)$.

□

Theorem 4.5. *Let* $F_j=\left(\left(\left[\underline{m}_j,\overline{m}_j\right],\left[\underline{n}_j,\overline{n}_j\right]\right),\left(t_j,\beta_j\right)\right); j=1,2,..,n$ *be a class of CFFNs, and let*

$$F^-=\left(\left(\left[\min_j \underline{m}_j,\min_j \overline{m}_j\right],\left[\max_j \underline{n}_j,\max_j \overline{n}_j\right]\right),\left(\min_j t_j,\max_j \beta_j\right)\right),$$

and

$$F^+=\left(\left(\left[\max_j \underline{m}_j,\max_j \overline{m}_j\right],\left[\min_j \underline{n}_j,\min_j \overline{n}_j\right]\right),\left(\max_j t_j,\min_j \beta_j\right)\right).$$

Then

$$F^-\leqslant \mathrm{CFFSuWPA}\left(F_1,F_2,...,F_n\right)\leqslant F^+. \tag{4.6}$$

Proof. Based on Theorems 4.3 and 4.4, one can easily derive the proof. □

Definition 4.6. Let $F_j=\left(\left(\left[\underline{m}_j,\overline{m}_j\right],\left[\underline{n}_j,\overline{n}_j\right]\right),\left(t_j,\beta_j\right)\right);\ j=1,2,...,n$ be a class of CFFNs. Then the CFF-SuWPWA operator is defined as follows:

$$\mathrm{CFFSuWPWA}\left(F_1,F_2,...,F_n\right)=\bigoplus_{j=1}^{n}\mathbb{M}_jF_j, \tag{4.7}$$

where $\mathbb{M}_j = \frac{\varpi_j(1+T(F_j))}{\sum_{j=1}^{n}(1+T(F_j))}$, $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ is the weight vector of F_j with $\varpi_j \in [0,1]$, $\sum_{j=1}^{n} \varpi_j = 1$ and $T(F_j) = \sum_{i=1}^{n} \text{Sup}(F_j, F_i)_{j \neq i}$.

Theorem 4.7. *For a class of CFFNs* $F_j = (([\underline{m}_j, \overline{m}_j], [\underline{n}_j, \overline{n}_j]), (t_j, \beta_j))$; $j = 1, 2, ..., n$, *the aggregated value of the CFFSuWPWA operator remains a CFFN, resulting in the following:*

$$\begin{aligned} &\text{CFFSuWPWA}(F_1, F_2, ..., F_n) \\ &= \left(\begin{array}{c} \left(\begin{array}{c} \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1 - \prod_{j=1}^{n}\left(1 - \underline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\mathbb{M}_j}\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1 - \prod_{j=1}^{n}\left(1 - \overline{m}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\mathbb{M}_j}\right)} \right], \\ \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds \underline{n}_j^3+1}{1+\pounds}\right)^{\mathbb{M}_j} - 1\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds \overline{n}_j^3+1}{1+\pounds}\right)^{\mathbb{M}_j} - 1\right)} \right] \end{array} \right), \\ \left(\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1 - \prod_{j=1}^{n}\left(1 - t_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\mathbb{M}_j}\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds \beta_j^3+1}{1+\pounds}\right)^{\mathbb{M}_j} - 1\right)} \right) \end{array} \right). \end{aligned} \tag{4.8}$$

Proof. Similar to Theorem 4.2. □

Example 4.8. Let $F_1 = (([0.32, 0.45], [0.30, 0.40]), (0.55, 0.46))$, $F_2 = (([0.34, 0.36], [0.43, 0.51]), (0.55, 0.38))$ and $F_3 = (([0.31, 0.47], [0.29, 0.42]), (0.54, 0.48))$ be three CFFNs, and let the weight vector be $\varpi = (0.4, 0.3, 0.3)$. Suppose $\pounds = 3$. Then, according to Definition 4.6 and Theorem 4.7, we have

$$\text{CFFSuWPWA}(F_1, F_2, F_3) = (([0.2249, 0.3034], [0.8141, 0.8308]), (0.3849, 0.8314)).$$

The following results can be efficiently proven using the CFFSuWPWA operator.

Theorem 4.9. *Let* $F_j = (([\underline{m}_j, \overline{m}_j], [\underline{n}_j, \overline{n}_j]), (t_j, \beta_j))$; $j = 1, 2, ..., n$, *be a class of CFFNs such that* $F_j = F \ \forall j$. *Then*

$$\text{CFFSuWPWA}(F_1, F_2, ..., F_n) = F. \tag{4.9}$$

Theorem 4.10. *Let* $F_j = (([\underline{m}_j, \overline{m}_j], [\underline{n}_j, \overline{n}_j]), (t_j, \beta_j))$ *and* $\breve{F}_j = (([\underline{\breve{m}}_j, \breve{\overline{m}}_j], [\underline{\breve{n}}_j, \breve{\overline{n}}_j]), (\breve{t}_j, \breve{\beta}_j))$; $j = 1, 2, ..., n$, *be two classes of CFFNs such that* $\underline{m}_j \leqslant \underline{\breve{m}}_j$, $\overline{m}_j \leqslant \breve{\overline{m}}_j$, $\underline{n}_j \geqslant \underline{\breve{n}}_j$, $\overline{n}_j \geqslant \breve{\overline{n}}_j$, $t_j \leqslant \breve{t}_j$ *and* $\beta_j \geqslant \breve{\beta}_j \ \forall j$. *Then*

$$\text{CFFSuWPWA}(F_1, F_2, ..., F_n) \leqslant \text{CFFSuWPWA}(\breve{F}_1, \breve{F}_2, ..., \breve{F}_n). \tag{4.10}$$

Theorem 4.11. *Let* $F_j = (([\underline{m}_j, \overline{m}_j], [\underline{n}_j, \overline{n}_j]), (t_j, \beta_j))$; $j = 1, 2, .., n$ *be a class of CFFNs, and let*

$$F^- = \left(\left(\left[\min_j \underline{m}_j, \min_j \overline{m}_j\right], \left[\max_j \underline{n}_j, \max_j \overline{n}_j\right]\right), \left(\min_j t_j, \max_j \beta_j\right)\right)$$

and

$$F^+ = \left(\left(\left[\max_j \underline{m}_j, \max_j \overline{m}_j\right], \left[\min_j \underline{n}_j, \min_j \overline{n}_j\right]\right), \left(\max_j t_j, \min_j \beta_j\right)\right).$$

Then

$$F^- \leqslant \text{CFFSuWPWA}(F_1, F_2, ..., F_n) \leqslant F^+. \tag{4.11}$$

4.2. Cubic Fermatean Fuzzy Sugeno–Weber geometric aggregation operators

Building on the operational laws of SuW t-norms, we develop new mathematical operators for CFF information, specifically the CFF Sugeno-Weber power geometric (CFFSuWPG) and CFF Sugeno-Weber power weighted geometric (CFFSuWPWG) operators.

Definition 4.12. Let $\mathsf{F}_{\mathrm{j}} = \left(\left(\left[\underline{m}_{\mathrm{j}}, \overline{m}_{\mathrm{j}}\right], \left[\underline{n}_{\mathrm{j}}, \overline{n}_{\mathrm{j}}\right]\right), \left(t_{\mathrm{j}}, \beta_{\mathrm{j}}\right)\right)$; $\mathrm{j} = 1, 2, ..., n$ be a class of CFFNs. Then, the CFFSuWPG operator is defined as follows:

$$\mathrm{CFFSuWPG}\left(\mathsf{F}_1, \mathsf{F}_2, ..., \mathsf{F}_n\right) = \bigotimes_{\mathrm{j}=1}^{n} \mathsf{F}_{\mathrm{j}}^{\Phi_{\mathrm{j}}}, \tag{4.12}$$

where $\Phi_{\mathrm{j}} = \frac{\left(1+\mathsf{T}\left(\mathsf{F}_{\mathrm{j}}\right)\right)}{\sum_{\mathrm{j}=1}^{n}\left(1+\mathsf{T}\left(\mathsf{F}_{\mathrm{j}}\right)\right)}$, and $\mathsf{T}\left(\mathsf{F}_{\mathrm{j}}\right) = \sum_{i=1}^{n} \mathrm{Sup}\left(\mathsf{F}_{\mathrm{j}}, \mathsf{F}_i\right)_{\mathrm{j} \neq i}$.

Theorem 4.13. *For a class of CFFNs* $\mathsf{F}_{\mathrm{j}} = \left(\left(\left[\underline{m}_{\mathrm{j}}, \overline{m}_{\mathrm{j}}\right], \left[\underline{n}_{\mathrm{j}}, \overline{n}_{\mathrm{j}}\right]\right), \left(t_{\mathrm{j}}, \beta_{\mathrm{j}}\right)\right)$; $\mathrm{j} = 1, 2, ..., n$, *the aggregated value of the CFFSuWPG operator remains a CFFN, resulting in the following:*

$$\begin{aligned}&\mathrm{CFFSuWPG}\left(\mathsf{F}_1, \mathsf{F}_2, ..., \mathsf{F}_n\right)\\ &= \left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1}{£}\left((1+£)\prod_{\mathrm{j}=1}^{n}\left(\frac{£\underline{m}_{\mathrm{j}}^3+1}{1+£}\right)^{\Phi_{\mathrm{j}}}-1\right)}, \sqrt[3]{\frac{1}{£}\left((1+£)\prod_{\mathrm{j}=1}^{n}\left(\frac{£\overline{m}_{\mathrm{j}}^3+1}{1+£}\right)^{\Phi_{\mathrm{j}}}-1\right)}\right],\\ \left[\sqrt[3]{\frac{1+£}{£}\left(1-\prod_{\mathrm{j}=1}^{n}\left(1-\underline{n}_{\mathrm{j}}^3\left(\frac{£}{1+£}\right)\right)^{\Phi_{\mathrm{j}}}\right)}, \sqrt[3]{\frac{1+£}{£}\left(1-\prod_{\mathrm{j}=1}^{n}\left(1-\overline{n}_{\mathrm{j}}^3\left(\frac{£}{1+£}\right)\right)^{\Phi_{\mathrm{j}}}\right)}\right]\end{array}\right),\\ \left(\sqrt[3]{\frac{1}{£}\left((1+£)\prod_{\mathrm{j}=1}^{n}\left(\frac{£t_{\mathrm{j}}^3+1}{1+£}\right)^{\Phi_{\mathrm{j}}}-1\right)}, \sqrt[3]{\frac{1+£}{£}\left(1-\prod_{\mathrm{j}=1}^{n}\left(1-\beta_{\mathrm{j}}^3\left(\frac{£}{1+£}\right)\right)^{\Phi_{\mathrm{j}}}\right)}\right)\end{array}\right).\end{aligned} \tag{4.13}$$

Proof. We can use the induction approach to show the preceding expression. For $n = 2$, we have

$$\mathsf{F}_1^{\Phi_1} = \left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1}{£}\left((1+£)\left(\frac{£\underline{m}_1^3+1}{1+£}\right)^{\Phi_1}-1\right)}, \sqrt[3]{\frac{1}{£}\left((1+£)\left(\frac{£\overline{m}_1^3+1}{1+£}\right)^{\Phi_1}-1\right)}\right],\\ \left[\sqrt[3]{\frac{1+£}{£}\left(1-\left(1-\underline{n}_1^3\left(\frac{£}{1+£}\right)\right)^{\Phi_1}\right)}, \sqrt[3]{\frac{1+£}{£}\left(1-\left(1-\overline{n}_1^3\left(\frac{£}{1+£}\right)\right)^{\Phi_1}\right)}\right]\end{array}\right),\\ \sqrt[3]{\frac{1}{£}\left((1+£)\left(\frac{£t_1^3+1}{1+£}\right)^{\Phi_1}-1\right)}, \left(\sqrt[3]{\frac{1+£}{£}\left(1-\left(1-\beta_1^3\left(\frac{£}{1+£}\right)\right)^{\Phi_1}\right)}\right)\end{array}\right),$$

$$\mathsf{F}_2^{\Phi_2} = \left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1}{£}\left((1+£)\left(\frac{£\underline{m}_2^3+1}{1+£}\right)^{\Phi_2}-1\right)}, \sqrt[3]{\frac{1}{£}\left((1+£)\left(\frac{£\overline{m}_2^3+1}{1+£}\right)^{\Phi_2}-1\right)}\right],\\ \left[\sqrt[3]{\frac{1+£}{£}\left(1-\left(1-\underline{n}_2^3\left(\frac{£}{1+£}\right)\right)^{\Phi_2}\right)}, \sqrt[3]{\frac{1+£}{£}\left(1-\left(1-\overline{n}_2^3\left(\frac{£}{1+£}\right)\right)^{\Phi_2}\right)}\right]\end{array}\right),\\ \left(\sqrt[3]{\frac{1}{£}\left((1+£)\left(\frac{£t_2^3+1}{1+£}\right)^{\Phi_2}-1\right)}, \sqrt[3]{\frac{1+£}{£}\left(1-\left(1-\beta_2^3\left(\frac{£}{1+£}\right)\right)^{\Phi_2}\right)}\right)\end{array}\right),$$

$$\begin{aligned}&\mathrm{CFFSuWPG}\left(\mathsf{F}_1, \mathsf{F}_2\right) = \bigotimes_{\mathrm{j}=1}^{2} \mathsf{F}_{\mathrm{j}}^{\Phi_{\mathrm{j}}} = \mathsf{F}_1^{\Phi_1} \otimes \mathsf{F}_2^{\Phi_2}\\ &= \left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1}{£}\left((1+£)\left(\frac{£\underline{m}_1^3+1}{1+£}\right)^{\Phi_1}-1\right)}, \sqrt[3]{\frac{1}{£}\left((1+£)\left(\frac{£\overline{m}_1^3+1}{1+£}\right)^{\Phi_1}-1\right)}\right],\\ \left[\sqrt[3]{\frac{1+£}{£}\left(1-\left(1-\underline{n}_1^3\left(\frac{£}{1+£}\right)\right)^{\Phi_1}\right)}, \sqrt[3]{\frac{1+£}{£}\left(1-\left(1-\overline{n}_1^3\left(\frac{£}{1+£}\right)\right)^{\Phi_1}\right)}\right]\end{array}\right),\\ \left(\sqrt[3]{\frac{1}{£}\left((1+£)\left(\frac{£t_1^3+1}{1+£}\right)^{\Phi_1}-1\right)}, \sqrt[3]{\frac{1+£}{£}\left(1-\left(1-\beta_1^3\left(\frac{£}{1+£}\right)\right)^{\Phi_1}\right)}\right)\end{array}\right)\end{aligned}$$

$$
\otimes\left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\underline{m}_2^3+1}{1+\pounds}\right)^{\Phi_2}-1\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\overline{m}_2^3+1}{1+\pounds}\right)^{\Phi_2}-1\right)}\right],\\ \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\underline{n}_2^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_2}\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\overline{n}_2^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_2}\right)}\right]\end{array}\right),\\ \left(\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds t_2^3+1}{1+\pounds}\right)^{\Phi_2}-1\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\beta_2^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_2}\right)}\right)\end{array}\right)
$$

$$
=\left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{2}\left(\frac{\pounds\underline{m}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{2}\left(\frac{\pounds\overline{m}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}\right],\\ \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{2}\left(1-\underline{n}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{2}\left(1-\overline{n}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}\right]\end{array}\right),\\ \left(\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{2}\left(\frac{\pounds t_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{2}\left(1-\beta_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}\right)\end{array}\right).
$$

Assume that Eq. (4.13) is valid for $n = k$.

$$
CFFSuWPG\left(F_1,F_2,\ldots,F_k\right)=\bigotimes_{j=1}^{k}F_j^{\Phi_j}=
$$

$$
\left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k}\left(\frac{\pounds\underline{m}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k}\left(\frac{\pounds\overline{m}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}\right],\\ \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k}\left(1-\underline{n}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k}\left(1-\overline{n}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}\right]\end{array}\right),\\ \left(\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k}\left(\frac{\pounds t_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k}\left(1-\beta_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}\right)\end{array}\right). \tag{4.14}
$$

Following this, we can verify for $n = k+1$.

$$
CFFSuWPG\left(F_1,F_2,\ldots,F_{k+1}\right)=\bigotimes_{j=1}^{k}F_j^{\Phi_j}\otimes F_{k+1}^{\Phi_{k+1}}=
$$

$$
\left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k}\left(\frac{\pounds\underline{m}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k}\left(\frac{\pounds\overline{m}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}\right],\\ \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k}\left(1-\underline{n}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k}\left(1-\overline{n}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}\right]\end{array}\right),\\ \left(\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k}\left(\frac{\pounds t_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k}\left(1-\beta_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}\right)\end{array}\right)
$$

$$
\otimes\left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\underline{m}_{k+1}^3+1}{1+\pounds}\right)^{\Phi_{k+1}}-1\right)},\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds\overline{m}_{k+1}^3+1}{1+\pounds}\right)^{\Phi_{k+1}}-1\right)}\right],\\ \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\underline{n}_{k+1}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_{k+1}}\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\overline{n}_{k+1}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_{k+1}}\right)}\right]\end{array}\right),\\ \left(\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds t_{k+1}^3+1}{1+\pounds}\right)^{\Phi_{k+1}}-1\right)},\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\beta_{k+1}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_{k+1}}\right)}\right)\end{array}\right)
$$

$$= \left(\left(\begin{array}{c} \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k+1}\left(\frac{\pounds \underline{m}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k+1}\left(\frac{\pounds \overline{m}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)} \right], \\ \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k+1}\left(1-\underline{n}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k+1}\left(1-\overline{n}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)} \right] \end{array} \right), \\ \left(\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{k+1}\left(\frac{\pounds t_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{k+1}\left(1-\beta_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)} \right) \right).$$

Thus, Theorem 4.13 is verified.

□

Theorem 4.14. *Let* $F_j = \left(\left(\left[\underline{m}_j, \overline{m}_j\right], \left[\underline{n}_j, \overline{n}_j\right]\right), (t_j, \beta_j)\right)$; $j = 1, 2, ..., n$, *be a class of CFFNs such that* $F_j = F\ \forall\ j$. *Then*

$$\mathrm{CFFSuWPG}(F_1, F_2, ..., F_n) = F. \tag{4.15}$$

Proof. Since $F_j = F\ \forall\ j$, based on Eq. (4.13), we get

$\mathrm{CFFSuWPG}(F_1, F_2, ..., F_n)$

$$= \left(\left(\begin{array}{c} \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds \underline{m}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds \overline{m}_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)} \right], \\ \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\underline{n}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\overline{n}_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)} \right] \end{array} \right), \\ \left(\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds t_j^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\beta_j^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)} \right) \right)$$

$$= \left(\left(\begin{array}{c} \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds \underline{m}^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds \overline{m}^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)} \right], \\ \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\underline{n}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\overline{n}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)} \right] \end{array} \right), \\ \left(\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod_{j=1}^{n}\left(\frac{\pounds t^3+1}{1+\pounds}\right)^{\Phi_j}-1\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod_{j=1}^{n}\left(1-\beta^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\Phi_j}\right)}, \right) \right)$$

$$= \left(\left(\begin{array}{c} \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds \underline{m}^3+1}{1+\pounds}\right)^{\sum_{j=1}^{n}\Phi_j}-1\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds \overline{m}^3+1}{1+\pounds}\right)^{\sum_{j=1}^{n}\Phi_j}-1\right)} \right], \\ \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\underline{n}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\sum_{j=1}^{n}\Phi_j}\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\overline{n}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\sum_{j=1}^{n}\Phi_j}\right)} \right] \end{array} \right), \\ \left(\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\left(\frac{\pounds t^3+1}{1+\pounds}\right)^{\sum_{j=1}^{n}\Phi_j}-1\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\left(1-\beta^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\sum_{j=1}^{n}\Phi_j}\right)} \right) \right)$$

$= \left(\left(\left[\underline{m}, \overline{m}\right], \left[\underline{n}, \overline{n}\right]\right), (t, \beta)\right) = F$. Thus, Theorem 4.14 is verified. □

Theorem 4.15. *Let* $F_j = \left(\left(\left[\underline{m}_j, \overline{m}_j\right], \left[\underline{n}_j, \overline{n}_j\right]\right), (t_j, \beta_j)\right)$ *and* $\breve{F}_j = \left(\left(\left[\underline{\breve{m}}_j, \breve{\overline{m}}_j\right], \left[\underline{\breve{n}}_j, \breve{\overline{n}}_j\right]\right), (\breve{t}_j, \breve{\beta}_j)\right)$; $j = 1, 2, ..., n$, *be two classes of CFFNs such that* $\underline{m}_j \leqslant \underline{\breve{m}}_j$, $\overline{m}_j \leqslant \breve{\overline{m}}_j$, $\underline{n}_j \geqslant \underline{\breve{n}}_j$, $\overline{n}_j \geqslant \breve{\overline{n}}_j$, $t_j \leqslant \breve{t}_j$ *and* $\beta_j \geqslant \breve{\beta}_j\ \forall\ j$. *Then*

$$\mathrm{CFFSuWPG}\left(F_1, F_2, ..., F_n\right) \leqslant \mathrm{CFFSuWPG}\left(\breve{F}_1, \breve{F}_2, ..., \breve{F}_n\right). \tag{4.16}$$

Proof. Since $\underline{m}_j \leqslant \underline{\breve{m}}_j$, $\overline{m}_j \leqslant \overline{\breve{m}}_j$, $\underline{n}_j \geqslant \underline{\breve{n}}_j$, $\overline{n}_j \geqslant \overline{\breve{n}}_j$, $t_j \leqslant \breve{t}_j$ and $\beta_j \geqslant \breve{\beta}_j$ $\forall$ j. From this, we get $\prod_{j=1}^{n}\left(\frac{£\underline{m}_j^3+1}{1+£}\right)^{\Phi_j} \leqslant \prod_{j=1}^{n}\left(\frac{£\underline{\breve{m}}_j^3+1}{1+£}\right)^{\Phi_j}$, $\Longrightarrow \left((1+£)\prod_{j=1}^{n}\left(\frac{£\underline{m}_j^3+1}{1+£}\right)^{\Phi_j}-1\right) \leqslant \left((1+£)\prod_{j=1}^{n}\left(\frac{£\underline{\breve{m}}_j^3+1}{1+£}\right)^{\Phi_j}-1\right)$, $\Longrightarrow$ $\sqrt[3]{\frac{1}{£}\left((1+£)\prod_{j=1}^{n}\left(\frac{£\underline{m}_j^3+1}{1+£}\right)^{\Phi_j}-1\right)} \leqslant \sqrt[3]{\frac{1}{£}\left((1+£)\prod_{j=1}^{n}\left(\frac{£\underline{\breve{m}}_j^3+1}{1+£}\right)^{\Phi_j}-1\right)}$. Similarly, we can get

$$\sqrt[3]{\frac{1}{£}\left((1+£)\prod_{j=1}^{n}\left(\frac{£\overline{m}_j^3+1}{1+£}\right)^{\Phi_j}-1\right)} \leqslant \sqrt[3]{\frac{1}{£}\left((1+£)\prod_{j=1}^{n}\left(\frac{£\overline{\breve{m}}_j^3+1}{1+£}\right)^{\Phi_j}-1\right)},$$

and

$$\sqrt[3]{\frac{1}{£}\left((1+£)\prod_{j=1}^{n}\left(\frac{£t_j^3+1}{1+£}\right)^{\Phi_j}-1\right)} \leqslant \sqrt[3]{\frac{1}{£}\left((1+£)\prod_{j=1}^{n}\left(\frac{£\breve{t}_j^3+1}{1+£}\right)^{\Phi_j}-1\right)}.$$

Next $\underline{n}_j^3\left(\frac{£}{1+£}\right) \geqslant \underline{\breve{n}}_j^3\left(\frac{£}{1+£}\right)$, $\Longrightarrow 1-\underline{n}_j^3\left(\frac{£}{1+£}\right) \leqslant 1-\underline{\breve{n}}_j^3\left(\frac{£}{1+£}\right) \Longrightarrow \prod_{j=1}^{n}\left(1-\underline{n}_j^3\left(\frac{£}{1+£}\right)\right)^{\Phi_j} \leqslant \prod_{j=1}^{n}\left(1-\underline{\breve{n}}_j^3\left(\frac{£}{1+£}\right)\right)^{\Phi_j}$ $\Longrightarrow \left(1-\prod_{j=1}^{n}\left(1-\underline{n}_j^3\left(\frac{£}{1+£}\right)\right)^{\Phi_j}\right) \geqslant \left(1-\prod_{j=1}^{n}\left(1-\underline{\breve{n}}_j^3\left(\frac{£}{1+£}\right)\right)^{\Phi_j}\right) \Longrightarrow \sqrt[3]{\frac{1+£}{£}\left(1-\prod_{j=1}^{n}\left(1-\underline{n}_j^3\left(\frac{£}{1+£}\right)\right)^{\Phi_j}\right)} \geqslant$ $\sqrt[3]{\frac{1+£}{£}\left(1-\prod_{j=1}^{n}\left(1-\underline{\breve{n}}_j^3\left(\frac{£}{1+£}\right)\right)^{\Phi_j}\right)}$. Similarly, we have

$$\sqrt[3]{\frac{1+£}{£}\left(1-\prod_{j=1}^{n}\left(1-\overline{n}_j^3\left(\frac{£}{1+£}\right)\right)^{\Phi_j}\right)} \geqslant \sqrt[3]{\frac{1+£}{£}\left(1-\prod_{j=1}^{n}\left(1-\overline{\breve{n}}_j^3\left(\frac{£}{1+£}\right)\right)^{\Phi_j}\right)},$$

and

$$\sqrt[3]{\frac{1+£}{£}\left(1-\prod_{j=1}^{n}\left(1-\beta_j^3\left(\frac{£}{1+£}\right)\right)^{\Phi_j}\right)} \geqslant \sqrt[3]{\frac{1+£}{£}\left(1-\prod_{j=1}^{n}\left(1-\breve{\beta}_j^3\left(\frac{£}{1+£}\right)\right)^{\Phi_j}\right)}.$$

This implies that

$$\left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1}{£}\left((1+£)\prod_{j=1}^{n}\left(\frac{£\underline{m}_j^3+1}{1+£}\right)^{\Phi_j}-1\right)}, \sqrt[3]{\frac{1}{£}\left((1+£)\prod_{j=1}^{n}\left(\frac{£\overline{m}_j^3+1}{1+£}\right)^{\Phi_j}-1\right)}\right], \\ \left[\sqrt[3]{\frac{1+£}{£}\left(1-\prod_{j=1}^{n}\left(1-\underline{n}_j^3\left(\frac{£}{1+£}\right)\right)^{\Phi_j}\right)}, \sqrt[3]{\frac{1+£}{£}\left(1-\prod_{j=1}^{n}\left(1-\overline{n}_j^3\left(\frac{£}{1+£}\right)\right)^{\Phi_j}\right)}\right]\end{array}\right), \\ \left(\sqrt[3]{\frac{1}{£}\left((1+£)\prod_{j=1}^{n}\left(\frac{£t_j^3+1}{1+£}\right)^{\Phi_j}-1\right)}, \sqrt[3]{\frac{1+£}{£}\left(1-\prod_{j=1}^{n}\left(1-\beta_j^3\left(\frac{£}{1+£}\right)\right)^{\Phi_j}\right)}\right)\end{array}\right)$$

$$\leqslant\left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1}{£}\left((1+£)\prod_{j=1}^{n}\left(\frac{£\breve{\underline{m}}_j^3+1}{1+£}\right)^{\Phi_j}-1\right)},\sqrt[3]{\frac{1}{£}\left((1+£)\prod_{j=1}^{n}\left(\frac{£\breve{\overline{m}}_j^3+1}{1+£}\right)^{\Phi_j}-1\right)}\right],\\ \left[\sqrt[3]{\frac{1+£}{£}\left(1-\prod_{j=1}^{n}\left(1-\breve{\underline{n}}_j^3\left(\frac{£}{1+£}\right)\right)^{\Phi_j}\right)},\sqrt[3]{\frac{1+£}{£}\left(1-\prod_{j=1}^{n}\left(1-\breve{\overline{n}}_j^3\left(\frac{£}{1+£}\right)\right)^{\Phi_j}\right)}\right]\end{array}\right),\\ \left(\sqrt[3]{\frac{1}{£}\left((1+£)\prod_{j=1}^{n}\left(\frac{£\breve{t}_j^3+1}{1+£}\right)^{\Phi_j}-1\right)},\sqrt[3]{\frac{1+£}{£}\left(1-\prod_{j=1}^{n}\left(1-\breve{\beta}_j^3\left(\frac{£}{1+£}\right)\right)^{\Phi_j}\right)}\right)\end{array}\right).$$

Hence, $CFFSuWPG(F_1,F_2,...,F_n)\leqslant CFFSuWPG(\breve{F}_1,\breve{F}_2,...,\breve{F}_n)$. □

Theorem 4.16. *Let* $F_j=(([\underline{m}_j,\overline{m}_j],[\underline{n}_j,\overline{n}_j]),(t_j,\beta_j));j=1,2,..,n$ *be a class of CFFNs, and let*

$$F^-=\left(\left(\left[\min_j\underline{m}_j,\min_j\overline{m}_j\right],\left[\max_j\underline{n}_j,\max_j\overline{n}_j\right]\right),\left(\min_j t_j,\max_j\beta_j\right)\right)$$

and

$$F^+=\left(\left(\left[\max_j\underline{m}_j,\max_j\overline{m}_j\right],\left[\min_j\underline{n}_j,\min_j\overline{n}_j\right]\right),\left(\max_j t_j,\min_j\beta_j\right)\right).$$

Then

$$F^-\leqslant CFFSuWPG(F_1,F_2,...,F_n)\leqslant F^+. \tag{4.17}$$

Proof. Based on Theorems 4.14 and 4.15, one can easily derive the proof. □

Definition 4.17. Let $F_j=(([\underline{m}_j,\overline{m}_j],[\underline{n}_j,\overline{n}_j]),(t_j,\beta_j));\ j=1,2,...,n$ be a class of CFFNs. Then the CFFSuWPWG operator is defined as follows:

$$CFFSuWPWG(F_1,F_2,...,F_n)=\bigotimes_{j=1}^{n}F_j^{\Lambda\!\!\Lambda_j}, \tag{4.18}$$

where $\Lambda\!\!\Lambda_j=\frac{\varpi_j(1+T(F_j))}{\sum_{j=1}^{n}(1+T(F_j))}$, $\varpi=(\varpi_1,\varpi_2,...,\varpi_n)^T$ is the weight vector of F_j with $\varpi_j\in[0,1]$, $\sum_{j=1}^{n}\varpi_j=1$ and $T(F_j)=\sum_{i=1}^{n}Sup(F_j,F_i)_{j\neq i}$.

Theorem 4.18. *For a class of CFFNs* $F_j=(([\underline{m}_j,\overline{m}_j],[\underline{n}_j,\overline{n}_j]),(t_j,\beta_j));j=1,2,...,n$, *the aggregated value of the CFFSuWPWG operator remains a CFFN, resulting in the following:*

$$CFFSuWPWG(F_1,F_2,...,F_n)$$
$$=\left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt[3]{\frac{1}{£}\left((1+£)\prod_{j=1}^{n}\left(\frac{£\underline{m}_j^3+1}{1+£}\right)^{\Lambda\!\!\Lambda_j}-1\right)},\sqrt[3]{\frac{1}{£}\left((1+£)\prod_{j=1}^{n}\left(\frac{£\overline{m}_j^3+1}{1+£}\right)^{\Lambda\!\!\Lambda_j}-1\right)}\right],\\ \left[\sqrt[3]{\frac{1+£}{£}\left(1-\prod_{j=1}^{n}\left(1-\underline{n}_j^3\left(\frac{£}{1+£}\right)\right)^{\Lambda\!\!\Lambda_j}\right)},\sqrt[3]{\frac{1+£}{£}\left(1-\prod_{j=1}^{n}\left(1-\overline{n}_j^3\left(\frac{£}{1+£}\right)\right)^{\Lambda\!\!\Lambda_j}\right)}\right]\end{array}\right),\\ \left(\sqrt[3]{\frac{1}{£}\left((1+£)\prod_{j=1}^{n}\left(\frac{£t_j^3+1}{1+£}\right)^{\Lambda\!\!\Lambda_j}-1\right)},\sqrt[3]{\frac{1+£}{£}\left(1-\prod_{j=1}^{n}\left(1-\beta_j^3\left(\frac{£}{1+£}\right)\right)^{\Lambda\!\!\Lambda_j}\right)}\right)\end{array}\right). \tag{4.19}$$

Proof. Similar to Theorem 4.13. □

Example 4.19. Let

$$\begin{aligned}
\mathrm{F}_1 =&(([0.2249, 0.3034], [0.8141, 0.8308]), (0.3849, 0.8314)),\\
\mathrm{F}_2 =&(([0.4311, 0.4574], [0.8113, 0.8469]), (0.2244, 0.7989)),\\
\mathrm{F}_3 =&(([0.4833, 0.6187], [0.8212, 0.8345]), (0.4171, 0.8332)),\\
\mathrm{F}_4 =&(([0.2820, 0.3221], [0.8163, 0.8209]), (0.3509, 0.8246))
\end{aligned}$$

be four CFFNs, and let the weight vector be $\varpi = (0.2367, 0.2626, 0.2989, 0.2018)$. Suppose $\pounds = 3$. Then, according to Definition 4.17 and Theorem 4.18, we have

$$\mathrm{CFFSuWPG}(\mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3, \mathrm{F}_4) = ([0.1050, 0.1201], [0.9905, 0.9907], (0.1309, 0.9909)).$$

The following results can be efficiently proven using the CFFSuWPWG operator.

Theorem 4.20. *Let* $\mathrm{F}_j = (([\underline{m}_j, \overline{m}_j], [\underline{n}_j, \overline{n}_j]), (t_j, \beta_j))$*;* $j = 1, 2, ..., n$*, be a class of CFFNs such that* $\mathrm{F}_j = \mathrm{F}\ \forall\ j$*. Then*

$$\mathrm{CFFSuWPWG}(\mathrm{F}_1, \mathrm{F}_2, ..., \mathrm{F}_n) = \mathrm{F}. \tag{4.20}$$

Theorem 4.21. *Let* $\mathrm{F}_j = (([\underline{m}_j, \overline{m}_j], [\underline{n}_j, \overline{n}_j]), (t_j, \beta_j))$ *and* $\check{\mathrm{F}}_j = (([\check{\underline{m}}_j, \check{\overline{m}}_j], [\check{\underline{n}}_j, \check{\overline{n}}_j]), (\check{t}_j, \check{\beta}_j))$ *;* $j = 1, 2, ..., n$*, be two classes of CFFNs such that* $\underline{m}_j \leqslant \check{\underline{m}}_j$*,* $\overline{m}_j \leqslant \check{\overline{m}}_j$*,* $\underline{n}_j \geqslant \check{\underline{n}}_j$*,* $\overline{n}_j \geqslant \check{\overline{n}}_j$*,* $t_j \leqslant \check{t}_j$ *and* $\beta_j \geqslant \check{\beta}_j\ \forall\ j$*. Then*

$$\mathrm{CFFSuWPWG}(\mathrm{F}_1, \mathrm{F}_2, ..., \mathrm{F}_n) \leqslant \mathrm{CFFSuWPWG}(\check{\mathrm{F}}_1, \check{\mathrm{F}}_2, ..., \check{\mathrm{F}}_n). \tag{4.21}$$

Theorem 4.22. *Let* $\mathrm{F}_j = (([\underline{m}_j, \overline{m}_j], [\underline{n}_j, \overline{n}_j]), (t_j, \beta_j))$*;* $j = 1, 2, .., n$ *be a class of CFFNs, and let*

$$\mathrm{F}^- = \left(\left(\left[\min_j \underline{m}_j, \min_j \overline{m}_j\right], \left[\max_j \underline{n}_j, \max_j \overline{n}_j\right]\right), \left(\min_j t_j, \max_j \beta_j\right)\right)$$

and

$$\mathrm{F}^+ = \left(\left(\left[\max_j \underline{m}_j, \max_j \overline{m}_j\right], \left[\min_j \underline{n}_j, \min_j \overline{n}_j\right]\right), \left(\max_j t_j, \min_j \beta_j\right)\right).$$

Then

$$\mathrm{F}^- \leqslant \mathrm{CFFSuWPWG}(\mathrm{F}_1, \mathrm{F}_2, ..., \mathrm{F}_n) \leqslant \mathrm{F}^+. \tag{4.22}$$

5. Proposed Cubic Fermatean Fuzzy Approach

In the present section, we develop an integrated group DM approach by cohesively combining the proposed SuW operators with maximizing deviation models within CFF setting.

5.1. Problem Statement

To outline the detailed procedures of the proposed cubic Fermatean group decision framework, several essential assumptions and concepts are introduced.

Consider a group decision problem involving o alternatives denoted as $A = \{A_1, A_2, ...A_i, ..., A_o\}$ and k criteria denoted as $C = \{C_1, C_2, ..., C_j, ..., C_k\}$. A panel of DEs, represented as $D = \{D_1, D_2, ...D_l, ..., D_e\}$, is invited to provide judgments to rank the o alternatives with respect to the k criteria. The weights of the DEs and criteria are denoted by $w = (w_1, w_2, ..., w_e)^T$ and $\varpi = (\varpi_1, \varpi_2, ..., \varpi_k)^T$,respectively, where $w_l, \varpi_j \in [0, 1]$, and $\sum_{l=1}^{e} w_l = 1$ and $\sum_{j=1}^{k} \varpi_j = 1$. The alternatives are evaluated by DEs under the given criteria within the CFF context $\mathrm{F}_{ij}^{(l)} = \left(\left(\left[\underline{m}_{ij}^{(l)}, \overline{m}_{ij}^{(l)}\right], \left[\underline{n}_{ij}^{(l)}, \overline{n}_{ij}^{(l)}\right]\right), \left(t_{ij}^{(l)}, \beta_{ij}^{(l)}\right)\right)$, ensuring that $0 \leqslant \left(\overline{m}_{ij}^{(l)}\right)^3 + \left(\overline{n}_{ij}^{(l)}\right)^3 \leqslant 1$ and $0 \leqslant \left(t_{ij}^{(l)}\right)^3 + \left(\beta_{ij}^{(l)}\right)^3 \leqslant 1$.

5.2. Global cubic Fermatean fuzzy assessment matrix

Step 1: Initial assessment matrices: Collect the CFF information data in the form of decision matrices

$$\mathcal{M}^{(l)} = \begin{array}{c} \\ A_1 \\ \vdots \\ A_i \\ \vdots \\ A_o \end{array} \begin{array}{c} \begin{array}{ccccc} C_1 & \cdots & C_j & \cdots & C_k \end{array} \\ \begin{pmatrix} F_{11}^{(l)} & \cdots & F_{1j}^{(l)} & \cdots & F_{1k}^{(l)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ F_{i1}^{(l)} & \cdots & F_{ij}^{(l)} & \cdots & F_{ik}^{(l)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ F_{o1}^{(l)} & \cdots & F_{oj}^{(l)} & \cdots & F_{ok}^{(l)} \end{pmatrix} \end{array}, \text{provided by the DEs } D_l;\ l = 1, 2, ..., k.$$

Step 2: Normalization: In the provided decision matrices $\mathcal{M}^{(l)} = \left[F_{ij}^{(l)}\right]_{o\times k}$, considering different types of criteria may lead to unreasonable outcomes. Therefore, to make the provided data regarding different criteria compatible for operation, we compute the normalized CFF assessment matrices $\widetilde{\mathcal{M}}^{(l)} = \left[\widetilde{F}_{ij}^{(l)}\right]_{o\times k}$ using Eq. (5.1).

$$\widetilde{F}_{ij}^{(l)} = \begin{cases} \left(\left(\left[\underline{m}_{ij}^{(l)}, \overline{m}_{ij}^{(l)}\right], \left[\underline{n}_{ij}^{(l)}, \overline{n}_{ij}^{(l)}\right]\right), \left(t_{ij}^{(l)}, \beta_{ij}^{(l)}\right)\right), & \text{if } C_j \text{ is benefit criteria} \\ \left(\left(\left[\underline{n}_{ij}^{(l)}, \overline{n}_{ij}^{(l)}\right], \left[\underline{m}_{ij}^{(l)}, \overline{m}_{ij}^{(l)}\right]\right), \left(\beta_{ij}^{(l)}, t_{ij}^{(l)}\right)\right), & \text{if } C_j \text{ is cost criteria.} \end{cases} \tag{5.1}$$

Step 3: Support determination across individual decision matrices: Based on Eq. (5.2), compute the total support $T\left(F_{ij}^{(l)}\right)$; $l = 1, 2, ..., e$.

$$T\left(F_{ij}^{(l)}\right) = \sum_{\pounds=1}^{e} Sup\left(F_{ij}^{(l)}, F_{ij}^{(\pounds)}\right)_{l\neq\pounds}, \tag{5.2}$$

where $Sup\left(F_{ij}^{(l)}, F_{ij}^{(\pounds)}\right) = 1 - d\left(F_{ij}^{(l)}, F_{ij}^{(\pounds)}\right)$, and d is the measure given in Eq. (2.8).

Step 4: Determination of power weights using DEs' weights: Determine the power weights $\sqcap_{ij}^{(l)}$; $l = 1, 2, .., e$ for each individual decision matrix with the aid of Eq. (5.3).

$$\sqcap_{ij}^{(l)} = \frac{w_l\left(1 + T\left(F_{ij}^{(l)}\right)\right)}{\sum\limits_{l=1}^{e}\left(1 + T\left(F_{ij}^{(l)}\right)\right)}, \tag{5.3}$$

$w = (w_1, w_2, ..., w_e)^T$ is the DEs' weights such that $w_l \in [0, 1]$, $\sum\limits_{l=1}^{e} \varpi_l = 1$.

Step 5: Aggregation of individual assessment matrices: The individual decision matrices are converted into single decision matrix by the formulation expressed in Eq. (5.4)

$$F_{ij} = CFFSuWPWA\left(F_{ij}^{(1)}, F_{ij}^{(2)}, ..., F_{ij}^{(e)}\right)$$
$$= \left(\left(\begin{array}{c} \left[\sqrt[3]{\frac{1+£}{£}\left(1-\prod_{l=1}^{e}\left(1-\underline{m}_{ij}^{(l)^3}\left(\frac{£}{1+£}\right)\right)^{\sqcap_{ij}^{(l)}}\right)}, \sqrt[3]{\frac{1+£}{£}\left(1-\prod_{l=1}^{e}\left(1-\overline{m}_{ij}^{(l)^3}\left(\frac{£}{1+£}\right)\right)^{\sqcap_{ij}^{(l)}}\right)} \right], \\ \left[\sqrt[3]{\frac{1}{£}\left((1+£)\prod_{l=1}^{e}\left(\frac{£\underline{n}_{ij}^{(l)^3}+1}{1+£}\right)^{\sqcap_{ij}^{(l)}}-1\right)}, \sqrt[3]{\frac{1}{£}\left((1+£)\prod_{l=1}^{e}\left(\frac{£\overline{n}_{ij}^{(l)^3}+1}{1+£}\right)^{\sqcap_{ij}^{(l)}}-1\right)} \right] \end{array} \right), \\ \left(\sqrt[3]{\frac{1+£}{£}\left(1-\prod_{l=1}^{e}\left(1-t_{ij}^{(l)^3}\left(\frac{£}{1+£}\right)\right)^{\sqcap_{ij}^{(l)}}\right)}, \sqrt[3]{\frac{1}{£}\left((1+£)\prod_{l=1}^{e}\left(\frac{£\beta_{ij}^{(l)^3}+1}{1+£}\right)^{\sqcap_{ij}^{(l)}}-1\right)} \right) \right). \tag{5.4}$$

5.3. *Criteria weight determination*

Step 6: This step includes the following two cases.

Case 1: Completely unknown weight information

Construct an optimization model based on the method of maximizing deviation to identify the optimal relative weights of criteria using CFFNs. For the criteria $C_j \in C$, the deviation of the alternative A_i in relation to all other alternatives can be expressed as follows:

$$D_{ij}(\varpi) = \sum_{k=1}^{m} d\left(F_{ij}, F_{kj}\right)\varpi_j; i = 1, 2, ..., m, j = 1, 2, ..., n$$

where

$$d\left(F_{ij}, F_{kj}\right) = \left(\frac{1}{6}\left(\left|\underline{m}_{ij}^3 - \underline{m}_{kj}^3\right|^2 + \left|\overline{m}_{ij}^3 - \overline{m}_{kj}^3\right|^2 + \left|\underline{n}_{ij}^3 - \underline{n}_{kj}^3\right|^2 + \left|\overline{n}_{ij}^3 - \overline{n}_{kj}^3\right|^2 + \left|t_{ij}^3 - t_{kj}^3\right|^2 + \left|\beta_{ij}^3 - \beta_{kj}^3\right|^2\right)\right)^{\frac{1}{2}}.$$

Let

$$D_j(\varpi) = \sum_{i=1}^{m} D_{ij}(\varpi)$$
$$= \sum_{i=1}^{m}\sum_{k=1}^{m}\varpi_j\left(\frac{1}{6}\left(\left|\underline{m}_{ij}^3 - \underline{m}_{kj}^3\right|^2 + \left|\overline{m}_{ij}^3 - \overline{m}_{kj}^3\right|^2 + \left|\underline{n}_{ij}^3 - \underline{n}_{kj}^3\right|^2 + \left|\overline{n}_{ij}^3 - \overline{n}_{kj}^3\right|^2 + \left|t_{ij}^3 - t_{kj}^3\right|^2 + \left|\beta_{ij}^3 - \beta_{kj}^3\right|^2\right)\right)^{\frac{1}{2}};$$
$$j = 1, 2, ..., n.$$

Then $D_j(\varpi)$ represents the deviation value of all alternatives relative to other alternatives for the criteria $C_j \in C$. Based on the aforementioned analysis, to select the weight vector ϖ that maximizes the deviation values for all criteria, a non-linear programming model is formulated as follows:

$$(M-1)\begin{cases} \max D(\varpi) = \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m}\varpi_j\left(\frac{1}{6}\left(\begin{array}{c}\left|\underline{m}_{ij}^3 - \underline{m}_{kj}^3\right|^2 + \left|\overline{m}_{ij}^3 - \overline{m}_{kj}^3\right|^2 + \left|\underline{n}_{ij}^3 - \underline{n}_{kj}^3\right|^2 \\ + \left|\overline{n}_{ij}^3 - \overline{n}_{kj}^3\right|^2 + \left|t_{ij}^3 - t_{kj}^3\right|^2 + \left|\beta_{ij}^3 - \beta_{kj}^3\right|^2\end{array}\right)\right)^{\frac{1}{2}}, \\ \text{s.t. } \varpi_j \geqslant 0, j = 1, 2, ...n, \sum\limits_{j=1}^{n}\varpi_j^2 = 1. \end{cases}$$

To solve the model described above, the Lagrange function is constructed as follows:

$$L(\varpi, \psi) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\frac{1}{6}\left(\begin{array}{c}\left|\underline{m}_{ij}^3 - \underline{m}_{kj}^3\right|^2 + \left|\overline{m}_{ij}^3 - \overline{m}_{kj}^3\right|^2 + \left|\underline{n}_{ij}^3 - \underline{n}_{kj}^3\right|^2 \\ + \left|\overline{n}_{ij}^3 - \overline{n}_{kj}^3\right|^2 + \left|t_{ij}^3 - t_{kj}^3\right|^2 + \left|\beta_{ij}^3 - \beta_{kj}^3\right|^2\end{array}\right)\right)^{\frac{1}{2}}\varpi_j + \frac{\psi}{2}\left(\sum_{j=1}^{n}\varpi_j^2 - 1\right),$$

where ψ is a real number that represents the Lagrange multiplier variable. We then compute the partial derivatives of L and set them as follows:

$$\frac{\partial L}{\partial \varpi_j} = \sum_{i=1}^{m}\sum_{k=1}^{m}\left(\frac{1}{6}\left(\left|\underline{m}_{ij}^3-\underline{m}_{kj}^3\right|^2+\left|\overline{m}_{ij}^3-\overline{m}_{kj}^3\right|^2+\left|\underline{n}_{ij}^3-\underline{n}_{kj}^3\right|^2+\left|\overline{n}_{ij}^3-\overline{n}_{kj}^3\right|^2+\left|t_{ij}^3-t_{kj}^3\right|^2+\left|\beta_{ij}^3-\beta_{kj}^3\right|^2\right)\right)^{\frac{1}{2}} + \psi\varpi_j = 0,$$

$$\frac{\partial L}{\partial \psi} = \frac{1}{2}\left(\sum_{j=1}^{n}\varpi_j^2 - 1\right) = 0.$$

By solving the equations above, an exact formula for determining the attribute weights can be derived as follows:

$$\varpi_j^* = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\frac{1}{6}\left(\left|\underline{m}_{ij}^3-\underline{m}_{kj}^3\right|^2+\left|\overline{m}_{ij}^3-\overline{m}_{kj}^3\right|^2+\left|\underline{n}_{ij}^3-\underline{n}_{kj}^3\right|^2+\left|\overline{n}_{ij}^3-\overline{n}_{kj}^3\right|^2+\left|t_{ij}^3-t_{kj}^3\right|^2+\left|\beta_{ij}^3-\beta_{kj}^3\right|^2\right)\right)^{\frac{1}{2}}}{\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\frac{1}{6}\left(\left|\underline{m}_{ij}^3-\underline{m}_{kj}^3\right|^2+\left|\overline{m}_{ij}^3-\overline{m}_{kj}^3\right|^2+\left|\underline{n}_{ij}^3-\underline{n}_{kj}^3\right|^2+\left|\overline{n}_{ij}^3-\overline{n}_{kj}^3\right|^2+\left|t_{ij}^3-t_{kj}^3\right|^2+\left|\beta_{ij}^3-\beta_{kj}^3\right|^2\right)\right)^{\frac{1}{2}}\right)^2}}.$$

Since the weights of the attributes must satisfy the normalization condition, we obtain the normalized criteria weights as follows:

$$\varpi_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\frac{1}{6}\left(\left|\underline{m}_{ij}^3-\underline{m}_{kj}^3\right|^2+\left|\overline{m}_{ij}^3-\overline{m}_{kj}^3\right|^2+\left|\underline{n}_{ij}^3-\underline{n}_{kj}^3\right|^2+\left|\overline{n}_{ij}^3-\overline{n}_{kj}^3\right|^2+\left|t_{ij}^3-t_{kj}^3\right|^2+\left|\beta_{ij}^3-\beta_{kj}^3\right|^2\right)\right)^{\frac{1}{2}}}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\frac{1}{6}\left(\left|\underline{m}_{ij}^3-\underline{m}_{kj}^3\right|^2+\left|\overline{m}_{ij}^3-\overline{m}_{kj}^3\right|^2+\left|\underline{n}_{ij}^3-\underline{n}_{kj}^3\right|^2+\left|\overline{n}_{ij}^3-\overline{n}_{kj}^3\right|^2+\left|t_{ij}^3-t_{kj}^3\right|^2+\left|\beta_{ij}^3-\beta_{kj}^3\right|^2\right)\right)^{\frac{1}{2}}}. \tag{5.5}$$

Case 2: Partially known weight information

However, in some cases, the information about the weight vector is only partially known rather than fully known. For these situations, based on the set of known weight information, Γ, a constrained optimization model can be designed as follows:

$$(M-2)\begin{cases} \max D(\varpi) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\varpi_j\left(\frac{1}{6}\left(\begin{array}{c}\left|\underline{m}_{ij}^3-\underline{m}_{kj}^3\right|^2+\left|\overline{m}_{ij}^3-\overline{m}_{kj}^3\right|^2+\left|\underline{n}_{ij}^3-\underline{n}_{kj}^3\right|^2\\ +\left|\overline{n}_{ij}^3-\overline{n}_{kj}^3\right|^2+\left|t_{ij}^3-t_{kj}^3\right|^2+\left|\beta_{ij}^3-\beta_{kj}^3\right|^2\end{array}\right)\right)^{\frac{1}{2}}, \\ \text{s.t. } \varpi\in\Gamma,\ \varpi_j \geqslant 0, j=1,2,...n,\ \sum_{j=1}^{n}\varpi_j = 1 \end{cases}$$

where Γ represents a set of constraint conditions that the weight values ϖ_j must satisfy according to real-world requirements. The model $(M-2)$ is a linear programming model. By solving this model, the optimal solution $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^t$ is obtained, which can then be used as the attribute weight vector.

5.4. Alternatives Ranking

Step 7: Support determination across global decision matrix: According to Eq. (5.6), compute the total support $T(F_{ij})$.

$$T(F_{ij}) = \sum_{\complement=1}^{k} Sup\left(F_{ij}, F_{i\complement}\right)_{j\neq\complement}, \tag{5.6}$$

where $Sup\left(F_{ij}, F_{i\complement}\right) = 1 - d\left(F_{ij}, F_{i\complement}\right)$, and d is the measure given in Eq. (2.8).

Step 8: Determination of power weights using criteria' weights: Calculate the power weights $\mathbb{M}_{ij}$ with the aid of Eq. (5.7).

$$\mathbb{M}_{ij} = \frac{\varpi_j \left(1 + T\left(F_{ij}\right)\right)}{\sum\limits_{j=1}^{k} \left(1 + T\left(F_{ij}\right)\right)}, \tag{5.7}$$

$\varpi = \left(\varpi_1, \varpi_2, ..., \varpi_k\right)^T$ is the criteria' weights such that $w_j \in [0,1]$, $\sum\limits_{j=1}^{k} \varpi_j = 1$.

Step 9: Aggregation: Aggregate the criteria values of the alternatives in light of Eq. (5.8)

$$\begin{aligned} F_i &= CFFSuWPWG\left(F_{i1}, F_{i2}, ..., F_{in}\right) \\ &= \left(\begin{array}{c} \left(\begin{array}{c} \left[\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod\limits_{j=1}^{n}\left(\frac{\pounds \underline{m}_{ij}^3+1}{1+\pounds}\right)^{\mathbb{M}_{ij}} - 1\right)}, \sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod\limits_{j=1}^{n}\left(\frac{\pounds \overline{m}_{ij}^3+1}{1+\pounds}\right)^{\mathbb{M}_{ij}} - 1\right)} \right], \\ \left[\sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod\limits_{j=1}^{n}\left(1-\underline{n}_{ij}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\mathbb{M}_{ij}}\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod\limits_{j=1}^{n}\left(1-\overline{n}_{ij}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\mathbb{M}_{ij}}\right)} \right] \end{array} \right), \\ \left(\sqrt[3]{\frac{1}{\pounds}\left((1+\pounds)\prod\limits_{j=1}^{n}\left(\frac{\pounds t_{ij}^3+1}{1+\pounds}\right)^{\mathbb{M}_{ij}} - 1\right)}, \sqrt[3]{\frac{1+\pounds}{\pounds}\left(1-\prod\limits_{j=1}^{n}\left(1-\beta_{ij}^3\left(\frac{\pounds}{1+\pounds}\right)\right)^{\mathbb{M}_{ij}}\right)} \right) \end{array} \right), \end{aligned} \tag{5.8}$$

where $i = 1, 2, ..., k$.

Step 10: Score values and Ranking: Calculate the score function $S\left(F_i\right)$, and ranked the alternatives based on $S\left(F_i\right)$ in decreasing order.

6. Case Study

The proposed group DM framework is empirically validated through a comprehensive case study focusing on the performance evaluation of project team members. To further highlight the adaptability and robustness of the model, a detailed parameter analysis is conducted, showcasing its responsiveness under various conditions.

6.1. Contextual Overview

Assessing the performance of project team members involves a multifaceted evaluation of their individual and collaborative contributions throughout the project lifecycle. Although assessment criteria may vary depending on the project type, organizational context, and specific evaluation objectives, several commonly accepted dimensions are consistently utilized. These include:

- **Quality of work:** Assesses the precision, thoroughness, and impact of the individual's deliverables. It considers factors such as attention to detail, compliance with project standards, and the degree to which the work meets or exceeds expectations.
- **Productivity:** Measures the individual's ability to deliver outcomes efficiently, focusing on the volume and timeliness of work produced relative to the resources allocated. It includes task completion rates, adherence to schedules, and consistency in output.
- **Teamwork and collaboration:** Evaluates the degree to which the team member engages constructively with colleagues, contributes to group goals, supports team dynamics, and fosters a cooperative atmosphere. It includes willingness to assist others and openness to feedback.

- **Problem-solving skills:** Focuses on analytical thinking and the capacity to address challenges effectively. This criterion examines how well an individual identifies problems, evaluates alternatives, applies judgment, and implements viable solutions during different project phases.

- **Initiative and proactivity:** Reflects the extent to which the individual demonstrates self-motivation, takes ownership of tasks, and proactively identifies opportunities for improvement or innovation. It also includes acting without direct supervision when appropriate.

- **Adaptability:** Gauges flexibility in response to changing project needs, including modifications in timelines, roles, or deliverables. It reflects the individual's resilience, openness to new ideas, and ability to quickly adjust to dynamic circumstances.

- **Communication skills:** Assesses the effectiveness and clarity with which the individual conveys ideas, instructions, and updates across various channels. It also considers listening skills, responsiveness, and the ability to tailor communication to different stakeholders.

- **Leadership abilities:** When relevant, examines the individual's capability to guide team efforts, delegate responsibilities, motivate peers, and align team goals with project objectives. This includes conflict mediation and strategic DM.

- **Project knowledge:** Measures the depth of understanding the individual holds regarding the project's goals, methodologies, technical requirements, and domain-specific knowledge. It also includes awareness of relevant tools and industry practices.

- **Time management:** Evaluates the individual's proficiency in organizing tasks, meeting deadlines, and balancing competing priorities. It involves planning, setting realistic schedules, and maintaining consistent progress without last-minute rushes.

- **Conflict resolution:** Reviews the individual's approach to handling disagreements or misunderstandings within the team. It assesses emotional intelligence, negotiation abilities, and the capacity to restore and maintain positive working relationships.

- **Creativity and innovation:** Assesses the individual's ability to think outside the box, offer original ideas, and implement novel approaches that contribute to project differentiation and efficiency. It values experimentation and calculated risk-taking.

- **Stakeholder relations:** Where applicable, evaluates the quality and professionalism of interactions with external or internal stakeholders, including clients, vendors, and senior management. Key indicators include responsiveness, transparency, and alignment with stakeholder expectations.

- **Ethical conduct:** Measures adherence to professional and organizational standards, including honesty, responsibility, fairness, and confidentiality. It also includes compliance with legal and regulatory frameworks relevant to the project.

- **Professional development:** Considers the individual's engagement in continuous learning, such as attending training sessions, earning certifications, or actively seeking mentorship to enhance skills and stay relevant in the project domain.

- **Overall contribution:** Provides a holistic view of the team member's net impact on the project's outcomes. It considers cumulative performance across all dimensions and reflects the perceived value added by the individual to the team and project success.

6.2. Overpopulation and Its Consequences

Overpopulation arises when the population surpasses the environmental carrying capacity, triggering a host of social, economic, and ecological challenges. The unsustainable use of natural resources, driven by population growth, accelerates the degradation of ecosystems and increases stress on essential services.

Economically, rapid urbanization and population increases exert immense pressure on infrastructure, employment, and service delivery. Without strategic interventions, this trajectory may lead to widespread scarcity, including food and water shortages, reduced quality of life, and potential conflict.

According to a study by the National Academy of Sciences, if global population trends persist, the world population may reach 30 billion by 2075. While growth rates fluctuate, Turkey, for instance, continues to exhibit upward population trends. In 2019, the growth rate was 1.39%, which declined to 0.55% in 2020, resulting in a population of 83,614,362 and a population density of 109 persons per square kilometer [29]. The median age also rose slightly, from 32.4 to 32.7 years over the same period.

To effectively respond to the challenges posed by population growth, a multi-criteria decision-making (MCDM) approach can be employed. This involves systematically identifying, categorizing, and analyzing evaluation criteria with the help of expert consultation. A specialized committee—comprising experts from the Ministry of Health, Ministry of Treasury and Finance, and Ministry of Family and Social Services—can collaborate to refine these criteria and formulate an informed assessment structure.

Key Evaluation Criteria for Population Challenges:

- **Unemployment (C_1):** An expanding labor force without proportional job creation results in rising unemployment, posing severe economic and social issues.
- **Disease Incidence (C_2):** Larger populations are more vulnerable to health risks and environmental pollutants. For instance, in the U.S., approximately 2.8 billion kilograms of toxic chemicals are released into the environment annually [11].
- **Malnutrition (C_3):** A common consequence of overpopulation, malnutrition includes the deficiency of essential nutrients like proteins, vitamins, and minerals [12].
- **Income Distribution (C_4):** Shifts in the socioeconomic structure—particularly the expansion of the middle class—play a critical role in national development and resource allocation.

Strategic Government Measures:

- A_1: Implement policies to decelerate population growth.
- A_2: Enhance disease surveillance and public health infrastructure.
- A_3: Invest in agricultural innovation to increase food security.
- A_4: Create employment opportunities that outpace population expansion.

Three government specialists D_1, D_2 and D_3 having weight vector $w = (0.4, 0.3, 0.3)^t$ from the Ministry of Health, Ministry of Treasury and Finance, and Ministry of Family and Social Services compiled and provided their observational data, which is detailed in $\mathcal{M}^{(1)}$, $\mathcal{M}^{(2)}$ and $\mathcal{M}^{(3)}$.

$\mathcal{M}^{(1)} =$

	C_1	C_2	C_3	C_4
A_1	$(([0.32, 0.45], [0.30, 0.40]), (0.55, 0.46))$	$(([0.65, 0.67], [0.34, 0.56]), (0.28, 0.13))$	$(([0.72, 0.88], [0.37, 0.48]), (0.59, 0.48))$	$(([0.27, 0.35], [0.28, 0.32]), (0.53, 0.44))$
A_2	$(([0.58, 0.64], [0.35, 0.46]), (0.40, 0.36))$	$(([0.30, 0.45], [0.50, 0.55]), (0.49, 0.39))$	$(([0.42, 0.57], [0.44, 0.52]), (0.72, 0.64))$	$(([0.68, 0.72], [0.32, 0.38]), (0.48, 0.40))$
A_3	$(([0.48, 0.53], [0.62, 0.65]), (0.45, 0.42))$	$(([0.70, 0.72], [0.58, 0.61]), (0.40, 0.30))$	$(([0.26, 0.54], [0.44, 0.49]), (0.50, 0.45))$	$(([0.15, 0.18], [0.33, 0.35]), (0.64, 0.40))$
A_4	$(([0.33, 0.36], [0.42, 0.52]), (0.54, 0.41))$	$(([0.28, 0.38], [0.28, 0.32]), (0.50, 0.35))$	$(([0.55, 0.57], [0.38, 0.50]), (0.52, 0.46))$	$(([0.70, 0.72], [0.35, 0.40]), (0.55, 0.30))$

$\mathcal{M}^{(2)} =$

	C_1	C_2	C_3	C_4
A_1	$(([0.34, 0.36], [0.43, 0.51]), (0.55, 0.38))$	$(([0.48, 0.50], [0.30, 0.33]), (0.40, 0.22))$	$(([0.52, 0.57], [0.46, 0.48]), (0.60, 0.35))$	$(([0.56, 0.60], [0.47, 0.50]), (0.42, 0.28))$
A_2	$(([0.70, 0.75], [0.44, 0.47]), (0.52, 0.40))$	$(([0.25, 0.38], [0.34, 0.36]), (0.54, 0.43))$	$(([0.60, 0.65], [0.40, 0.44]), (0.54, 0.35))$	$(([0.38, 0.47], [0.25, 0.30]), (0.47, 0.38))$
A_3	$(([0.58, 0.64], [0.24, 0.28]), (0.76, 0.70))$	$(([0.72, 0.75], [0.40, 0.45]), (0.54, 0.30))$	$(([0.43, 0.45], [0.17, 0.26]), (0.62, 0.34))$	$(([0.59, 0.60], [0.50, 0.56]), (0.48, 0.30))$
A_4	$(([0.37, 0.40], [0.13, 0.20]), (0.50, 0.35))$	$(([0.75, 0.78], [0.56, 0.57]), (0.55, 0.40))$	$(([0.47, 0.55], [0.15, 0.18]), (0.70, 0.42))$	$(([0.65, 0.70], [0.44, 0.48]), (0.56, 0.32))$

$\mathcal{M}^{(3)} =$

	C_1	C_2	C_3	C_4
A_1	$(([0.31, 0.47], [0.29, 0.42]), (0.54, 0.48))$	$(([0.64, 0.70], [0.33, 0.58]), (0.27, 0.15))$	$(([0.71, 0.90], [0.36, 0.42]), (0.58, 0.50))$	$(([0.26, 0.37], [0.32, 0.34]), (0.52, 0.46))$
A_2	$(([0.58, 0.66], [0.34, 0.48]), (0.40, 0.38))$	$(([0.30, 0.47], [0.49, 0.54]), (0.48, 0.41))$	$(([0.41, 0.59], [0.44, 0.54]), (0.72, 0.66))$	$(([0.67, 0.74], [0.31, 0.40]), (0.47, 0.42))$
A_3	$(([0.48, 0.55], [0.61, 0.67]), (0.50, 0.44))$	$(([0.69, 0.74], [0.57, 0.63]), (0.40, 0.32))$	$(([0.25, 0.56], [0.43, 0.51]), (0.50, 0.47))$	$(([0.14, 0.20], [0.32, 0.37]), (0.64, 0.42))$
A_4	$(([0.32, 0.38], [0.41, 0.54]), (0.53, 0.43))$	$(([0.27, 0.40], [0.27, 0.34]), (0.49, 0.37))$	$(([0.54, 0.60], [0.37, 0.52]), (0.51, 0.48))$	$(([0.58, 0.62], [0.34, 0.42]), (0.54, 0.32))$

6.3. *Decision analysis*

Step 1: The CFF data provided by the three experts are listed in matrices $\mathcal{M}^{(1)}$, $\mathcal{M}^{(2)}$ and $\mathcal{M}^{(3)}$.

Step 2: Since all the criteria are of the same nature, there is no need for normalization.

Step 3: Based on Eq. (5.2), the total support values $T\left(F_{ij}^{(l)}\right)$; $l = 1, 2, 3$ are determined as shown in matrices $T\left(F_{ij}^{(1)}\right)$, $T\left(F_{ij}^{(2)}\right)$ and $T\left(F_{ij}^{(3)}\right)$.

$$T\left(F_{ij}^{(1)}\right) = \begin{pmatrix} 1.9472 & 1.8653 & 1.7474 & 1.8766 \\ 1.8917 & 1.9268 & 1.8399 & 1.8311 \\ 1.7464 & 1.8931 & 1.9069 & 1.8446 \\ 1.9257 & 1.7402 & 1.8819 & 1.8779 \end{pmatrix},$$

$$T\left(F_{ij}^{(2)}\right) = \begin{pmatrix} 1.9100 & 1.7573 & 1.5308 & 1.7735 \\ 1.8144 & 1.8731 & 1.7030 & 1.6868 \\ 1.5411 & 1.8187 & 1.8296 & 1.7008 \\ 1.8658 & 1.4955 & 1.7888 & 1.9006 \end{pmatrix},$$

$$T\left(F_{ij}^{(3)}\right) = \begin{pmatrix} 1.9440 & 1.8525 & 1.7308 & 1.8798 \\ 1.8983 & 1.9292 & 1.8330 & 1.8246 \\ 1.7548 & 1.8900 & 1.9003 & 1.8453 \\ 1.9206 & 1.7424 & 1.8750 & 1.8588 \end{pmatrix}.$$

Step 4: According to Eq. (5.3), the power weights $\sqcap_{ij}^{(l)}$; $l = 1, 2, 3$ are calculated and listed matrices $\sqcap_{ij}^{(1)}$, $\sqcap_{ij}^{(2)}$ and $\sqcap_{ij}^{(3)}$.

$$\sqcap_{ij}^{(1)} = \begin{pmatrix} 0.1343 & 0.1363 & 0.1395 & 0.1357 \\ 0.1350 & 0.1345 & 0.1369 & 0.1371 \\ 0.1385 & 0.1352 & 0.1353 & 0.1369 \\ 0.1348 & 0.1398 & 0.1357 & 0.1332 \end{pmatrix},$$

$$\sqcap_{ij}^{(2)} = \begin{pmatrix} 0.0988 & 0.0963 & 0.0917 & 0.0962 \\ 0.0971 & 0.0981 & 0.0950 & 0.0947 \\ 0.0917 & 0.0974 & 0.0974 & 0.0947 \\ 0.0980 & 0.0901 & 0.0968 & 0.1011 \end{pmatrix},$$

$$\sqcap_{ij}^{(3)} = \begin{pmatrix} 0.1005 & 0.1015 & 0.1037 & 0.1020 \\ 0.1016 & 0.1010 & 0.1023 & 0.1025 \\ 0.1044 & 0.1012 & 0.1011 & 0.1027 \\ 0.1009 & 0.1050 & 0.1014 & 0.0989 \end{pmatrix}.$$

Step 5: Following Eq. (5.4), the single decision matrix $\mathcal{M}$ is obtained as follows:

$\mathcal{M} =$

	C_1	C_2
A_1	$(([0.2249, 0.3034], [0.8141, 0.8308]), (0.3849, 0.8314))$	$(([0.4311, 0.4574], [0.8113, 0.8469]), (0.2244, 0.7989))$
A_2	$(([0.4406, 0.4880], [0.8188, 0.8364]), (0.3094, 0.8190))$	$(([0.1995, 0.3065], [0.8340, 0.8437]), (0.3528, 0.8242))$
A_3	$(([0.3597, 0.4031], [0.8546, 0.8649]), (0.4189, 0.8475))$	$(([0.5038, 0.5296], [0.8513, 0.8628]), (0.3156, 0.8090))$
A_4	$(([0.2367, 0.2638], [0.8175, 0.8359]), (0.3694, 0.8228))$	$(([0.3682, 0.4039], [0.8195, 0.8246]), (0.3595, 0.8171))$

	C_3	C_4
A_1	$(([0.4833, 0.6187], [0.8212, 0.8345]), (0.4171, 0.8332))$	$(([0.2820, 0.3221], [0.8163, 0.8209]), (0.3509, 0.8246))$
A_2	$(([0.3405, 0.4255], [0.8279, 0.8446]), (0.4868, 0.8644))$	$(([0.4422, 0.4856], [0.8080, 0.8170]), (0.3321, 0.8226))$
A_3	$(([0.2284, 0.3683], [0.8206, 0.8319]), (0.3813, 0.8281))$	$(([0.2788, 0.2873], [0.8203, 0.8279]), (0.4281, 0.8195))$
A_4	$(([0.3699, 0.4050], [0.8127, 0.8322]), (0.4136, 0.8334))$	$(([0.4654, 0.4916], [0.8192, 0.8289]), (0.3871, 0.8100))$

Step 6: In this step, we determine the criteria weights, which contains the following two cases:

Case I: The information about the criteria weights is completely unknown. Then, based on Eq. (5.5), we get the following weight vector.

$\varpi = (0.2367, 0.2626, 0.2989, 0.2018)$.

Step 7: With Eq. (5.6), the total support $T\left(F_{ij}\right)$ are determined in matrix $T\left(F_{ij}\right)$.

$$T\left(F_{ij}\right) = \begin{pmatrix} 2.8874 & 2.8674 & 2.7589 & 2.8905 \\ 2.8434 & 2.8441 & 2.8382 & 2.8865 \\ 2.8387 & 2.8138 & 2.8480 & 2.8808 \\ 2.8869 & 2.8714 & 2.8684 & 2.8786 \end{pmatrix}.$$

Step 8: Using Eq. (5.7), the power weights are computed in matrix $\sqcap_{ij}$.

$$\sqcap_{ij} = \begin{pmatrix} 0.0599 & 0.0660 & 0.0723 & 0.0512 \\ 0.0590 & 0.0655 & 0.0743 & 0.0511 \\ 0.0590 & 0.0649 & 0.0748 & 0.0511 \\ 0.0594 & 0.0655 & 0.0745 & 0.0505 \end{pmatrix}.$$

Step 9: The aggregated values of alternatives are obtained using Eq. (5.8) as follows:

$$F_1 = ([0.1050, 0.1201], [0.9905, 0.9907], (0.1309, 0.9909)),$$
$$F_2 = ([0.1658, 0.1827], [0.9901, 0.9905], (0.1237, 0.9908)),$$
$$F_3 = ([0.1037, 0.1069], [0.9907, 0.9911], (0.1604, 0.9907)),$$
$$F_4 = ([0.1742, 0.1844], [0.9908, 0.9913], (0.1441, 0.9903)).$$

Step 10: The score values and ranking of the alternatives is given as follows:
$S(F_1) = -0.2354$, $S(F_2) = -0.2342$, $S(F_3) = -0.2350$, $S(F_4) = -0.2339$.
Thus, the ranking of alternatives is $A_4 > A_2 > A_3 > A_1$.

Case II: The information about the criteria weights are partially known, and the partial information are listed as follows:

$$\Gamma = \{0.15 \leqslant \varpi_1 \leqslant 0.25, 0.14 \leqslant \varpi_2 \leqslant 0.20, 0.26 \leqslant \varpi_3 \leqslant 0.35, 0.30 \leqslant \varpi_1 \leqslant 0.55\}.$$

Using model $(M-2)$, we derive the following single-objective programming model.

$$(M-2)\begin{cases}\max D(\varpi) = 0.5436\varpi_1 + 0.6033\varpi_2 + 0.6865\varpi_3 + 0.4636\varpi_4,\\ \text{s.t. } \varpi \in \Gamma,\ \varpi_j \geqslant 0, j = 1,2,3,4,\ \sum_{j=1}^{4} \varpi_j = 1\end{cases}$$

By solving the above model through Lingo software, we get the following weight vector

$$\varpi = (0.15, 0.20, 0.35, 0.30).$$

Step 7: With Eq. (5.6), the total support $T(F_{ij})$ are determined as follows:

$$T(F_{ij}) = \begin{pmatrix} 2.8874 & 2.8674 & 2.7589 & 2.8905\\ 2.8434 & 2.8441 & 2.8382 & 2.8865\\ 2.8387 & 2.8138 & 2.8480 & 2.8808\\ 2.8869 & 2.8714 & 2.8684 & 2.8786 \end{pmatrix},$$

Step 8: Using Eq. (5.7), the power weights are computed as shown in the below matrix.

$$\sqcap_{ij} = \begin{pmatrix} 0.0380 & 0.0503 & 0.0847 & 0.0760\\ 0.0374 & 0.0498 & 0.0870 & 0.0759\\ 0.0374 & 0.0494 & 0.0876 & 0.0759\\ 0.0376 & 0.0499 & 0.0873 & 0.0751 \end{pmatrix}.$$

Step 9: The aggregated values of alternatives are obtained using Eq. (5.8) as follows:

$$F_1 = (0.1198, 0.1370, 0.9859, 0.9862, 0.1494, 0.9865),$$
$$F_2 = (0.1891, 0.2084, 0.9853, 0.9860, 0.1412, 0.9864),$$
$$F_3 = (0.1184, 0.1220, 0.9862, 0.9868, 0.1830, 0.9861),$$
$$F_4 = (0.1987, 0.2103, 0.9863, 0.9870, 0.1644, 0.9856).$$

Step 10: The score values and ranking of the alternatives is given as follows:

$$S(F_1) = -0.2283, S(F_2) = -0.2266, S(F_3) = -0.2278, S(F_4) = -0.2262.$$

Thus, the ranking of alternatives is $A_4 > A_2 > A_3 > A_1$.

6.4. Parameter Analysis

This section presents a sensitivity analysis of the parameter £, illustrating its impact on the ranking outcomes of the proposed approach. Likewise, in Section 6.2, we have employed the proposed algorithm (outlined in Section 5) on the dataset under varying values of the parameter £ to examine its effect on ranking behavior. The corresponding score values and rankings across different £ values are reported in

Table 1, with a visual representation provided in Fig. 3. These collectively demonstrate the robustness of the model under varying conditions.

A consistent observation across all tested values of £ is the stability of the top-ranked alternative: A_4 remains the optimal choice throughout, indicating the reliability of the DM process. Additionally, the trend shows that as £ increases, the score values for all alternatives rise proportionally. Notably, the difference between the scores of A_3 and A_1 narrows with increasing £, eventually leading to identical scores for these two alternatives when £ $\geqslant 135$.

Figure 3 visually supports these findings. The score trajectories of each alternative gradually converge, particularly between A_1 and A_3, reflecting the sensitivity of their relative performance to changes in £. This convergence suggests that while the overall ranking remains mostly stable, the parameter can influence the closeness between alternatives with similar performance.

While the analysis confirms the method's robustness, a potential limitation is that extreme values of £ may reduce discrimination between closely ranked alternatives, as seen with A_1 and A_3. Future work may explore adaptive or data-driven tuning of £ to enhance discrimination when needed.

Table 1: Ranking results for different values of SuW parameter

£	Score values of alternatives				Ranking
	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	
1	-0.2378	-0.2367	-0.2374	-0.2364	$A_4 > A_2 > A_3 > A_1$
3	-0.2354	-0.2342	-0.2350	-0.2339	$A_4 > A_2 > A_3 > A_1$
5	-0.2340	-0.2328	-0.2336	-0.2325	$A_4 > A_2 > A_3 > A_1$
25	-0.2298	-0.2285	-0.2296	-0.2281	$A_4 > A_2 > A_3 > A_1$
45	-0.2288	-0.2274	-0.2286	-0.2269	$A_4 > A_2 > A_3 > A_1$
70	-0.2282	-0.2268	-0.2281	-0.2263	$A_4 > A_2 > A_3 > A_1$
85	-0.2280	-0.2267	-0.2279	-0.2262	$A_4 > A_2 > A_3 > A_1$
135	-0.2277	-0.2264	-0.2277	-0.2258	$A_4 > A_2 > A_3 = A_1$
185	-0.2276	-0.2263	-0.2276	-0.2257	$A_4 > A_2 > A_3 = A_1$
300	-0.2275	-0.2262	-0.2275	-0.2256	$A_4 > A_2 > A_3 = A_1$

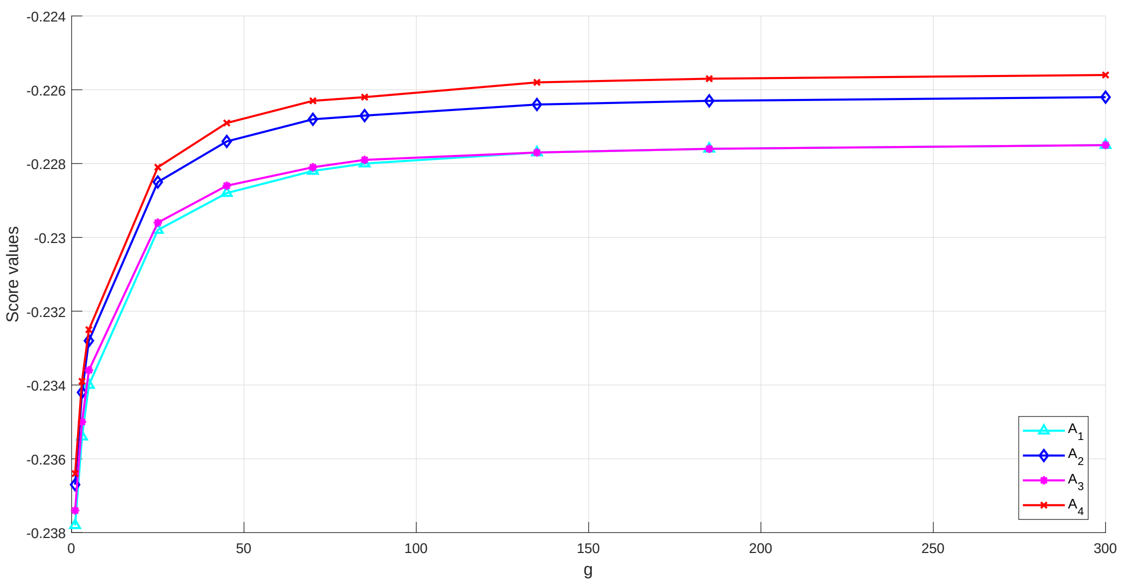

Figure 3: Sensitivity analysis with respect to SuW parameter

7. Comparison with Existing Operators

To demonstrate the efficacy and practical relevance of the proposed model, a comparative analysis is conducted against several well-established AOs from the literature [14, 15, 41, 53]. These existing methods are applied to the same case study previously described, allowing for a direct performance comparison. The computed results, along with the resulting rankings of the alternative options, are summarized in Table 2. This comparison underscores the strengths and advantages of the proposed method in handling complex group DM scenarios.

Table 2: Comparative analysis results

AOs	Score values	Ranking
CCFFWA [15]	-0.9922,-1.0381,-1.0418,-0.9993	$A_1 > A_4 > A_2 > A_3$
CCFFWG [15]	-1.0106,-1.0389,-1.0560,-1.0085	$A_4 > A_1 > A_2 > A_3$
CFEFWG [14]	0.0020,0.0013,0.0011,0.0009	$A_1 > A_2 > A_3 > A_4$
FCFWA [41]	-0.0067,-0.0066,-0.0092,-0.0066	$A_4 = A_2 > A_1 > A_3$
FCFWG [41]	-0.0067,-0.0065,-0.0094,-0.0066	$A_2 > A_4 > A_1 > A_3$
FCWHMA [53]	-0.1766,-0.1770,-0.1800,-0.1794	$A_1 > A_2 > A_4 > A_3$
Proposed approach	-0.2283,-0.2266,-0.2278,-0.2262.	$A_4 > A_2 > A_3 > A_1$

CCFFWA: Confidence cubic fermatean fuzzy weighted averaging, CCFFWG: Confidence cubic fermatean fuzzy weighted geometric, CFEFWG: Cubic fermatean Einstein fuzzy weighted geometric, FCFWA: Fermatean cubic fuzzy-weighted averaging, FCFWG: Fermatean cubic fuzzy-weighted geometric, FCWHMA: Fermatean cubic weighted Heronian mean aggregation.

I). A comparative evaluation of the results derived from Garg et al.'s [15] AOs and those from the proposed approach reveals significant discrepancies in the resulting rankings. Specifically, the CCFFWA and CCFFWG operators produce rankings of $A_1 > A_4 > A_2 > A_3$ and $A_4 > A_1 > A_2 > A_3$, respectively. These differ notably from the rankings generated by our method—particularly in the case of the CCFFWA operator, which ranks A_1 as the most preferred option, whereas it is considered the least favorable in our approach. Two main factors contribute to this divergence. First, the proposed method incorporates power-weighted aggregation, which effectively dampens the influence of extreme values, ensuring a more balanced evaluation. In contrast, Garg et al.'s operators rely exclusively on the weight of the criteria and do not incorporate any mechanism to mitigate outlier impact. Second, the underlying score computation formulas differ fundamentally between the two approaches, which naturally leads to variation in ranking outcomes. It is worth noting, however, that Garg et al.'s operators have the advantage of explicitly integrating confidence levels into the DM process. Although the current case assumes full confidence among DEs (with confidence levels set to 1), their method may prove advantageous in situations involving uncertain or varied confidence levels among decision participants.

II). The CFEFWG operator proposed by Fahmi et al. [14], as defined in their Theorem 1, contains several critical technical issues. Most notably, the formulation applies the same computational expression to both the membership and non-membership intervals—an approach that is logically inconsistent. Furthermore, the use of the Einstein parameter is mishandled, being inappropriately substituted with the criteria weights. These design flaws compromise the reliability of the aggregation process and, by extension, the accuracy of the final rankings. As illustrated in Table 2, the method ranks alternative A_4—which is consistently among the top performers in most other approaches—as the lowest-ranked option. Such inconsistencies underscore the importance of mathematical rigor in the formulation of AOs, particularly in high-stakes DM scenarios.

III). The rankings produced using the FCFWA and FCFWG operators by Riaz et al. [41] are largely in agreement with those obtained through the proposed methodology. However, the FCFWA operator fails to adequately differentiate between A_4 and A_2, and the FCFWG operator yields nearly identical score values for these two alternatives. In contrast, the proposed approach exhibits a stronger discriminatory capacity, successfully distinguishing between all alternatives. Additionally, the FCWHMA operator introduced by Wang et al. [53] yields some variation in the ranking outcomes. This is due to the operator's ability to account for interdependencies among alternatives—an important strength of that method. Nonetheless, our proposed model offers several unique advantages, including its use of power weights to minimize the effect of extreme values, and its generalization through the SuW parameter £, which enhances its adaptability. Notably, many existing operators [41] can be derived as special cases of the proposed model, further affirming its comprehensive and flexible design.

A crucial aspect in decision analysis involves assessing the similarity between two sets of rankings. A basic method is to evaluate their consistency or inconsistency directly. However, this approach is often limited in effectiveness, especially when comparing more than two or three rankings [43]. A more robust strategy involves using coefficients that measure monotonic dependence between rankings. In this context, rankings of the considered alternatives are treated as variables, and various statistical coefficients have been proposed to evaluate the degree of similarity between them.

Some widely used coefficients are outlined below:

- **Weighted Spearman's rank correlation** by Costa and Soares [39]:
$$\tau_w = 1 - \frac{6\sum_{t=1}^{p}(x_t - y_t)^2\left((N - x_t + 1)(N - y_t + 1)\right)}{N(N^3 + N^2 - N - 1)}.$$
- **WS coefficient** by Salabun and Urbaniak [44]:
$$WS = 1 - \sum_{t=1}^{p}\left(2^{-x_t} \cdot \frac{|x_t - y_t|}{\max(|1 - x_t|, |1 - y_t|)}\right).$$
- **Spearman's rank correlation** by Kizielevicz et al. [24]:
$$r_s = 1 - \frac{6\sum_{t=1}^{p}(x_t - y_t)^2}{N(N^2 - 1)}.$$
- **Blest's measure** [10]:
$$\vartheta = 1 - \frac{12\sum_{t=1}^{p}(N + 1 - x_t)^2 y_t - N(N + 1)^2(N + 2)}{N(N + 1)^2(N - 1)}.$$

Here, x_t and y_t represent the ranks or positions of the t-th alternative in the reference and comparative rankings, respectively, and N is the total number of ranked alternatives.

Table 3: Correlation coefficients of various methods with the reference ranking

Method	τ_w	WS	r_s	ϑ
CCFFWA	-0.1200	0.1667	-0.2000	-0.3600
CCFFWG	0.6000	NaN	0.4000	0.4800
CFEFWG	-0.4400	0.4375	-0.8000	-0.8000
FCFWA	0.6800	NaN	0.7000	0.3200
FCFWG	0.4400	0.1875	0.6000	0.6000
FCWHMA	-0.4000	0.3542	-0.4000	-0.5600
Proposed	1.0000	1.0000	1.0000	1.0000

Table 3 presents the correlation coefficients of various existing methods with the reference ranking generated by the proposed method. One can observe that the correlation values vary significantly across different measures and methods. For instance, methods such as FCFWA and CCFFWG show moderate to strong agreement with the proposed ranking in terms of τ_w and r_s, whereas methods like CFEFWG and FCWHMA exhibit negative correlations under several measures, indicating disagreement in the ranking preferences.

Furthermore, the corresponding heatmap visualizations for each correlation measure are shown in Figures 4–7, illustrating the pairwise similarity between all methods, with particular focus on their alignment with the proposed approach.

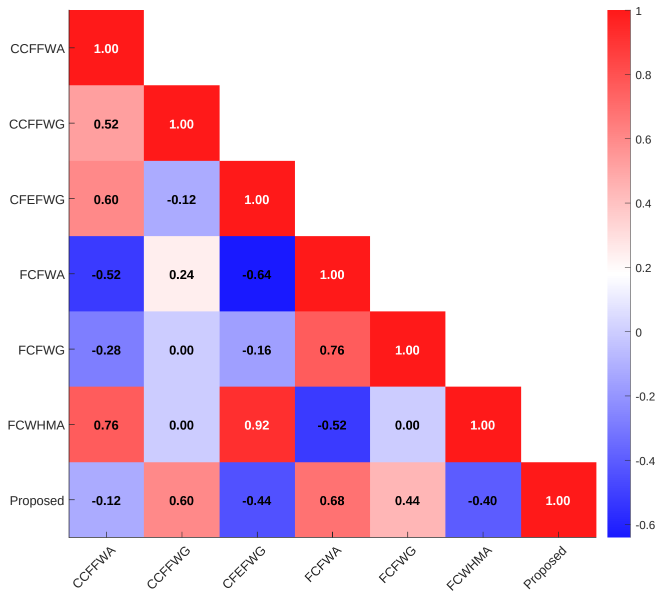

Figure 4: Heatmap visualization of the Weighted Spearman coefficient (τ_w)

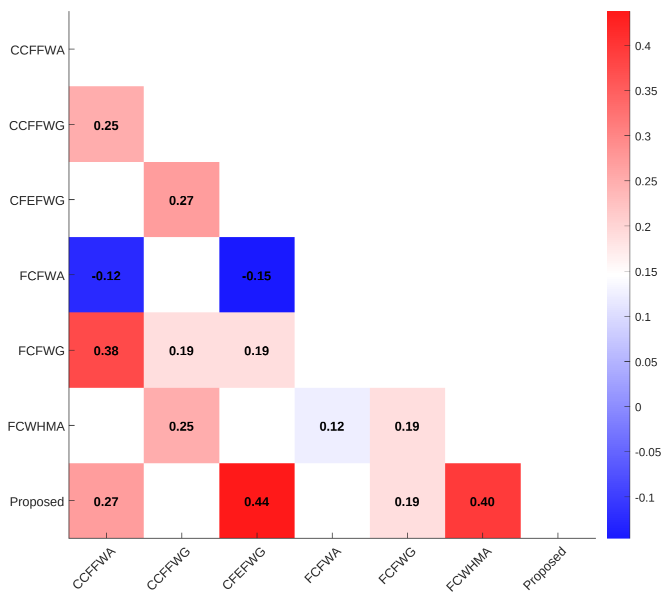

Figure 5: Heatmap visualization of the WS coefficient (WS)

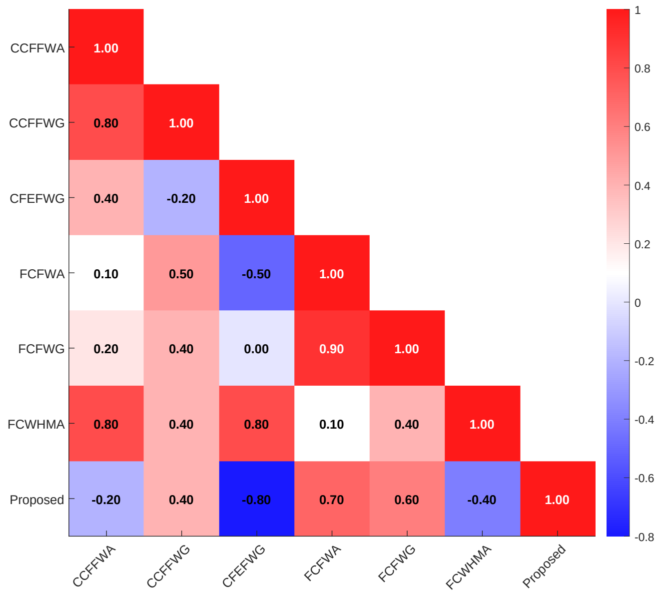

Figure 6: Heatmap visualization of the Spearman's rank correlation (r_s)

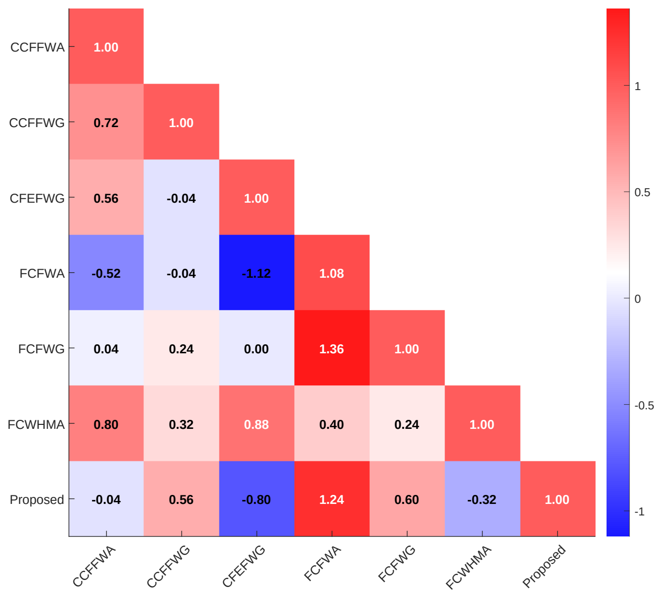

Figure 7: Heatmap visualization of the Blest's measure (ϑ)

Some main merits of the developed methodology are enumerated as follows:

i) Unlike existing methods [14, 15, 41], the proposed approach incorporates maximizing deviation models to effectively address scenarios where weight information is either partially or entirely unknown.

ii) The methodology is built upon SuW operational laws, enhancing its flexibility. DEs can adjust the Sugeno parameter £ based on their specific needs and requirements, making the approach adaptable to diverse contexts.

iii) In contrast to existing techniques [14, 15, 41], the proposed method is specifically designed for group DM, allowing any finite number of DEs to participate in problem-solving while sharing their individual perspectives.

iv) A key advantage of the proposed technique lies in its use of both criteria weights and power weights within the AOs. The utilization of power weights mitigate the impact of extreme data, preventing biases or inaccuracies stemming from either limited knowledge or experience by the DEs.

While the proposed methodology exhibits strong performance across a range of scenarios, several limitations should be acknowledged. First, the current formulation relies on a fixed power value (3) for CFF structures. In certain decision environments, particularly where input data lacks consistency or exhibits extreme fuzziness, this fixed exponent may lead to computational instability or less reliable results. Second, the sensitivity analysis reveals that at higher values of the SuW parameter £, score values across some alternatives begin to converge. This convergence can diminish the method's ability to clearly distinguish between alternatives, potentially impacting its effectiveness in fine-grained DM tasks. Finally, the model does not currently incorporate the psychological behavior or cognitive biases of DEs, which can play a critical role in real-world group DM.

8. Conclusions

- A novel MCGDM framework was developed based on CFFSs and power-based AOs derived from SuW t-norms, offering a flexible and mathematically sound approach to uncertain decision environments.
- Two new operators, CFFSuWPWA and CFFSuWPWG, were introduced, enhancing the aggregation capabilities of CFFNs by enabling more adaptable and reliable information fusion.
- A maximizing deviation-based algorithm was incorporated to compute criteria weights effectively under both partially known and completely unknown weight scenarios.
- The proposed framework was validated through a real-world case study, demonstrating its effectiveness in evaluating alternatives across multiple dimensions and highlighting its ability to produce discriminative and robust decisions.
- Comparative analysis confirmed the model's superiority in terms of accuracy, computational efficiency, and resilience, as compared to existing aggregation approaches.
- Overall, the methodology contributes a scalable and resilient tool for handling complex, group-based decision-making problems under uncertainty, with strong potential for application in dynamic, real-world environments.
- The strength of the proposed solution lies in its ability to combine interval-valued and crisp assessments while incorporating hesitation, which many classical frameworks overlook. This integration allows for nuanced modeling in high-uncertainty situations, particularly in domains where DEs face vague, conflicting, or incomplete information.
- Additionally, the adoption of SuW-based AOs has introduced controllable flexibility into the aggregation process, enabling DEs to adjust the fusion behavior according to the nature of the evaluation. This dynamic adaptability makes the solution not only theoretically appealing but also practically robust across varying real-world contexts.

Looking ahead, several research avenues can further extend the applicability and robustness of the proposed approach. One promising direction involves the development of adaptive mechanisms for parameter tuning—especially for the SuW parameter—based on data-driven insights or machine learning techniques, which may prevent score convergence in extreme settings. Another important extension lies in the integration of cognitive and behavioral DM models, which would allow the framework to better reflect the psychological tendencies and biases of real-world experts. Moreover, employing partition-type or hybrid AOs may enhance the ability to manage large-scale decision matrices with more structured information grouping. Finally, the methodology could be tested on higher-dimensional and real-time datasets across domains such as supply chain management, smart cities, and healthcare systems, which would help verify its scalability, generalizability, and practical effectiveness.

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