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A study on generalized dynamic inequalities of Hilbert-type on time scales

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Abstract

In this paper, we establish new generalizations of certain Hilbert-type dynamic inequalities on an arbitrary time scale $\mathbb{T}$, by constructing a general kernel in n-dimensions for Hilbert-type inequalities. The proofs of our results are based on applications of Hölder's inequality, Fubini's theorem, the integration by parts formula in the nabla calculus on time scales, and the mean inequality. As special cases of our results (corresponding to $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = \mathbb{R}$), one recovers the classical discrete and continuous inequalities previously established by Yang [23]. In the setting of quantum calculus (when $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$), the resulting inequalities are essentially new.

Keywords: Hilbert-type inequalities, time scales nabla calculus, Hölder's inequality, Fubini's theorem, mean inequality.

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1. Introduction

David Hilbert first introduced the double-series inequality in the context of his 1907-1908 lectures on integral equations (later published in [11]). The classical form asserts that if $\{a_m\}$ and $\{b_n\}$ are real sequences satisfying

$$0 < \sum_{m=1}^{\infty} a_m^2 < \infty \quad \text{and} \quad 0 < \sum_{n=1}^{\infty} b_n^2 < \infty,$$

then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} \leqslant C\Big(\sum_{m=1}^{\infty} a_m^2\Big)^{1/2}\Big(\sum_{n=1}^{\infty} b_n^2\Big)^{1/2}. \tag{1.1}$$

Hilbert originally obtained $C = 2\pi$, and in 1911 Issai Schur [20] refined the result to determine the exact best constant $C = \pi$. This conclusion is now known as Hilbert's double-series inequality in its sharp form. In 1911, Schur [20] also established the continuous analogue of (1.1), showing that if f, g are real (nonnegative) functions satisfying

$$0 < \int_0^{\infty} f^2(x)dx < \infty \quad \text{and} \quad 0 < \int_0^{\infty} g^2(y)dy < \infty,$$

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then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \leqslant \pi\Big(\int_0^\infty f^2(x)dx\Big)^{1/2}\Big(\int_0^\infty g^2(y)dy\Big)^{1/2}$$

with the constant π again shown to be best possible. Hardy further extended these results in 1925 by introducing a pair of conjugate exponents (p,η) with $p,\eta > 1$ and $1/p + 1/\eta = 1$. He proved [10] that for nonnegative sequences $\{a_m\}$ and $\{b_n\}$ satisfying

$$0 < \sum_{m=1}^{\infty} a_m^p < \infty \quad \text{and} \quad 0 < \sum_{n=1}^{\infty} b_n^\eta < \infty,$$

the following generalization of Hilbert's discrete inequality holds:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} \leqslant \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^\eta\right)^{\frac{1}{\eta}},$$

where the constant $\pi/\sin(\pi/p)$ is sharp. For the continuous analogue, Hardy and Riesz [11] established that if f and g are nonnegative functions satisfying

$$0 < \int_0^\infty f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^\eta(y)dy < \infty,$$

then the integral inequality

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \leqslant \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^\eta(y)dy\right)^{\frac{1}{\eta}}$$

also holds, with the same sharp constant $\pi/\sin(\pi/p)$. In 1998, Pachpatte [15] introduced a discrete Hilbert-type inequality involving forward differences. Let $a(s) : \{0,1,\dots,p\} \to \mathbb{R}$, $b(\vartheta) : \{0,1,\dots,\eta\} \to \mathbb{R}$, with $a(0) = b(0) = 0$. Then

$$\sum_{s=1}^{p}\sum_{\vartheta=1}^{\eta}\frac{|a_s||b_\vartheta|}{s+\vartheta} \leqslant C(p,\eta)\left(\sum_{s=1}^{p}(p-s+1)|\nabla a_s|^2\right)^{1/2}\left(\sum_{\vartheta=1}^{\eta}(\eta-\vartheta+1)|\nabla b_\vartheta|^2\right)^{1/2},$$

where $\nabla a_s = a_s - a_{s-1}$ and $C(p,\eta) = \frac{1}{2}\sqrt{p\eta}$. In 2000, Pachpatte [16] further generalized this result using conjugate exponents $p,\eta > 1$ with $1/p + 1/\eta = 1$. For sequences defined on $\{0,1,\dots,m\}$ and $\{0,1,\dots,n\}$, respectively, with $a(0) = b(0) = 0$, he proved:

$$\sum_{s=1}^{m}\sum_{\vartheta=1}^{n}\frac{|a_s||b_\vartheta|}{\eta s^{p-1}+p\vartheta^{\eta-1}} \leqslant C(p,\eta,m,n)\left(\sum_{s=1}^{m}(m-s+1)|\nabla a_s|^p\right)^{1/p}\left(\sum_{\vartheta=1}^{n}(n-\vartheta+1)|\nabla b_\vartheta|^\eta\right)^{1/\eta}, \tag{1.2}$$

where $\nabla a_s = a_s - a_{s-1}$ and $C(p,\eta,m,n) = \frac{1}{p\eta}m^{\frac{p-1}{p}}n^{\frac{\eta-1}{\eta}}$. In 2002, Kim et al. [12] removed the conjugacy condition $1/p + 1/\eta = 1$ from the inequality (1.2), replacing the denominator kernel with $\mu s^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}$, and proved that for $\lambda,\mu > 1$, and real sequences $a(s) : \{0,1,\dots,p\} \to \mathbb{R}$, $b(\vartheta) : \{0,1,\dots,\eta\} \to \mathbb{R}$ satisfying $a(0) = b(0) = 0$, the following inequality holds:

$$\begin{aligned}\sum_{s=1}^{p}\sum_{\vartheta=1}^{\eta}&\frac{|a_s||b_\vartheta|}{\mu s^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda\vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\\ &\leqslant D^*(\lambda,\mu,p,\eta)\left(\sum_{s=1}^{p}(p-s+1)|\nabla a_s|^\lambda\right)^{1/\lambda}\left(\sum_{\vartheta=1}^{\eta}(\eta-\vartheta+1)|\nabla b_\vartheta|^\mu\right)^{1/\mu},\end{aligned} \tag{1.3}$$

where $\nabla a_s = a_s - a_{s-1}$ and $D^*(\lambda,\mu,p,\eta) = \frac{1}{\lambda+\mu} p^{\frac{\lambda-1}{\lambda}} \eta^{\frac{\mu-1}{\mu}}$. Kim et al. also derived a continuous analogue of (1.3): for $\lambda,\mu > 1$, real continuous functions f and g on $(0,x)$ and $(0,y)$, respectively, with $f(0) = g(0) = 0$ and $x, y > 0$, they proved

$$\begin{aligned} &\int_0^x \int_0^y \frac{|f(s)||g(t)|}{\mu s^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda t^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}} ds dt \\ &\quad \leqslant \frac{1}{\lambda+\mu} x^{\frac{\lambda-1}{\lambda}} y^{\frac{\mu-1}{\mu}} \left(\int_0^x (x-s)|f'(s)|^\lambda ds \right)^{1/\lambda} \left(\int_0^y (y-t)|g'(t)|^\mu dt \right)^{1/\mu}. \end{aligned} \tag{1.4}$$

The kernel $\mu s^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda \vartheta^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}$ appearing in inequalities (1.3) and (1.4) can be regarded as a special case of a more general n-variable structure. Specifically, this kernel can be generalized to the form $\sum_{i=1}^n \beta_i s_i^{(p_i-1)\alpha}$, where $n \in \mathbb{N}$ with $n \geqslant 2$, $p_i > 1$ for $i = 1, 2, \ldots, n$, $\alpha = \sum_{i=1}^n \frac{1}{p_i}$, and $\beta_i = \prod_{\substack{j=1 \\ j \neq i}}^n p_j$. For $n = 2$, with $p_1 = \lambda$ and $p_2 = \mu$, we recover $\beta_1 = p_2 = \mu$, $\beta_2 = p_1 = \lambda$, $\alpha = \frac{1}{\lambda} + \frac{1}{\mu}$, so that

$$\sum_{i=1}^2 \beta_i s_i^{(p_i-1)\alpha} = \mu s_1^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}} + \lambda s_2^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}},$$

which coincides exactly with the kernel in (1.3) and (1.4). Using this approach, Yang [23] established the following n-dimensional Hilbert-type inequality. Let $n \in \mathbb{N}$ with $n \geqslant 2$, and let $a_i(s_i) : \{0, 1, \ldots, m_i\} \to \mathbb{R}$ be real-valued functions satisfying $a_i(0) = 0$ for $i = 1, \ldots, n$, where $m_i \in \mathbb{N}$ and $p_i > 1$. Define

$$\beta_i = \prod_{\substack{j=1 \\ j \neq i}}^n p_j, \qquad \alpha = \sum_{i=1}^n \frac{1}{p_i}. \tag{1.5}$$

Then the inequality

$$\sum_{s_1=1}^{m_1} \cdots \sum_{s_n=1}^{m_n} \frac{\prod_{i=1}^n |a_i(s_i)|}{\sum_{i=1}^n \beta_i s_i^{(p_i-1)\alpha}} \leqslant \frac{1}{\sum_{i=1}^n \beta_i} \prod_{i=1}^n m_i^{\frac{p_i-1}{p_i}} \prod_{i=1}^n \left(\sum_{s_i=1}^{m_i} (m_i - s_i + 1) |\nabla a_i(s_i)|^{p_i} \right)^{\frac{1}{p_i}} \tag{1.6}$$

holds, where $\nabla a_i(s_i) = a_i(s_i) - a_i(s_i - 1)$ for each $i = 1, \ldots, n$ (see [23, Theorem 2.1]). Furthermore, Yang derived the continuous analogue of (1.6) as follows. Let $n \in \mathbb{N}$ with $n \geqslant 2$ and let $f_i : [0, x_i) \to \mathbb{R}$ be differentiable functions satisfying $f_i(0) = 0$ and $x_i \in (0, \infty)$ for each $i = 1, \ldots, n$. Then

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |f_i(s_i)|}{\sum_{i=1}^n \beta_i s_i^{(p_i-1)\alpha}} ds_n \cdots ds_1 \leqslant \frac{1}{\sum_{i=1}^n \beta_i} \prod_{i=1}^n x_i^{\frac{p_i-1}{p_i}} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i) |f_i'(s_i)|^{p_i} ds_i \right)^{\frac{1}{p_i}}, \tag{1.7}$$

as given in [23, Theorem 3.1]. In addition, Yang [23] extended the discrete inequality (1.6) by replacing the univariate functions $a_i(s_i)$ with bivariate functions $a_i(s_i, t_i)$ and introducing a new two-dimensional structure in the kernel. Let $n \in \mathbb{N}$ with $n \geqslant 2$, $p_i > 1$, and define (1.5). Suppose $a_i(s_i, t_i) : \{0, 1, \ldots, m_i\} \times \{0, 1, \ldots, n_i\} \to \mathbb{R}$ satisfies $a_i(s_i, 0) = a_i(0, t_i) = 0$ for all $i = 1, \ldots, n$, where $m_i, n_i \in \mathbb{N}$. Then the following inequality holds:

$$\begin{aligned} &\sum_{s_1=1}^{m_1} \sum_{t_1=1}^{n_1} \cdots \sum_{s_n=1}^{m_n} \sum_{t_n=1}^{n_n} \frac{\prod_{i=1}^n |a_i(s_i, t_i)|}{\sum_{i=1}^n \beta_i (s_i t_i)^{(p_i-1)\alpha}} \\ &\leqslant \frac{1}{\sum_{i=1}^n \beta_i} \prod_{i=1}^n (m_i n_i)^{\frac{p_i-1}{p_i}} \prod_{i=1}^n \left(\sum_{s_i=1}^{m_i} \sum_{t_i=1}^{n_i} (m_i - s_i + 1)(n_i - t_i + 1) |\nabla_2 \nabla_1 a_i(s_i, t_i)|^{p_i} \right)^{\frac{1}{p_i}}, \end{aligned} \tag{1.8}$$

where ∇_1 and ∇_2 denote forward differences with respect to the first and second variables, respectively (see [23, Theorem 2.2]). Yang also provided the continuous analogue of (1.8), extending (1.7) to the setting of functions of two variables. Let $f_i : [0, x_i) \times [0, y_i) \to \mathbb{R}$ be twice continuously differentiable functions such that $f_i(s_i, 0) = f_i(0, t_i) = 0$ for all $i = 1, \dots, n$, where $x_i, y_i \in (0, \infty)$. Then the following inequality holds:

$$\begin{aligned}&\int_0^{x_1}\int_0^{y_1}\cdots\int_0^{x_n}\int_0^{y_n}\frac{\prod_{i=1}^n |f_i(s_i,t_i)|}{\sum_{i=1}^n \beta_i (s_i t_i)^{(p_i-1)\alpha}} ds_n dt_n \cdots ds_1 dt_1\\ &\quad\leqslant \frac{1}{\sum_{i=1}^n \beta_i}\prod_{i=1}^n (x_i y_i)^{\frac{p_i-1}{p_i}}\prod_{i=1}^n\left(\int_0^{x_i}\int_0^{y_i}(x_i-s_i)(y_i-t_i)\left|\frac{\partial^2 f_i}{\partial s_i \partial t_i}(s_i,t_i)\right|^{p_i} ds_i dt_i\right)^{\frac{1}{p_i}},\end{aligned} \tag{1.9}$$

as given in [23, Theorem 3.2]. This naturally motivates researchers to seek a unification of the continuous inequality (1.4) and its discrete counterpart (1.3) into a single, more general inequality applicable across both discrete and continuous domains. Following up on the response of this idea, by using a general domain (a time scale $\mathbb{T}$), Zakarya et al. [24] established the time scale version of inequalities (1.3) and (1.4) and proved that if $a, x, y \in \mathbb{T}$, λ, $\mu > 1$ and $f \in C_{ld}([a, y)_{\mathbb{T}}, \mathbb{R})$, $g \in C_{ld}([a, x)_{\mathbb{T}}, \mathbb{R})$ are ∇-differentiable such that $f(a) = g(a) = 0$, then for x, $y \in (a, \infty)_{\mathbb{T}}$,

$$\begin{aligned}&\int_a^x\int_a^y \frac{|f(s)|\,|g(t)|}{\mu(s-a)^{\frac{(\lambda-1)(\lambda+\mu)}{\lambda\mu}}+\lambda(t-a)^{\frac{(\mu-1)(\lambda+\mu)}{\lambda\mu}}}\nabla s\nabla t\\ &\leqslant \frac{1}{\lambda+\mu}(y-a)^{\frac{\lambda-1}{\lambda}}(x-a)^{\frac{\mu-1}{\mu}}\left(\int_a^x (x-\rho(s))\left|f^{\nabla}(s)\right|^{\lambda}\nabla s\right)^{\frac{1}{\lambda}}\left(\int_a^y (y-\rho(t))\left|g^{\nabla}(t)\right|^{\mu}\nabla t\right)^{\frac{1}{\mu}}.\end{aligned} \tag{1.10}$$

As special cases of (1.10), one recovers: the classical continuous inequality (1.4) when $\mathbb{T} = \mathbb{R}$, the discrete version (1.3) when $\mathbb{T} = \mathbb{N}$, and the corresponding formula in quantum calculus when $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$. For further developments and foundational tools in dynamic inequalities on time scales, we refer the reader to the book by Agarwal et al. [1] and recent papers including [2, 4–7, 13, 17–19].

Our aim in this paper is to extend the Hilbert-type inequalities (1.6), (1.7), (1.8), and (1.9) to their dynamic analogues on time scales. This is accomplished by employing tools such as the mean inequality, Hölder's inequality, Fubini's theorem, and integration by parts in the nabla calculus. The organization of the paper as follows. Section 2 collects preliminary definitions and lemmas from time scale calculus that will be needed in proving the main results. Section 3 contains our main theorems, along with corollaries, illustrative examples, and relevant remarks.

2. Basic lemmas

In this section, we collect several important results from nabla calculus and the theory of time scales that will be used in our main results. For background, notation, and further details on time scale calculus, we refer the reader to the monograph by Bohner and Peterson [8], which offers a comprehensive treatment of the subject.

On a time scale $\mathbb{T}$, the *backward jump* is $\rho : \mathbb{T} \to \mathbb{T}$ with $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ (set $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$), and the *graininess* is $\nu(t) = t - \rho(t)$. For any function v, write $v_\rho(t) := v(\rho(t))$.

Definition 2.1 (ld-continuity, [8, Definition 8.43]). A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *ld-continuous* if it is continuous at each left-dense point in $\mathbb{T}$ and the right-hand limit $\lim_{s \to t^+} f(s)$ exists and is finite for all right-dense points $t \in \mathbb{T}$. The space of all ld-continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

Lemma 2.2 (Chain rule, [9, Lemma 3.1]). *Let $u : \mathbb{R} \to \mathbb{R}$ be continuously differentiable, and let $v : \mathbb{T} \to \mathbb{R}$ be nabla differentiable. Then the composition $u \circ v$ is also nabla differentiable, and the derivative satisfies*

$$(u \circ v)^{\nabla}(t) = v^{\nabla}(t)\int_0^1 u'\big(v^{\rho}(t) + h\nu(t)v^{\nabla}(t)\big)dh,$$

where $\nu(t) = t - \rho(t)$ *denotes the graininess function.*

Theorem 2.3 (Properties of nabla integration, [8, Theorem 8.47]). *Let* $a, b, c \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$, *and suppose* $u, v : \mathbb{T} \to \mathbb{R}$ *are ld-continuous. Then the following properties hold:*

1. $\int_a^b [\alpha u(t) + \beta v(t)] \nabla t = \alpha \int_a^b u(t)\nabla t + \beta \int_a^b v(t)\nabla t$;
2. $\int_a^b u(t)\nabla t = -\int_b^a u(t)\nabla t$;
3. $\int_a^c u(t)\nabla t = \int_a^b u(t)\nabla t + \int_b^c u(t)\nabla t$;
4. $\int_a^a u(t)\nabla t = 0$;
5. *the integration by parts formula is given by*

$$\int_a^b u(t)v^{\nabla}(t)\nabla t = u(t)v(t)|_a^b - \int_a^b u^{\nabla}(t)v^{\rho}(t)\nabla t. \tag{2.1}$$

Lemma 2.4 (Hölder's inequality, [14, Theorem 3.1]). *Let* $a, b \in \mathbb{T}$, *and let* $f, g \in C_{ld}([a, b]_{\mathbb{T}}, \mathbb{R})$. *Then for any* $p > 1$ *with conjugate exponent* $p^* = \frac{p}{p-1}$, *the following inequality holds:*

$$\int_a^b |f(\tau)g(\tau)|\nabla\tau \leqslant \left(\int_a^b |f(\tau)|^p \nabla\tau\right)^{1/p} \left(\int_a^b |g(\tau)|^{p^*} \nabla\tau\right)^{1/p^*}. \tag{2.2}$$

Lemma 2.5 (Fubini's theorem, [3]). *Let* $a, b, c, d \in \mathbb{T}$, *and suppose* $f \in CC_{ld}([a, b]_{\mathbb{T}} \times [c, d]_{\mathbb{T}}, \mathbb{R})$ *is* ∇-*integrable. Then*

$$\int_a^b \left(\int_c^d f(x, y)\nabla_2 y\right) \nabla_1 x = \int_c^d \left(\int_a^b f(x, y)\nabla_1 x\right) \nabla_2 y.$$

Lemma 2.6 (Mean inequality, [11]). *Let* $\delta_i, \epsilon_i > 0$ *for* $i = 1, 2, \ldots, n$. *Then*

$$\prod_{i=1}^{n} \delta_i^{\epsilon_i} \leqslant \left(\frac{\sum_{i=1}^n \delta_i \epsilon_i}{\sum_{i=1}^n \epsilon_i}\right)^{\sum_{i=1}^n \epsilon_i}.$$

Before stating the following lemma in two dimensions, we introduce some notation and continuity conditions. Let $\mathbb{T}_1$, $\mathbb{T}_2$ be time scales, and let $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be a real-valued function. We say that f is ld-continuous in t_2 if, for every fixed $\alpha_1 \in \mathbb{T}_1$, the function $t_2 \mapsto f(\alpha_1, t_2)$ is ld-continuous on $\mathbb{T}_2$. Similarly, f is ld-continuous in t_1 if, for every fixed $\alpha_2 \in \mathbb{T}_2$, the function $t_1 \mapsto f(t_1, \alpha_2)$ is ld-continuous on $\mathbb{T}_1$. We define CC_{ld} as the space of functions $f(t_1, t_2)$ on $\mathbb{T}_1 \times \mathbb{T}_2$ that are ld-continuous in both t_1 and t_2. Furthermore, let CC^1_{ld} denote the subclass of CC_{ld} consisting of functions for which both partial nabla derivatives $\nabla_1 f$ and $\nabla_2 f$ exist and belong to CC_{ld} (essentially $CC^2_{ld} := \{f \in CC^1_{ld} : \nabla_1\nabla_2 f$ exists and $\nabla_1\nabla_2 f \in CC_{ld}\}$). As a special case of [21, Theorem 3.3] (taking $\alpha = 0$ and $h(\tau, \xi) = 1$), we obtain the following inequality.

Lemma 2.7 (Hölder's inequality, [21, Theorem 3.3]). *Let* $\varepsilon, \delta \in \mathbb{T}$ *with* $\varepsilon < \delta$, *and let* $f, g \in CC_{ld}([\varepsilon, \delta]_{\mathbb{T}} \times [\varepsilon, \delta]_{\mathbb{T}}, \mathbb{R})$. *Then, for any* $p > 1$ *with conjugate exponent* $p^* = \frac{p}{p-1}$,

$$\int_\varepsilon^\delta \int_\varepsilon^\delta |f(\tau, \xi)g(\tau, \xi)|\nabla_1\tau\nabla_2\xi \leqslant \left(\int_\varepsilon^\delta \int_\varepsilon^\delta |f(\tau, \xi)|^p \nabla_1\tau\nabla_2\xi\right)^{1/p} \left(\int_\varepsilon^\delta \int_\varepsilon^\delta |g(\tau, \xi)|^{p^*} \nabla_1\tau\nabla_2\xi\right)^{1/p^*}. \tag{2.3}$$

3. Main results

Throughout this section, we assume all integrals considered exist. We now establish the time scale analogues of inequalities (1.6) and (1.7).

Theorem 3.1. *Let* $\mathbb{T}$ *be a time scale,* $a_i, m_i \in \mathbb{T}$ *with* $a_i < m_i$*,* $p_i > 1$*,* $\alpha := \sum_{i=1}^{n} \frac{1}{p_i}$*,* $\beta_i := \prod_{j \neq i} p_j$*,* $i = 1, \dots, n$*, and* $f_i \in C_{ld}([a_i, m_i]_{\mathbb{T}}, \mathbb{R})$ *with* $f_i(a_i) = 0$ *for* $i = 1, \dots, n$*. Then*

$$\begin{aligned} &\int_{a_n}^{m_n} \cdots \int_{a_1}^{m_1} \frac{\prod_{i=1}^{n} |f_i(x_i)|}{\sum_{i=1}^{n} \beta_i (x_i - a_i)^{\alpha(p_i-1)}} \nabla x_1 \cdots \nabla x_n \\ &\quad \leqslant \frac{1}{\sum_{i=1}^{n} \beta_i} \prod_{i=1}^{n} (m_i - a_i)^{\frac{p_i-1}{p_i}} \left(\int_{a_i}^{m_i} (m_i - \rho(x_i)) |f_i^{\nabla}(x_i)|^{p_i} \nabla x_i \right)^{\frac{1}{p_i}}. \end{aligned} \tag{3.1}$$

Proof. Since $f_i(a_i) = 0$, one can observe that

$$\int_{a_i}^{x_i} f_i^{\nabla}(t_i) \nabla t_i = f_i(x_i) - f_i(a_i) = f_i(x_i). \tag{3.2}$$

Then by applying Hölder's inequality (2.2) on the left-hand side of (3.2) with $p_i > 1$ and $p_i^* = p_i/(p_i - 1)$, we have

$$|f_i(x_i)| = \left| \int_{a_i}^{x_i} f_i^{\nabla}(t_i) \nabla t_i \right| \leqslant \int_{a_i}^{x_i} \left| f_i^{\nabla}(t_i) \right| \nabla t_i \leqslant (x_i - a_i)^{\frac{p_i-1}{p_i}} \left(\int_{a_i}^{x_i} \left| f_i^{\nabla}(t_i) \right|^{p_i} \nabla t_i \right)^{\frac{1}{p_i}}. \tag{3.3}$$

Using Lemma 2.6 with $\delta_i = (x_i - a_i)^{(p_i-1)\sum_{i=1}^{n} \frac{1}{p_i}}$ and $\epsilon_i = \frac{1}{p_i}$, we observe that

$$\left(\prod_{i=1}^{n} (x_i - a_i)^{\frac{p_i-1}{p_i}} \right)^{\sum_{j=1}^{n} \frac{1}{p_j}} = \prod_{i=1}^{n} (x_i - a_i)^{\frac{p_i-1}{p_i} \sum_{j=1}^{n} \frac{1}{p_j}} \leqslant \left(\frac{\sum_{i=1}^{n} \frac{1}{p_i} (x_i - a_i)^{(p_i-1)\sum_{j=1}^{n} \frac{1}{p_j}}}{\sum_{i=1}^{n} \frac{1}{p_i}} \right)^{\sum_{i=1}^{n} \frac{1}{p_i}}.$$

Therefore,

$$\prod_{i=1}^{n} (x_i - a_i)^{\frac{p_i-1}{p_i}} \leqslant \frac{\sum_{i=1}^{n} \frac{1}{p_i} (x_i - a_i)^{(p_i-1)\sum_{j=1}^{n} \frac{1}{p_j}}}{\sum_{i=1}^{n} \frac{1}{p_i}} = \frac{\sum_{i=1}^{n} \frac{1}{p_i} (x_i - a_i)^{\alpha(p_i-1)}}{\alpha}. \tag{3.4}$$

Note that

$$\begin{aligned} &\frac{\sum_{i=1}^{n} \frac{1}{p_i} (x_i - a_i)^{\alpha(p_i-1)}}{\alpha} \\ &\quad = \frac{1}{\left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} \right)} \left(\frac{1}{p_1} (x_1 - a_1)^{\alpha(p_1-1)} + \frac{1}{p_2} (x_2 - a_2)^{\alpha(p_2-1)} + \cdots + \frac{1}{p_n} (x_n - a_n)^{\alpha(p_n-1)} \right) \\ &\quad = \frac{p_1 p_2 \cdots p_n}{p_2 \cdots p_n + p_1 p_3 \cdots p_n + \cdots + p_1 p_2 \cdots p_{n-1}} \\ &\quad\quad \times \left(\frac{1}{p_1} (x_1 - a_1)^{\alpha(p_1-1)} + \frac{1}{p_2} (x_2 - a_2)^{\alpha(p_2-1)} + \cdots + \frac{1}{p_n} (x_n - a_n)^{\alpha(p_n-1)} \right) \\ &\quad = \frac{1}{p_2 \cdots p_n + p_1 p_3 \cdots p_n + \cdots + p_1 p_2 \cdots p_{n-1}} \\ &\quad\quad \times \left(p_2 \cdots p_n (x_1 - a_1)^{\alpha(p_1-1)} + p_1 p_3 \cdots p_n (x_2 - a_2)^{\alpha(p_2-1)} + \cdots + p_1 p_2 \cdots p_{n-1} (x_n - a_n)^{\alpha(p_n-1)} \right) \\ &\quad = \frac{1}{\sum_{i=1}^{n} \beta_i} \sum_{i=1}^{n} \beta_i (x_i - a_i)^{\alpha(p_i-1)}. \end{aligned} \tag{3.5}$$

From (3.3), (3.4), and (3.5), one can obtain

$$\begin{aligned}\prod_{i=1}^{n}|f_i(x_i)| &\leqslant \prod_{i=1}^{n}(x_i-a_i)^{\frac{p_i-1}{p_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{x_i}\left|f_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}\\ &\leqslant \frac{\sum_{i=1}^{n}\frac{1}{p_i}(x_i-a_i)^{\alpha(p_i-1)}}{\alpha}\prod_{i=1}^{n}\left(\int_{a_i}^{x_i}\left|f_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}\\ &=\frac{1}{\sum_{i=1}^{n}\beta_i}\left[\sum_{i=1}^{n}\beta_i(x_i-a_i)^{\alpha(p_i-1)}\right]\prod_{i=1}^{n}\left(\int_{a_i}^{x_i}\left|f_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}\end{aligned}$$

and then

$$\begin{aligned}&\int_{a_n}^{m_n}\cdots\int_{a_1}^{m_1}\frac{\prod_{i=1}^{n}|f_i(x_i)|}{\sum_{i=1}^{n}\beta_i(x_i-a_i)^{\alpha(p_i-1)}}\nabla x_1\cdots\nabla x_n\\ &\leqslant \int_{a_n}^{m_n}\cdots\int_{a_1}^{m_1}\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}\left(\int_{a_i}^{x_i}\left|f_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}\nabla x_1\cdots\nabla x_n\\ &=\frac{1}{\sum_{i=1}^{n}\beta_i}\int_{a_n}^{m_n}\cdots\int_{a_1}^{m_1}\prod_{i=1}^{n}\left(\int_{a_i}^{x_i}\left|f_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}\nabla x_1\cdots\nabla x_n\\ &=\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}\int_{a_i}^{m_i}\left(\int_{a_i}^{x_i}\left|f_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}\nabla x_i.\end{aligned}\tag{3.6}$$

Applying Hölder's inequality (2.2) on the right-hand side of (3.6) with $p_i > 1$ and $p_i^* = p_i/(p_i-1)$, we get

$$\int_{a_i}^{m_i}\left(\int_{a_i}^{x_i}\left|f_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\right)^{\frac{1}{p_i}}\nabla x_i \leqslant (m_i-a_i)^{\frac{p_i-1}{p_i}}\left(\int_{a_i}^{m_i}\int_{a_i}^{x_i}\left|f_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\nabla x_i\right)^{\frac{1}{p_i}},$$

thus (3.6) becomes

$$\begin{aligned}&\int_{a_n}^{m_n}\cdots\int_{a_1}^{m_1}\frac{\prod_{i=1}^{n}|f_i(x_i)|}{\sum_{i=1}^{n}\beta_i(x_i-a_i)^{\alpha(p_i-1)}}\nabla x_1\cdots\nabla x_n\\ &\leqslant \frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}(m_i-a_i)^{\frac{p_i-1}{p_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{m_i}\left[\int_{a_i}^{x_i}\left|f_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\right]\nabla x_i\right)^{\frac{1}{p_i}}.\end{aligned}\tag{3.7}$$

Applying the integration by parts formula (2.1) on the right-hand side of (3.7) with $u(x_i) = \int_{a_i}^{x_i}\left|f_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i$ and $v(x_i) = x_i - m_i$, one can obtain

$$\begin{aligned}\int_{a_i}^{m_i}\left[\int_{a_i}^{x_i}\left|f_i^{\nabla}(t_i)\right|^{p_i}\nabla t_i\right]\nabla x_i &= \int_{a_i}^{m_i}u(x_i)v^{\nabla}(x_i)\nabla x_i\\ &= u(m_i)v(m_i)-u(a_i)v(a_i)-\int_{a_i}^{m_i}u^{\nabla}(x_i)v^{\rho}(x_i)\nabla x_i\\ &= -\int_{a_i}^{m_i}u^{\nabla}(x_i)v^{\rho}(x_i)\nabla x_i = \int_{a_i}^{m_i}(m_i-\rho(x_i))\left|f_i^{\nabla}(x_i)\right|^{p_i}\nabla x_i,\end{aligned}$$

so, the inequality (3.7) gives

$$\begin{aligned}&\int_{a_n}^{m_n}\cdots\int_{a_1}^{m_1}\frac{\prod_{i=1}^{n}|f_i(x_i)|}{\sum_{i=1}^{n}\beta_i(x_i-a_i)^{\alpha(p_i-1)}}\nabla x_1\cdots\nabla x_n\\ &\leqslant \frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}(m_i-a_i)^{\frac{p_i-1}{p_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{m_i}(m_i-\rho(x_i))\left|f_i^{\nabla}(x_i)\right|^{p_i}\nabla x_i\right)^{\frac{1}{p_i}},\end{aligned}$$

which is the desired (3.1). □

Remark 3.2. As a special case of Theorem 3.1 with $n = 2$, the inequality (3.1) reduces to the dynamic inequality (1.10) proved by Zakarya et al. [24].

Remark 3.3. If $\mathbb{T} = \mathbb{N}_0$ and $a_i = 0$ for $i = 1, 2, \ldots, n$, then (3.1) reduces to the discrete inequality (1.6) proved by Yang [23].

Remark 3.4. If $\mathbb{T} = \mathbb{R}$ and $a_i = 0$ for $i = 1, 2, \ldots, n$, then (3.1) reduces to the classical integral inequality (1.7) proved by Yang [23].

Corollary 3.5. *If $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, $a_i = q^{\ell_i}$, $m_i = q^{\delta_i}$ with $\ell_i, \delta_i \in \mathbb{N}_0$, $p_i > 1$, $\alpha = \sum_{i=1}^{n} \frac{1}{p_i}$, $\beta_i = \prod_{j \neq i} p_j$, and f_i are real-valued functions on $[a_i, m_i]$ with $f_i(a_i) = 0$ for $i = 1, \ldots, n$, then*

$$\sum_{k_n=\ell_n+1}^{\delta_n} \cdots \sum_{k_1=\ell_1+1}^{\delta_1} \frac{\prod_{i=1}^{n} \left(1 - \frac{1}{q}\right) q^{k_i} \left|f_i(q^{k_i})\right|}{\sum_{i=1}^{n} \beta_i \left(q^{k_i} - q^{\ell_i}\right)^{\alpha(p_i-1)}} \leqslant \frac{1}{\sum_{i=1}^{n} \beta_i} \prod_{i=1}^{n} \left(q^{\delta_i} - q^{\ell_i}\right)^{\frac{p_i-1}{p_i}} \prod_{i=1}^{n} \left(\sum_{k_i=\ell_i+1}^{\delta_i} \left(1 - \frac{1}{q}\right) q^{k_i} \left(q^{\delta_i} - q^{k_i-1}\right) \left|\nabla_q f_i(q^{k_i})\right|^{p_i} \right)^{\frac{1}{p_i}}, \tag{3.8}$$

where $\nabla_q f_i(q^{k_i}) = \dfrac{f_i(q^{k_i}) - f_i(q^{k_i-1})}{q^{k_i} - q^{k_i-1}}$.

In the following, we present some numerical examples to verify our results in two and three dimensions.

Example 3.6. Let $\mathbb{T} = \mathbb{N}_0$, $n = 2$, $a_1 = a_2 = 0$, $m_1 = 2$, $m_2 = 3$, $p_1 = 2$, $p_2 = 3$, $\beta_1 = 3$, $\beta_2 = 2$, $\alpha = \frac{1}{p_1} + \frac{1}{p_2} = \frac{5}{6}$, and $f_1(x) = f_2(x) = x$. Then the left-hand side of (3.1) is given by

$$\sum_{x_2=1}^{3} \sum_{x_1=1}^{2} \frac{f_1(x_1) f_2(x_2)}{p_2 x_1^{\alpha(p_1-1)} + p_1 x_2^{\alpha(p_2-1)}} = \sum_{x_2=1}^{3} \sum_{x_1=1}^{2} \frac{x_1 x_2}{3 x_1^{\frac{5}{6}} + 2 x_2^{\frac{5}{3}}} = 1.5588$$

and the right-hand side of (3.1) gives

$$\begin{aligned} &\frac{1}{p_1 + p_2} m_1^{\frac{p_1-1}{p_1}} m_2^{\frac{p_2-1}{p_2}} \left(\sum_{x_1=1}^{2} (m_1 - x_1 + 1) \right)^{\frac{1}{p_1}} \left(\sum_{x_2=1}^{3} (m_2 - x_2 + 1) \right)^{\frac{1}{p_2}} \\ &= \frac{1}{5} 2^{\frac{1}{2}} 3^{\frac{2}{3}} \left(\sum_{x_1=1}^{2} (3 - x_1) \right)^{\frac{1}{2}} \left(\sum_{x_2=1}^{3} (4 - x_2) \right)^{\frac{1}{3}} = 1.8526. \end{aligned}$$

Hence, the inequality (3.1) holds.

Example 3.7. Taking $\mathbb{T} = \mathbb{R}$, $n = 2$ with $a_1 = a_2 = 0$, $m_1 = m_2 = 1$, $p_1 = p_2 = 2$, $\beta_1 = p_2 = 2$, $\beta_2 = p_1 = 2$, $\alpha = \sum_{i=1}^{2} \frac{1}{p_i} = 1$, and $f_i(x_i) = x_i$ for $i = 1, 2$, then the left-hand side of (3.1) gives

$$\int_{a_2}^{m_2} \int_{a_1}^{m_1} \frac{\prod_{i=1}^{2} |f_i(x_i)|}{\sum_{i=1}^{2} \beta_i (x_i - a_i)^{\alpha(p_i-1)}} \nabla x_1 \nabla x_2 = \int_0^1 \int_0^1 \frac{x_1 x_2}{2(x_1 + x_2)} dx_1 dx_2 = 0.1023$$

and the right-hand side of (3.1) includes

$$\frac{1}{\sum_{i=1}^{2} \beta_i} \prod_{i=1}^{2} (m_i - a_i)^{\frac{p_i-1}{p_i}} \prod_{i=1}^{2} \left(\int_{a_i}^{m_i} (m_i - \rho(x_i)) \left|f_i^{\nabla}(x_i)\right|^{p_i} \nabla x_i \right)^{\frac{1}{p_i}}$$

$$= \frac{1}{\sum_{i=1}^{2}\beta_i}\prod_{i=1}^{2}(m_i-a_i)^{\frac{p_i-1}{p_i}}\prod_{i=1}^{2}\left(\int_{a_i}^{m_i}(m_i-x_i)\left|f_i'(x_i)\right|^{p_i}dx_i\right)^{\frac{1}{p_i}}$$
$$= \frac{1}{\sum_{i=1}^{2}\beta_i}\prod_{i=1}^{2}(m_i-a_i)^{\frac{p_i-1}{p_i}}\prod_{i=1}^{2}\left(\int_{a_i}^{m_i}(m_i-x_i)dx_i\right)^{\frac{1}{p_i}} = 0.125,$$

thus the inequality (3.1) is achieved.

Example 3.8. If $\mathbb{T}=2^{\mathbb{N}_0}$, $q=2$, $n=2$, $\ell_1=\ell_2=0$, $\delta_1=\delta_2=2$, $p_1=p_2=2$, $\alpha=1$, and $\beta_1=p_2=2$, $\beta_2=p_1=2$, and $f_i(x_i)=x_i-1$, then the left-hand side of (3.8) becomes

$$\sum_{k_2=\ell_2+1}^{\delta_2}\sum_{k_1=\ell_1+1}^{\delta_1}\frac{\prod_{i=1}^{2}\left(1-\frac{1}{q}\right)q^{k_i}\left|f_i(q^{k_i})\right|}{\sum_{i=1}^{2}\beta_i\left(q^{k_i}-q^{\ell_i}\right)^{\alpha(p_i-1)}} = \sum_{k_2=1}^{2}\sum_{k_1=1}^{2}\frac{2^{k_1+k_2-2}\left(2^{k_1}-1\right)\left(2^{k_2}-1\right)}{2\left(2^{k_1}-1\right)+2\left(2^{k_2}-1\right)} = 4.75$$

and the right-hand side of (3.8) gives

$$\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}\left(q^{\delta_i}-q^{\ell_i}\right)^{\frac{p_i-1}{p_i}}\prod_{i=1}^{n}\left(\sum_{k_i=\ell_i+1}^{\delta_i}\left(1-\frac{1}{q}\right)q^{k_i}\left(q^{\delta_i}-q^{k_i-1}\right)\left|\nabla_q f_i(q^{k_i})\right|^{p_i}\right)^{\frac{1}{p_i}}$$
$$= \frac{1}{4}\prod_{i=1}^{2}\left(2^{\delta_i}-2^{\ell_i}\right)^{\frac{p_i-1}{p_i}}\prod_{i=1}^{2}\left(\sum_{k_i=\ell_i+1}^{\delta_i}2^{k_i-1}\left(2^{\delta_i}-2^{k_i-1}\right)\right)^{\frac{1}{p_i}} = 5.25.$$

Thus, the inequality (3.8) holds.

Example 3.9. Taking $\mathbb{T}=\mathbb{N}_0$, $n=3$ with $a_1=0$, $a_2=1$, $a_3=0$, $m_1=2$, $m_2=2$, $m_3=3$, $p_1=2$, $p_2=3$, $p_3=6$, $\alpha=\sum_{i=1}^{3}\frac{1}{p_i}=1$, $\beta_1=p_2p_3=18$, $\beta_2=p_1p_3=12$, $\beta_3=p_1p_2=6$, $f_1(x_1)=x_1$, $f_2(x_2)=x_2-1$, and $f_3(x_3)=1-x_3$ with $|\nabla f_i(x_i)|=1$ for $i=1,2,3$, then the left-hand side of (3.1) becomes

$$\sum_{x_3=1+a_3}^{m_3}\sum_{x_2=1+a_2}^{m_2}\sum_{x_1=1+a_1}^{m_1}\frac{\prod_{i=1}^{3}|f_i(x_i)|}{\sum_{i=1}^{3}\beta_i(x_i-a_i)^{\alpha(p_i-1)}} = \frac{311}{18480} = 0.016829$$

and the right-hand side of (3.1) gives

$$\frac{1}{\sum_{i=1}^{3}\beta_i}\prod_{i=1}^{3}(m_i-a_i)^{\frac{p_i-1}{p_i}}\prod_{i=1}^{3}\left(\sum_{x_i=1+a_i}^{m_i}(m_i-x_i+1)\left|\nabla f_i(x_i)\right|^{p_i}\right)^{\frac{1}{p_i}}$$
$$= \frac{1}{\sum_{i=1}^{3}\beta_i}\prod_{i=1}^{3}(m_i-a_i)^{\frac{p_i-1}{p_i}}\prod_{i=1}^{3}\left(\sum_{x_i=1+a_i}^{m_i}(m_i-x_i+1)\right)^{\frac{1}{p_i}} = 0.22917.$$

Hence, the inequality (3.1) is satisfied.

The next result, which generalizes Theorem 3.1, provides a time scale version of inequalities (1.8) and (1.9).

Theorem 3.10. *Let $\mathbb{T}$ be a time scale, and let $a_i, m_i, n_i \in \mathbb{T}$ with $a_i < \min\{m_i, n_i\}$, and $p_i > 1$ for $i = 1, \ldots, n$. Define $\alpha := \sum_{i=1}^{n}\frac{1}{p_i}$, $\beta_i := \prod_{j\neq i}p_j$. Assume $f_i \in CC^2_{ld}([a_i, n_i]\times[a_i, m_i], \mathbb{R})$ satisfies $f_i(x_i, a_i) = f_i(a_i, y_i) =$*

0 for all $x_i \in [a_i, n_i]_{\mathbb{T}}$, $y_i \in [a_i, m_i]_{\mathbb{T}}$. *Then,*

$$\begin{aligned}
&\int_{a_n}^{m_n}\int_{a_n}^{n_n}\cdots\int_{a_1}^{m_1}\int_{a_1}^{n_1} \frac{\prod_{i=1}^{n}|f_i(x_i,y_i)|}{\sum_{i=1}^{n}\beta_i[(y_i-a_i)(x_i-a_i)]^{\alpha(p_i-1)}}\nabla x_1\nabla y_1\cdots\nabla x_n\nabla y_n \\
&\quad\leqslant \frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}[(m_i-a_i)(n_i-a_i)]^{\frac{p_i-1}{p_i}} \\
&\quad\times\prod_{i=1}^{n}\left[\int_{a_i}^{m_i}\int_{a_i}^{n_i}(m_i-\rho(y_i))(n_i-\rho(x_i))\left|f_i^{\nabla_2\nabla_1}(x_i,y_i)\right|^{p_i}\nabla x_i\nabla y_i\right]^{\frac{1}{p_i}}.
\end{aligned} \tag{3.9}$$

Here, ∇_1 *and* ∇_2 *denote the nabla derivatives with respect to* x_i *and* y_i*, respectively.*

Proof. Since $f_i(x_i, a_i) = f_i(a_i, y_i) = 0$ for $x_i \in [a_i, n_i]$, $y_i \in [a_i, m_i]$, we see that

$$\begin{aligned}
\int_{a_i}^{y_i}\int_{a_i}^{x_i} f_i^{\nabla_2\nabla_1}(s_i,t_i)\nabla s_i\nabla t_i &= \int_{a_i}^{y_i}\left(\int_{a_i}^{x_i}(f_i^{\nabla_2}(s_i,t_i))^{\nabla_1}\nabla s_i\right)\nabla t_i \\
&= \int_{a_i}^{y_i}[f_i^{\nabla_2}(x_i,t_i)-f_i^{\nabla_2}(a_i,t_i)]\nabla t_i \\
&= f_i(x_i,y_i)-f_i(x_i,a_i)-f_i(a_i,y_i)+f_i(a_i,a_i) = f_i(x_i,y_i).
\end{aligned} \tag{3.10}$$

Applying Hölder's inequality (2.3) on the left-hand side of (3.10) with $p_i > 1$ and $p_i^* = p_i/(p_i-1)$, we have

$$\begin{aligned}
|f_i(x_i,y_i)| &= \left|\int_{a_i}^{y_i}\int_{a_i}^{x_i} f_i^{\nabla_2\nabla_1}(s_i,t_i)\nabla s_i\nabla t_i\right| \\
&\leqslant \int_{a_i}^{y_i}\int_{a_i}^{x_i}\left|f_i^{\nabla_2\nabla_1}(s_i,t_i)\right|\nabla s_i\nabla t_i \\
&\leqslant (y_i-a_i)^{\frac{p_i-1}{p_i}}(x_i-a_i)^{\frac{p_i-1}{p_i}}\left(\int_{a_i}^{y_i}\int_{a_i}^{x_i}\left|f_i^{\nabla_2\nabla_1}(s_i,t_i)\right|^{p_i}\nabla s_i\nabla t_i\right)^{\frac{1}{p_i}}.
\end{aligned} \tag{3.11}$$

Using Lemma 2.6 with $\delta_i = [(y_i-a_i)(x_i-a_i)]^{(p_i-1)\sum_{j=1}^{n}\frac{1}{p_j}}$ and $\epsilon_i = \frac{1}{p_i}$, we observe that

$$\begin{aligned}
\left(\prod_{i=1}^{n}[(y_i-a_i)(x_i-a_i)]^{\frac{(p_i-1)}{p_i}}\right)^{\sum_{j=1}^{n}\frac{1}{p_j}} &= \prod_{i=1}^{n}[(y_i-a_i)(x_i-a_i)]^{\frac{p_i-1}{p_i}\sum_{j=1}^{n}\frac{1}{p_j}} \\
&\leqslant \left(\frac{\sum_{i=1}^{n}\frac{1}{p_i}[(y_i-a_i)(x_i-a_i)]^{(p_i-1)\sum_{j=1}^{n}\frac{1}{p_j}}}{\sum_{i=1}^{n}\frac{1}{p_i}}\right)^{\sum_{j=1}^{n}\frac{1}{p_j}},
\end{aligned}$$

then by using the same technique in (3.5), one can get

$$\begin{aligned}
&\prod_{i=1}^{n}[(y_i-a_i)(x_i-a_i)]^{\frac{p_i-1}{p_i}} \\
&\quad\leqslant \frac{\sum_{i=1}^{n}\frac{1}{p_i}[(y_i-a_i)(x_i-a_i)]^{\alpha(p_i-1)}}{\alpha} = \frac{1}{\sum_{i=1}^{n}\beta_i}\sum_{i=1}^{n}\beta_i[(y_i-a_i)(x_i-a_i)]^{\alpha(p_i-1)}.
\end{aligned} \tag{3.12}$$

From (3.11) and (3.12), one can obtain

$$\begin{aligned}
\prod_{i=1}^{n}|f_i(x_i,y_i)| &\leqslant \prod_{i=1}^{n}(y_i-a_i)^{\frac{p_i-1}{p_i}}(x_i-a_i)^{\frac{p_i-1}{p_i}}\prod_{i=1}^{n}\left(\int_{a_i}^{y_i}\int_{a_i}^{x_i}\left|f_i^{\nabla_2\nabla_1}(s_i,t_i)\right|^{p_i}\nabla s_i\nabla t_i\right)^{\frac{1}{p_i}} \\
&\leqslant \frac{1}{\sum_{i=1}^{n}\beta_i}\left[\sum_{i=1}^{n}\beta_i[(y_i-a_i)(x_i-a_i)]^{\alpha(p_i-1)}\right]\prod_{i=1}^{n}\left(\int_{a_i}^{y_i}\int_{a_i}^{x_i}\left|f_i^{\nabla_2\nabla_1}(s_i,t_i)\right|^{p_i}\nabla s_i\nabla t_i\right)^{\frac{1}{p_i}},
\end{aligned}$$

thus by applying Fubini's theorem and Hölder's inequality (2.3), we get

$$\begin{aligned}
&\int_{a_n}^{m_n}\int_{a_n}^{n_n}\cdots\int_{a_1}^{m_1}\int_{a_1}^{n_1}\frac{\prod_{i=1}^{n}|f_i(x_i,y_i)|}{\sum_{i=1}^{n}\beta_i[(y_i-a_i)(x_i-a_i)]^{\alpha(p_i-1)}}\nabla x_1\nabla y_1\cdots\nabla x_n\nabla y_n\\
&=\int_{a_n}^{n_n}\int_{a_n}^{m_n}\cdots\int_{a_1}^{n_1}\int_{a_1}^{m_1}\frac{\prod_{i=1}^{n}|f_i(x_i,y_i)|}{\sum_{i=1}^{n}\beta_i[(y_i-a_i)(x_i-a_i)]^{\alpha(p_i-1)}}\nabla y_1\nabla x_1\cdots\nabla y_n\nabla x_n\\
&\leqslant\int_{a_n}^{n_n}\int_{a_n}^{m_n}\cdots\int_{a_1}^{n_1}\int_{a_1}^{m_1}\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}\left(\int_{a_i}^{y_i}\int_{a_i}^{x_i}\left|f_i^{\nabla_2\nabla_1}(s_i,t_i)\right|^{p_i}\nabla s_i\nabla t_i\right)^{\frac{1}{p_i}}\nabla y_1\nabla x_1\cdots\nabla y_n\nabla x_n\\
&=\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}\int_{a_i}^{n_i}\int_{a_i}^{m_i}\left(\int_{a_i}^{y_i}\int_{a_i}^{x_i}\left|f_i^{\nabla_2\nabla_1}(s_i,t_i)\right|^{p_i}\nabla s_i\nabla t_i\right)^{\frac{1}{p_i}}\nabla y_i\nabla x_i\\
&\leqslant\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}[(n_i-a_i)(m_i-a_i)]^{\frac{p_i-1}{p_i}}\prod_{i=1}^{n}\left[\int_{a_i}^{n_i}\int_{a_i}^{m_i}\left(\int_{a_i}^{y_i}\int_{a_i}^{x_i}\left|f_i^{\nabla_2\nabla_1}(s_i,t_i)\right|^{p_i}\nabla s_i\nabla t_i\right)\nabla y_i\nabla x_i\right]^{\frac{1}{p_i}}.
\end{aligned}\tag{3.13}$$

Applying the integration by parts formula (2.1) with

$$u(y_i)=\int_{a_i}^{y_i}\int_{a_i}^{x_i}\left|f_i^{\nabla_2\nabla_1}(s_i,t_i)\right|^{p_i}\nabla s_i\nabla t_i\quad\text{and}\quad v(y_i)=y_i-m_i,$$

we see that

$$\begin{aligned}
\int_{a_i}^{m_i}\left(\int_{a_i}^{y_i}\left[\int_{a_i}^{x_i}\left|f_i^{\nabla_2\nabla_1}(s_i,t_i)\right|^{p_i}\nabla s_i\right]\nabla t_i\right)\nabla y_i&=\int_{a_i}^{m_i}u(y_i)v^{\nabla}(y_i)\nabla y_i\\
&=u(m_i)v(m_i)-u(a_i)v(a_i)-\int_{a_i}^{m_i}u^{\nabla}(y_i)v^{\rho}(y_i)\nabla y_i\\
&=\int_{a_i}^{m_i}\left(\int_{a_i}^{x_i}\left|f_i^{\nabla_2\nabla_1}(s_i,y_i)\right|^{p_i}\nabla s_i\right)(m_i-\rho(y_i))\nabla y_i,
\end{aligned}$$

and then by applying Fubini's theorem, we have

$$\begin{aligned}
&\int_{a_i}^{n_i}\int_{a_i}^{m_i}\left(\int_{a_i}^{y_i}\left[\int_{a_i}^{x_i}\left|f_i^{\nabla_2\nabla_1}(s_i,t_i)\right|^{p_i}\nabla s_i\right]\nabla t_i\right)\nabla y_i\nabla x_i\\
&\quad=\int_{a_i}^{n_i}\int_{a_i}^{m_i}\left(\int_{a_i}^{x_i}\left|f_i^{\nabla_2\nabla_1}(s_i,y_i)\right|^{p_i}\nabla s_i\right)(m_i-\rho(y_i))\nabla y_i\nabla x_i\\
&\quad=\int_{a_i}^{m_i}(m_i-\rho(y_i))\int_{a_i}^{n_i}\left(\int_{a_i}^{x_i}\left|f_i^{\nabla_2\nabla_1}(s_i,y_i)\right|^{p_i}\nabla s_i\right)\nabla x_i\nabla y_i.
\end{aligned}\tag{3.14}$$

Again by applying the integration by parts (2.1), we observe that

$$\int_{a_i}^{n_i}\left(\int_{a_i}^{x_i}\left|f_i^{\nabla_2\nabla_1}(s_i,y_i)\right|^{p_i}\nabla s_i\right)\nabla x_i=\int_{a_i}^{n_i}(n_i-\rho(x_i))\left|f_i^{\nabla_2\nabla_1}(x_i,y_i)\right|^{p_i}\nabla x_i,$$

thus (3.14) gives

$$\begin{aligned}
&\int_{a_i}^{n_i}\int_{a_i}^{m_i}\left(\int_{a_i}^{y_i}\left[\int_{a_i}^{x_i}\left|f_i^{\nabla_2\nabla_1}(s_i,t_i)\right|^{p_i}\nabla s_i\right]\nabla t_i\right)\nabla y_i\nabla x_i\\
&\quad=\int_{a_i}^{m_i}\int_{a_i}^{n_i}(m_i-\rho(y_i))(n_i-\rho(x_i))\left|f_i^{\nabla_2\nabla_1}(x_i,y_i)\right|^{p_i}\nabla x_i\nabla y_i.
\end{aligned}\tag{3.15}$$

Substituting (3.15) into (3.13), we see that

$$\int_{a_n}^{m_n}\int_{a_n}^{n_n}\cdots\int_{a_1}^{m_1}\int_{a_1}^{n_1}\frac{\prod_{i=1}^{n}|f_i(x_i,y_i)|}{\sum_{i=1}^{n}\beta_i[(y_i-a_i)(x_i-a_i)]^{\alpha(p_i-1)}}\nabla x_1\nabla y_1\cdots\nabla x_n\nabla y_n$$
$$\leqslant\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}[(m_i-a_i)(n_i-a_i)]^{\frac{p_i-1}{p_i}}\prod_{i=1}^{n}\left[\int_{a_i}^{m_i}\int_{a_i}^{n_i}(m_i-\rho(y_i))(n_i-\rho(x_i))\left|f_i^{\nabla_2\nabla_1}(x_i,y_i)\right|^{p_i}\nabla x_i\nabla y_i\right]^{\frac{1}{p_i}},$$

as required by inequality (3.9). □

Remark 3.11. If $\mathbb{T}=\mathbb{N}_0$ and $a_i=0$ for all i, then inequality (3.9) reduces to the discrete inequality (1.8) established by Yang [23].

Remark 3.12. If $\mathbb{T}=\mathbb{R}$ and $a_i=0$ for all i, then inequality (3.9) reduces to the classical integral inequality (1.9) also due to Yang [23].

Corollary 3.13. *Let $\mathbb{T}=q^{\mathbb{N}_0}$ with $q>1$, and let $a_i=q^{\ell_i}$, $m_i=q^{\delta_i}$, $n_i=q^{h_i}$, where $\ell_i,\delta_i,h_i\in\mathbb{N}_0$, $p_i>1$, $\alpha=\sum_{i=1}^{n}\frac{1}{p_i}$, and $\beta_i=\prod_{j\neq i}p_j$ for $i=1,\dots,n$. Assume that $f_i:[a_i,n_i]\times[a_i,m_i]\to\mathbb{R}$ satisfies $f_i(x_i,a_i)=f_i(a_i,y_i)=0$ for all $i=1,\dots,n$. Then*

$$\sum_{s_n=\ell_n+1}^{\delta_n}\sum_{k_n=\ell_n+1}^{h_n}\cdots\sum_{s_1=\ell_1+1}^{\delta_1}\sum_{k_1=\ell_1+1}^{h_1}\frac{\prod_{i=1}^{n}\left(1-\frac{1}{q}\right)^2q^{k_i+s_i}\left|f_i(q^{k_i},q^{s_i})\right|}{\sum_{i=1}^{n}\beta_i[(q^{s_i}-q^{\ell_i})(q^{k_i}-q^{\ell_i})]^{\alpha(p_i-1)}}$$
$$\leqslant L\prod_{i=1}^{n}\left[\sum_{s_i=\ell_i+1}^{\delta_i}\sum_{k_i=\ell_i+1}^{h_i}\left(1-\frac{1}{q}\right)^2q^{k_i+s_i}\left(q^{\delta_i}-q^{s_i-1}\right)\left(q^{h_i}-q^{k_i-1}\right)\left|\nabla_{q2}\nabla_{q1}f_i(q^{k_i},q^{s_i})\right|^{p_i}\right]^{\frac{1}{p_i}},\tag{3.16}$$

where

$$L=\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}[(q^{\delta_i}-q^{\ell_i})(q^{h_i}-q^{\ell_i})]^{\frac{p_i-1}{p_i}}.$$

Here, ∇_{q1} and ∇_{q2} denote the q-difference operators with respect to the first and second variables, respectively, defined by

$$\nabla_{q1}f(x,y)=\frac{f(x,y)-f\left(\frac{x}{q},y\right)}{(1-\frac{1}{q})x},\quad\nabla_{q2}f(x,y)=\frac{f(x,y)-f\left(x,\frac{y}{q}\right)}{(1-\frac{1}{q})y}.$$

The next numerical examples will be examined to the results in the continuous, difference, and quantum cases.

Example 3.14. Taking $\mathbb{T}=\mathbb{N}_0$, $n=2$ with $a_1=a_2=0$, $m_1=2$, $m_2=3$, $n_1=2$, $n_2=2$, $p_1=2$, $p_2=3$, $\beta_1=p_2=3$, $\beta_2=p_1=2$, $\alpha=\sum_{i=1}^{2}\frac{1}{p_i}=\frac{5}{6}$, and $f_i(x_i,y_i)=x_iy_i$ for $i=1,2$, where $\nabla_2\nabla_1f_i(x_i,y_i)=1$, then the left-hand side of (3.9) can be written as follows

$$\int_{a_2}^{m_2}\int_{a_2}^{n_2}\int_{a_1}^{m_1}\int_{a_1}^{n_1}\frac{\prod_{i=1}^{2}|f_i(x_i,y_i)|}{\sum_{i=1}^{2}\beta_i[(y_i-a_i)(x_i-a_i)]^{\alpha(p_i-1)}}\nabla x_1\nabla y_1\nabla x_2\nabla y_2$$
$$=\sum_{y_2=1}^{3}\sum_{x_2=1}^{2}\sum_{y_1=1}^{2}\sum_{x_1=1}^{2}\frac{\prod_{i=1}^{2}|f_i(x_i,y_i)|}{\sum_{i=1}^{2}\beta_i[x_iy_i]^{\alpha(p_i-1)}}=7.932$$

and the right-hand side of (3.9) becomes

$$\frac{1}{\sum_{i=1}^{2}\beta_i}\prod_{i=1}^{2}[(m_i-a_i)(n_i-a_i)]^{\frac{p_i-1}{p_i}}$$

$$\times \prod_{i=1}^{2}\left[\int_{a_i}^{m_i}\int_{a_i}^{n_i}(m_i-\rho(y_i))(n_i-\rho(x_i))\left|\nabla_2\nabla_1 f_i(x_i,y_i)\right|^{p_i}\nabla x_i\nabla y_i\right]^{\frac{1}{p_i}}$$

$$=\frac{1}{\sum_{i=1}^{2}\beta_i}\prod_{i=1}^{2}[m_in_i]^{\frac{p_i-1}{p_i}}\prod_{i=1}^{2}\left[\sum_{y_i=a_i+1}^{m_i}\sum_{x_i=a_i+1}^{n_i}(m_i+1-y_i)(n_i+1-x_i)\right]^{\frac{1}{p_i}}=10.384$$

and this shows that (3.9) holds.

Example 3.15. If $\mathbb{T}=\mathbb{R}$ with $n=2$, $a_1=a_2=0$, $m_1=n_1=1$, $m_2=n_2=2$, $p_1=2$, $p_2=4$, $\beta_1=p_2=4$, $\beta_2=p_1=2$, $\alpha=\frac{3}{4}$, and $f_i(x_i,y_i)=x_iy_i$ for $i=1,2$, then the left-hand side of (3.9) becomes

$$\int_{a_2}^{m_2}\int_{a_2}^{n_2}\int_{a_1}^{m_1}\int_{a_1}^{n_1}\frac{\prod_{i=1}^{2}|f_i(x_i,y_i)|}{\sum_{i=1}^{2}\beta_i[(y_i-a_i)(x_i-a_i)]^{\alpha(p_i-1)}}\nabla x_1\nabla y_1\nabla x_2\nabla y_2$$

$$=\int_0^2\int_0^2\int_0^1\int_0^1\frac{x_1y_1x_2y_2}{4(x_1y_1)^{3/4}+2(x_2y_2)^{9/4}}dx_1dy_1dx_2dy_2=0.1736575$$

and the right-hand side of (3.9) satisfies

$$\frac{1}{\sum_{i=1}^{2}\beta_i}\prod_{i=1}^{2}[(m_i-a_i)(n_i-a_i)]^{\frac{p_i-1}{p_i}}\prod_{i=1}^{2}\left[\int_{a_i}^{m_i}\int_{a_i}^{n_i}(m_i-\rho(y_i))(n_i-\rho(x_i))\left|f_i^{\nabla_2\nabla_1}(x_i,y_i)\right|^{p_i}\nabla x_i\nabla y_i\right]^{\frac{1}{p_i}}$$

$$=\frac{1}{6}4^{3/4}\prod_{i=1}^{2}\left[\int_{a_i}^{m_i}\int_{a_i}^{n_i}(m_i-y_i)(n_i-x_i)\left|\frac{\partial^2}{\partial x_i\partial y_i}f_i(x_i,y_i)\right|^{p_i}dx_idy_i\right]^{\frac{1}{p_i}}$$

$$=\frac{1}{6}4^{3/4}\prod_{i=1}^{2}\left[\int_{a_i}^{m_i}\int_{a_i}^{n_i}(m_i-y_i)(n_i-x_i)dx_idy_i\right]^{\frac{1}{p_i}}=0.33333$$

and thus the inequality (3.9) is achieved.

Example 3.16. If $\mathbb{T}=2^{\mathbb{N}_0}$ for $q=2$, $n=2$, $\ell_1=\ell_2=0$, $\delta_1=\delta_2=2$, $h_1=h_2=1$, $p_1=p_2=2$, $\beta_1=p_2=2$, $\beta_2=p_1=2$, $\alpha=1$, and $f_i(x_i,y_i)=(x_i-2^{\ell_i})(y_i-2^{\ell_i})$, then the left-hand side of (3.16) becomes

$$\sum_{s_n=\ell_n+1}^{\delta_n}\sum_{k_n=\ell_n+1}^{h_n}\cdots\sum_{s_1=\ell_1+1}^{\delta_1}\sum_{k_1=\ell_1+1}^{h_1}\frac{\prod_{i=1}^{n}\left(1-\frac{1}{q}\right)^2q^{k_i+s_i}\left|f_i(q^{k_i},q^{s_i})\right|}{\sum_{i=1}^{n}\beta_i[(q^{s_i}-q^{\ell_i})(q^{k_i}-q^{\ell_i})]^{\alpha(p_i-1)}}$$

$$=\sum_{s_2=1}^{2}\sum_{k_n=1}^{1}\sum_{s_1=1}^{2}\sum_{k_1=1}^{1}\left\{\prod_{i=1}^{2}\left[\left(1-\frac{1}{2}\right)^2 2^{k_i+s_i}\left|(2^{k_i}-2^{\ell_i})(2^{s_i}-2^{\ell_i})\right|\right]\right.$$

$$\left.\times\frac{1}{2[(2^{s_1}-2^{\ell_1})(2^{k_1}-2^{\ell_1})]^{\alpha(p_1-1)}+2[(2^{s_2}-2^{\ell_2})(2^{k_2}-2^{\ell_2})]^{\alpha(p_2-1)}}\right\}=\frac{19}{4},$$

while $\nabla_{q2}\nabla_{q1}f_i(q^{k_i},q^{s_i})=1$ and the right-hand side of (3.16) gives

$$\frac{1}{\sum_{i=1}^{n}\beta_i}\prod_{i=1}^{n}[(q^{\delta_i}-q^{\ell_i})(q^{h_i}-q^{\ell_i})]^{\frac{p_i-1}{p_i}}$$

$$\times\prod_{i=1}^{n}\left[\sum_{s_i=\ell_i+1}^{\delta_i}\sum_{k_i=\ell_i+1}^{h_i}\left(1-\frac{1}{q}\right)^2q^{k_i+s_i}\left(q^{\delta_i}-q^{s_i-1}\right)\left(q^{h_i}-q^{k_i-1}\right)\left|\nabla_{q2}\nabla_{q1}f_i(q^{k_i},q^{s_i})\right|^{p_i}\right]^{\frac{1}{p_i}}$$

$$=\frac{1}{4}\prod_{i=1}^{2}[(2^{\delta_i}-2^{\ell_i})(2^{h_i}-2^{\ell_i})]^{\frac{p_i-1}{p_i}}\prod_{i=1}^{2}\left[\sum_{s_i=\ell_i+1}^{\delta_i}\sum_{k_i=\ell_i+1}^{h_i}2^{k_i+s_i-2}\left(2^{\delta_i}-2^{s_i-1}\right)\left(2^{h_i}-2^{k_i-1}\right)\right]^{\frac{1}{p_i}}=\frac{21}{4}$$

and this shows that (3.16) holds.

4. Conclusion

In this paper, we have established novel dynamic inequalities of Hilbert-type on time scales, thereby extending classical discrete and continuous results. Utilizing the theory of nabla calculus, we employed Hölder's inequality and Fubini's theorem on time scales, and an appropriate form of the mean inequality to derive our main results.

By considering functions of multiple variables and applying partial nabla differentiation, we obtained generalizations of known inequalities in the continuous and discrete settings. The results presented not only encompass previously known inequalities as special cases but also furnish new inequalities applicable to a broader class of time scales, including quantum and nonuniform cases.

These contributions enhance the existing body of work on dynamic inequalities and provide a foundational framework for further investigations in the qualitative theory of dynamic equations. In particular, the results may have potential applications in the study of stability, boundedness, and asymptotic behavior of solutions to dynamic systems defined on arbitrary time scales.

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