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# Solution of mixed B.V.P including a first order three dimensional P.D.E with nonlocal and global boundary conditions

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### Abstract

In this paper solution of mixed complex boundary value problem of first order is considered. The basic term in the problem with respect to space variables, has Cauchy-Riemann operator. We first use Laplace transformation to introduce spectral problem. Then we investigate corresponding for Fredholm's type.

The spectral problem here is different from classical boundary value problems. Here boundary conditions are nonlocal and global and dependent functionals to boundary conditions are in general linear.

At the end for the solution of spectral problem which depends on unknown complex parameter, we find asymptotic expansion. With the help of this asymptotic expansion we prove existance and uniqueness of mixed problem.

**Keywords:** Mixed problem, Nonlocal and global boundary conditions, Singularity, Dependent boundary value conditions to general complex functionals, Regularization, Fredholm's type, Asymptotic expansion.

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# 1 Introduction

Let time $t \in (0,\infty)$, $x = (x_1, x_2) \in I\!R^2$ space variables and $D \subset I\!R^2$ to be a bounded connected region. $\Gamma = \overline{D} \setminus D$ considered as a boundary of $D$, where every vertical line to $x_1$ axis, at most can intersect the boundary in two points. Here we assume that $\Gamma$ is a Lyapunov curve [3]. Let each vertical line to $x_1$ axis divide $\Gamma$ in two curves $\Gamma_1$, $\Gamma_2$ respectively. Equations for $\Gamma_1$ and $\Gamma_2$ are given by

$$x_1 \in [a_1, b_1],\ \ x_2 = \gamma_k(x_1)\ ; \qquad k = 1, 2.$$

Consider the following complex, linear partial differential equation

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \frac{\partial u(x,t)}{\partial x_2} + i\frac{\partial u(x,t)}{\partial x_1} + a(x)u(x,t) + f(x,t) \\ &\qquad x \in D \subset I\!R^2\ ,\ \ t > 0 \end{aligned} \tag{1}$$

with boundary condition

$$\begin{aligned} &\alpha_1(x_1)u(x_1,\gamma_1(x_1),t) + \alpha_2(x_1)u(x_1,\gamma_2(x_1),t) \\ &\quad + \int_{a_1}^{b_1} [K_1(x_1,\xi_1)u(\xi_1,\gamma_1(\xi_1),t) + K_2(x_1,\xi_1)u(\xi_1,\gamma_2(\xi_1),t)]d\xi_1 \\ &\quad + \int_D K(x_1,\xi)u(\xi,t)d\xi = \alpha(x_1,t) \qquad x_1 \in [a_1,b_1] \subset I\!R\ ,\ t \geq 0 \end{aligned} \tag{2}$$

and initial value

$$u(x,0) = \varphi(x), \qquad x \in \overline{D}, \tag{3}$$

where $a(x)$, $f(x,t)$, $\alpha_j(x_1)$, $j = 1,2$, $\varphi(x)$ and $\alpha(x_1,t)$ are given condtinuous functions. $K_j(x_1,\xi_1)$ for $j = 1,2$ and $K(x_1,\xi)$ are given continuous kernels or they contain weak singularity. $\Gamma$ the boundary of $D$ is Lyapunov curve or picewise Lyapunov. Let $x_2 = \gamma_k(x_1);\ k = 1,2$ as a curve $\Gamma_k$ to be differentiable and its differential is of Holder class.

Mixed partial differential equations basicaly considered for parabolic (Heat equation) and hyperbolic (wave equation) kinds of problems ([4, 5, 6]). In these cases number of boundary conditions (for local conditions) is the half of highest order of derivative of unknown function with respect to space variables, (for even orders), ([4, 5, 6]).

The problem of investigation in this paper is the basical part of elliptic equation i.e. Cauchy-Riemann equation. Note that in classical mathematical physics problems, the simplest canonical elliptic equations are Laplace equation, which is of second order. Here our problem is of first order with mixed boundary conditions.

Here, if the boundary condition is Dirichlet (with any unknown equation in all over the boundary $\Gamma$) then the problem has no solution. i.e. problem is not well defined. In this problem, boundary condition is nonlocal and the number of boundary conditions is equal to highest order of derivative with respect to space variables ([7, 8, 9]). This phenomenon is similar to ordinary differential equations.

## 2 Spectral Problem

Applying Laplace transformation ([4, 9]) to mixed problem (1-3) gives:

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\frac{\partial u(x,t)}{\partial t}dt &= \int_0^\infty e^{-\lambda t}\frac{\partial u(x,t)}{\partial x_2} + i\int_0^\infty e^{-\lambda t}\frac{\partial u(x,t)}{\partial x_1}dt \\ &+ \int_0^\infty e^{-\lambda t}a(x)u(x,t)dt + \int_0^\infty e^{-\lambda t}f(x,t)dt\end{aligned}$$

where $\lambda = c + i\tau$ is a complex parameter, $c > 0$ and $c,\ \tau \in I\!R$. Now, considering (3) and integrating by parts from left hand side of above equation and by accepting

$$\int_0^\infty e^{-\lambda t}u(x,t)dt = \widetilde{u}\ (x,\lambda), \quad x \in D \tag{4}$$

$$\int_0^\infty e^{-\lambda t}f(x,t)dt = \widetilde{f}\ (x,\lambda), \quad x \in D \tag{5}$$

we can write

$$\ell\ \widetilde{u} \equiv \frac{\partial\ \widetilde{u}\ (x,\lambda)}{\partial x_2} + i\frac{\partial\ \widetilde{u}\ (x,\lambda)}{\partial x_1} - \lambda\ \widetilde{u}\ (x,\lambda) = F(x,\lambda), \quad x \in D \tag{6}$$

where

$$F(x,\lambda) = -\stackrel{\sim}{f}(x,\lambda) - \varphi(x) - a(x)\stackrel{\sim}{u}(x,\lambda), \quad x \in D \tag{7}$$

Similarly we may find from boundary condition (2):

$$\begin{aligned}&\alpha_1(x_1)\stackrel{\sim}{u}(x_1,\gamma_1(x_1),t) + \alpha_2(x_1)u(x_1,\gamma_2(x_1),t)\\&+\int_{a_1}^{b_1}[K_1(x_1,\xi_1)\stackrel{\sim}{u}(\xi_1,\gamma_1(\xi_1),t) + K_2(x_1,\xi_1)\stackrel{\sim}{u}(\xi_1,\gamma_2(\xi_1),t)]d\xi_1\\&+\int_D K(x_1,\xi)\stackrel{\sim}{u}(\xi,t)d\xi = \stackrel{\sim}{\alpha}(x_1,t), \qquad x_1 \in [a_1,b_1]\end{aligned} \tag{8}$$

where

$$\stackrel{\sim}{\alpha}(x_1,\lambda) = \int_0^\infty e^{-\lambda t}\alpha(x_1,t) \tag{9}$$

The problem found from (6) and (8) is called spectral problem dependent to mixed problem (1-3). Since in this problem we do not have any derivative with respect to $\lambda$ we call $\lambda$ as a spectral parameter.

**Remark 1:** In the spectral problem (6) and (8) continuity of kernels $K(x_1,\xi)$ and $K_j(x_1,\xi_1),\ j = 1,2$ is of great importance. If they are not continuous, they are dependent to linear combination of Dirac delta function. Then it is possible to determine minimal and maximal operators for (6) in region $D$ [9]. In this case we may find arbitrary operator between minimal and maximal operator from (6) and (8). In the other words arbitrary operator may be given with help of boundary condition (8) by squeezing down the domain of maximal operator. This is in contradition with what ever we assume in functional analysis, when we are working with theory of operators dependent to differential equations. i.e. we assume that it is not possible to find ordinary operator between maximal and minimal operators ([9, 10, 16]).

# 3 Finding Necessary Conditions

It is obvious that equation for adjoint problem of spectral problem (6) and (8) is in the form

$$\ell^* V \equiv -\frac{\partial V(x,\lambda)}{\partial x_2} + i\frac{\partial V(x,\lambda)}{\partial x_1} - \overline{\lambda} V(x,\lambda) = G(x,\lambda) \quad , \quad x \in D \tag{10}$$

where $G$ is an arbitrary function ([3, 4]).

In order to find fundamental solution for equation (10), we use Fourier transformation ([3, 11]), then we have

$$V(x-\xi,\lambda) = \frac{1}{2\pi i} \cdot \frac{e^{-ic(x_1-\xi_1)+i\tau(x_2-\xi_2)}}{x_1-\xi_1+i(x_2-\xi_2)} \tag{11}$$

where

$$-\frac{\partial V(x-\xi,\lambda)}{\partial x_2} + i\frac{\partial V(x-\xi,\lambda)}{\partial x_1} - \overline{\lambda} V(x-\xi,\lambda) = \delta(x-\xi) \tag{12}$$

while $\delta(x-\xi)$ is a two dimensional Dirac delta function.

Following the technique in [7, 8] we try to find arbitrary solution for (6) in a given region $D$. We are looking for those necessary conditions which the solution itself satisfy in them. Multiplying both sides of (6) into $\overline{V}(x-\xi,\lambda)$ and integrating over region $D$ we may write

$$\int_D \frac{\partial \tilde{u}\,(x,\lambda)}{\partial x_2}\overline{V}(x-\xi,\lambda)dx + i\int_D \frac{\partial \tilde{u}\,(x,\lambda)}{\partial x_1}\overline{V}(x-\xi,\lambda)dx$$
$$-\lambda \int_D \tilde{u}\,(x,\lambda)\overline{V}(x-\xi,\lambda)dx = \int_D F(x,\lambda)\overline{V}(x-\xi,\lambda)dx$$

Applying Astrogradesky-Gauss formula to the first two terms of above equation, or integrating by parts ([3], [4]) gives

$$\int_\Gamma \tilde{u}\,(x,\lambda)\overline{V}(x-\xi,\lambda)[\cos(n,x_2)+i\cos(n,x_1)]dx$$
$$+\int_D \tilde{u}\,(x-\xi,\lambda)\left[-\overline{\frac{\partial V(x-\xi,\lambda)}{\partial x_2} + i\frac{\partial V(x-\xi,\lambda)}{\partial x_1} - \overline{\lambda}(x-\xi,\lambda)}\right]dx$$
$$= \int_D F(x,\lambda)\overline{V}(x-\xi,\lambda)dx$$

where $n$ is the outer normal vector at point $x$ on $\Gamma$. Now using (12) and properties of Dirac delta function ([3], [7]) we will find

$$\begin{aligned}\int_D F(x,\lambda)\overline{V}(x-\xi,\lambda)dx &= \int_\Gamma \tilde{u}\,(x,\lambda)\overline{V}(x-\xi,\lambda)[\cos(n,x_2)+i\cos(n,x_1)]dx \\ &= \begin{cases} \tilde{u}\,(\xi,\lambda) & ;\quad \xi\in D \\ \frac{1}{2}\,\tilde{u}\,(\xi,\lambda) & ;\quad \xi\in\Gamma. \end{cases}\end{aligned} \tag{13}$$

From these relations we may write down the following theorem:

**Theorem 1:** Let region $D$ to be a connected bounded area in $I\!R^2$, and its boundary $\Gamma$ is of Lyapunov curve kinds (or it is picewise Lyapunov curve). Assume every vertical line to $x_1$ axis, intersect the boundary $\Gamma$ in at most two points. Moreover if $F(x,\lambda)$ is a continuous function, then every solution to (6) in $\overline{D}$ is satisfying (13).

Now we try to change boundary $\Gamma$ with $\Gamma_1$ and $\Gamma_2$ introduced in section 1. Consider the last term of (13), we can write

$$\begin{aligned}&\frac{1}{2}\,\tilde{u}\,(\xi_1,\gamma_k(\xi_1),\lambda) = \int_D F(x,\lambda)\overline{V}(x_1-\xi_1,x_2-\gamma_k(\xi_1),\lambda)dx \\ &+\int_{a_1}^{b_1}\tilde{u}\,(x_1,\gamma_1(x_1),\lambda)\overline{V}(x_1-\xi_1,\gamma_1(x_1)-\gamma_k(\xi_1,\lambda)[1-i\gamma_1'(x_1)]dx_1 \\ &-\int_{a_1}^{b_1}\tilde{u}\,(x_1,\gamma_2(x_1),\lambda)\overline{V}(x_1-\xi_1,\gamma_2(x_1)-\gamma_k(\xi_1,\lambda)[1-i\gamma_2'(x_1)]dx_1 \\ &\qquad\qquad k=1,2 \quad , \quad \xi_1\in[a_1,b_1]\end{aligned} \tag{14}$$

If the fundamental solution (11) is substituted in (14), in the resulting equality, for $k=1$, the integral kernel of the second term and for $k=2$, the integral kernel of third term has singularity. Considering these terms we will find

$$\begin{cases} \tilde{u}\,(\xi_1,\gamma_1(\xi_1),\lambda) = \frac{i}{\pi}\int_{a_1}^{b_1}\frac{\tilde{u}(x_1,\gamma_1(x_1),\lambda)}{x_1-\xi_1}dx_1 + B_1(\xi_1,\lambda) \\ \tilde{u}\,(\xi_1,\gamma_2(\xi_1),\lambda) = -\frac{i}{\pi}\int_{a_1}^{b_1}\frac{\tilde{u}(x_1,\gamma_2(x_1),\lambda)}{x_1-\xi_1}dx_1 + B_2(\xi_1,\lambda) \end{cases} \tag{15}$$

where $B_k(\xi_1,\lambda),\ k=1,2$ are the functions with weak singular terms.

**Remark 2:** For those who are intersted in numerical approach, one may write $B_1(\xi_1,\lambda)$ and

$B_2(\xi_1, \lambda)$ in the following forms:

$$\begin{aligned} B_1(\xi_1,\lambda) &= 2\int_D F(x,\lambda)\overline{V}(x_1-\xi_1, x_2-\gamma_1(\xi_1),\lambda) \\ &- 2\int_{a_1}^{b_1} \widetilde{u}\,(x_1,\gamma_2(x_1),\lambda)\overline{V}(x_1-\xi_1,\gamma_2(x_1)-\gamma_1(\xi_1),\lambda)(1-i\gamma_2'(x_1))dx_1 \\ &- \frac{1}{\pi}\int_{a_1}^{b_1} \frac{\gamma_1'(\sigma_1(x_1,\xi_1))-\gamma_1'(x_1)}{1-i\gamma_1'(\sigma_1(x_1,\xi_1)} \cdot \frac{\widetilde{u}\,(x_1,\gamma_1(x_1),\lambda)}{x_1-\xi_1}dx_1 \\ &+ \frac{i}{\pi}\int_{a_1}^{b_1} \frac{1-i\gamma_1'(x_1)}{1-i\gamma_1'(\sigma_1(x_1,\xi_1))}\left\{e^{-i(x_1-\xi_1)[\tau\gamma_1'(\sigma_1(x_1,\xi_1)-c]}-1\right\}\widetilde{u}\,(x_1,\gamma_1(x_1),\lambda) \\ &\cdot \frac{dx_1}{x_1-\xi_1} \end{aligned}$$

we also have

$$\begin{aligned} B_2(\xi_1,\lambda) &= 2\int_D F(x,\lambda)\overline{V}(x_1-\xi_1, x_2-\gamma_2(\xi_1),\lambda) \\ &+ 2\int_{a_1}^{b_1} \widetilde{u}\,(x_1,\gamma_1(x_1),\lambda)\overline{V}(x_1-\xi_1,\gamma_1(x_1)-\gamma_2(\xi_1),\lambda)(1-i\gamma_1'(x_1))dx_1 \\ &+ \frac{1}{\pi}\int_{a_1}^{b_1} \frac{\gamma_2'(\sigma_2(x_1,\xi_1))-\gamma_2'(x_1)}{1-i\gamma_2'(\sigma_2(x_1,\xi_1)} \cdot \frac{\widetilde{u}\,(x_1,\gamma_2(x_1),\lambda)}{x_1-\xi_1}dx_1 \\ &- \frac{i}{\pi}\int_{a_1}^{b_1} \frac{1-i\gamma_2'(x_1)}{1-i\gamma_2'(\sigma_2(x_1,\xi_1))}\left\{e^{-i(x_1-\xi_1)[\tau\gamma_2'(\sigma_2(x_1,\xi_1)-c]}-1\right\}\widetilde{u}\,(x_1,\gamma_2(x_1),\lambda) \\ &\cdot \frac{dx_1}{x_1-\xi_1} \end{aligned}$$

where $\sigma_1(x_1, \xi_1)$, $\sigma_2(x_1, \xi_1)$ are points between $x_1$, $\xi_1$.

Multiplying first and second terms of (15) into $\alpha_1(\xi_1)$ and$-\alpha_2(\xi_1)$ respectively and add two resulted equations. The right hand side of obtained equation have terms with strong singularity. Multipliers of these singularities (in the integral) are nonlocal terms in boundary condition (8). We determine these terms from (8) and by substitution we obtain

$$\begin{aligned} &\alpha_1(\xi_1)\,\widetilde{u}\,(\xi_1,\gamma_1(\xi_1),\lambda)-\alpha_2(\xi_1)\,\widetilde{u}\,(\xi_1,\gamma_2(\xi_1),\lambda) \\ &= \frac{i}{\pi}\int_{a_1}^{b_1}\left\{[\alpha_1(\xi_1)-\alpha_1(x_1)]\,\widetilde{u}\,(x_1,\gamma_1(x_1),\lambda)+[\alpha_2(\xi_1)-\alpha_2(x_1)]\,\widetilde{u}\,(x_1,\gamma_2(x_1),\lambda)\right\} \\ &\cdot\frac{dx_1}{x_1-\xi_1}+\alpha_1(\xi_1)B_1(\xi_1,\lambda)-\alpha_2(\xi_1)B_2(\xi_1,\lambda) \\ &+\frac{i}{\pi}\int_{a_1}^{b_1}\left[\alpha_1(x_1)\,\widetilde{u}\,(x_1,\gamma_1(x_1),\lambda)+\alpha_2(x_1)\,\widetilde{u}\,(x_1,\gamma_2(x_1),\lambda)\right]\frac{dx_1}{x_1-\xi_1} \end{aligned}$$

$$= \frac{i}{\pi}\int_{a_1}^{b_1}\frac{dx_1}{x_1-\xi_1}\Big\{\widetilde{\alpha}\,(x_1,\lambda)-\int_{a_1}^{b_1}\Big[K_1(x_1,\eta_1)\,\widetilde{u}\,(\eta_1,\gamma_1(\eta_1),\lambda)$$
$$+K_2(x_1,\eta_1)\,\widetilde{u}\,(\eta_1,\gamma_2(\eta_1),\lambda)\Big]\,d\eta_1-\int_D K(x_1,\eta)\,\widetilde{u}\,(\eta,\lambda)d\eta\Big\}$$
$$+\alpha_1(\xi_1)B_1(\xi_1,\lambda)-\alpha_2(\xi_1)B_2(\xi_1,\lambda)+\frac{i}{\pi}\int_{a_1}^{b_1}\Big\{\,(\alpha_1(\xi_1)-\alpha_1(x_1))$$
$$\cdot\,\widetilde{u}\,(x_1,\gamma_1(x_1),\lambda)+[\alpha_2(\xi_1)-\alpha_2(x_1)]\,\widetilde{u}\,(x_1,\gamma_2(x_1),\lambda)\Big\}\,\frac{dx_1}{x_1-\xi_1} \qquad (16)$$

Let us assume

**($i$)** The right hand side of boundary condition (8), $\widetilde{\alpha}\,(x_1,\lambda)$ is differentiable [of class $C^{(1)}(a_1,b_1)$] and vanishes at $a_1$ and $b_1$, i.e. $\widetilde{\alpha}\,(a_1,\lambda)=\widetilde{\alpha}\,(b_1,\lambda)=0$.

**($ii$)** $\alpha_1(x_1),\ \alpha_2(x_1)$ in boundary condition (8) are of Holder class.

**($iii$)** $K_1(x_1,\xi_1),\ K_2(x_1,\xi_1)$ the kernels inside the integral term of (8) are continuous or they contain weak singularity.

**Theorem 2:** If assumptions **($i$)**, **($ii$)**, **($iii$)** and conditions in Theorem 1 hold, then solution to boundary value problem (6) and (8) satisfies in regularization relations (16).

**Remark 3:** It is abvious that to keep right hand side of (16) regular, assumptions **($i$)-($iii$)** are sufficient. The first term of right hand side in (16), needs **($i$)**, second and third term need **($iii$)** and the last term needs **($ii$)** to be hold.

# 4 B.V.P (8) Is of Feredholm's Type

Regularization relation (16) plus boundary condition (8) make a system of integral equations with unknowns $\widetilde{u}\,(x_1,\gamma_k(x_1),\lambda);\ k=1,2.$ If the following condition plus conditions **($i$)-($iii$)** satisfies

**($iv$)** In boundary condition (8) the multipliers $\alpha_k(x_1);\ k=1,2$ where $x_1\in[a_1,b_1]$ are nonzero, i.e. $\alpha_k(x_1)\neq 0$, then an above system of integral equations can transform to the normal (canonical) form, where kernels of these integrals contain weak singularity. Hence the following theorem holds:

**Theorem 3:** If additional to conditions in theorem 2, assumption **(**$iv$**)** holds, then the boundary value problem (6) and (8) is of Fredholm's type.

Indeed system (8) and (16) with respect to function $\tilde{u}\ (x_1, \gamma_k(x_1), \lambda);\ k = 1, 2$ is a linear system and is linear dependent with respect to function $\tilde{u}\ (x_1, \lambda),\ x \in D$. If this system of equations has unique solution it is in the form $\tilde{u}\ (x_1, \gamma_k(x_1), \lambda)$ and it is linearly dependent with respect to function $\tilde{u}\ (x, \lambda),\ x \in D$. Now consider solution obtained form above discussion and substitute it into the first relation of (13), hence $\tilde{u}\ (x, \lambda)$ which is a Fredholm integral equation of second type, is obtained. This is a proof that boundary condition problem (6) and (8) is of Fredholm's type.

**Remark 4:** Aliev and Ebadpour [1] considered the boundary value problem (6) and (8) for the special case $\lambda = 0$. They proved for $\lambda = 0$ problem (6) and (8) is of Fredholm's type.

# 5 Solution to B.V.P and It's Asymptotic

In section 3, we explained how the solution of boundary value problem (6) and (8) will obtain from the first relation of (13). i.e.

$$\begin{aligned}
\tilde{u}\ (\xi, \lambda) &= -\int_D a(x)\overline{V}(x-\xi,\lambda)\ \tilde{u}\ (x,\lambda)dx - \int_D \varphi(x)\overline{V}(x-\xi,\lambda)dx \\
&- \int_D \tilde{f}\ (x,\lambda)\overline{V}(x-\xi,\lambda)dx \\
&+ \int_{a_1}^{b_1} \tilde{u}\ (x_1,\gamma_1(x_1),\lambda)\overline{V}(x_1-\xi_1,\gamma_1(x_1)-\xi_2,\lambda)\cdot(1-i\gamma_1'(x_1))dx_1 \\
&- \int_{a_1}^{b_1} \tilde{u}\ (x_1,\gamma_2(x_1),\lambda)\overline{V}(x_1-\xi_1,\gamma_2(x_1)-\xi_2,\lambda)(1-i\gamma_2'(x_1))dx_1 \\
&\qquad\qquad \xi \in D. \qquad (17)
\end{aligned}$$

Substituting fundamental solution (11) in (17), we obtain asymptotics of problem (6) and (8). Without lose of generality, we assume

$$\varphi(x) \equiv 0, \qquad\qquad x \in D, \tag{18}$$

this forces problem (1) and (2) not to be homogeneus. This implies that the second term in (17) vanishes.

To get asymptotics, from third term of (17), first we write (5) in the following way:

$$\begin{aligned}
\tilde{f}(x,\lambda) &= \int_0^\infty e^{-\lambda t} f(x,t)dt = f(x,t)\frac{e^{-\lambda t}}{-\lambda}\Big|_{t=0}^{\infty} \\
&+ \frac{1}{\lambda}\int_0^\infty e^{-\lambda t}\frac{\partial f(x,t)}{\partial t}dt = \frac{f(x,0)}{\lambda} + \frac{1}{\lambda}\left[\frac{\partial f(x,t)}{\partial t}\cdot\frac{e^{-\lambda t}}{-\lambda}\Big|_{t=0}^{\infty}\right. \\
&+ \left.\frac{1}{\lambda}\int_0^\infty e^{-\lambda t}\frac{\partial^2 f(x,t)}{\partial t^2}dt\right] = \frac{f(x,0)}{\lambda} + \frac{1}{\lambda^2}\frac{\partial f(x,t)}{\partial t}\Big|_{t=0} \\
&+ \frac{1}{\lambda^2}\left[\frac{\partial^2 f(x,t)}{\partial t^2}\cdot\frac{e^{-\lambda t}}{-\lambda}\Big|_{t=0}^{\infty} + \frac{1}{\lambda}\int_0^\infty e^{-\lambda t}\frac{\partial^3 f(x,t)}{\partial t^3}dt\right]
\end{aligned}$$

therefore we can write

**($v$)** Function $f(x,t)$ is three times differentiable with respect to t.

If we also assume

$$f(x,0) = \frac{\partial f(x,t)}{\partial t}\Big|_{t=0} = 0,$$

then, third term in (17) for large enough $|\lambda|$ satisfies in

$$\left|\int_D \tilde{f}(x,\lambda)\overline{V}(x-\xi,\lambda)dx\right| = O\left(|\lambda|^{-3}\right). \tag{19}$$

For largest absolute value of $\lambda$, when $\lambda$ has the most distance in Laplace line, if $|\tilde{u}(x,\lambda)|$ is bounded, then the first term in R.H.S. of (17), converges to zero with the speed of Fourier multipliers. Similarly the last two terms in R.H.S. of (17) converges to zero with the same speed. If $|\tilde{u}(x_1,\gamma_k(x_1),\lambda)|$ for $k = 1,2$ is bounded then from (17) we obtain:

$$\lim_{\tau\to+\infty}\tilde{u}(x,\lambda) = 0, \qquad\qquad x \in D. \tag{20}$$

Now if we ignore all those terms with singularity in (14), we may write

$$\lim_{\tau\to+\infty} \tilde{u}\,(x_1,\gamma_k(x_1),\lambda)=0 \qquad , \qquad x_1\in[a_1,b_1] \quad k=1,2 \tag{21}$$

thus the following theorem satisfies.

**Theorem 4:** Let conditions in theorem 3 holds, if $a(x)$ is continuous, $\varphi(x)\equiv 0$ and functions $f(x,t)$, $\alpha(x,t)$ satisfy in condition $(v)$ then for large enough absolute values of $\lambda$, solution of problem (6) and (8) exists and for this solution, the following asymptotic will obtain:

$$\left|\tilde{u}\,(x,\lambda)\right| = O\left(|\lambda|^{-3}\right), \qquad x\in\overline{D}. \tag{22}$$

**Remark 5:** Existence and uniquness of the solution to Schrodinger equation (an equation which is independent of time and dependent to parameter) with nonlocal boundary condition is proved by Kavei and Aliev [13].

# 6 Solution to Complex Problem

With the help of Laplace transformation, solution to complex problem (1-3) may be written as follows ([4, 11]):

$$u(x,t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t}\,\tilde{u}\,(x,\lambda)d\lambda \tag{23}$$

From (22), we considered that the solution of (23) and first derivatives (with respect to $x_1$, $x_2$ and $t$) exist. i.e. in (23) we may let derivative to appear inside integral. Substituting (23) into equation (1) gives([12]):

$$\begin{aligned}\frac{\partial u}{\partial t}-\frac{\partial u}{\partial x_2}-i\frac{\partial u}{\partial x_1} &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-\lambda t}\left\{\lambda\,\tilde{u}-\frac{\partial\,\tilde{u}}{\partial x_2}-i\frac{\partial\,\tilde{u}}{\partial x_1}\right\}d\lambda\\ &= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-\lambda t}F(x,\lambda)d\lambda\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-\lambda t}\,\tilde{f}\,(x,\lambda)d\lambda\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t} a(x) \overset{\sim}{u}(x,\lambda)d\lambda \\
&= \quad f(x,t) + a(x)u(x,t).
\end{aligned}
$$

This means that relation (23) satisfies in (1).

Now substitute (23) in the left hand side of (2), we then obtain:

$$
\begin{aligned}
&\alpha_1(x_1)u(x_1,\gamma_1(x_1),t) + \alpha_2(x_1)u(x_1,\gamma_2(x_1),t) \\
&+\int_{a_1}^{b_1}[K_1(x_1,\xi_1)u(\xi_1,\gamma_1(\xi_1),t) + K_2(x_1,\xi_1)u(\xi_1,\gamma_2(\xi_1),t)d\xi_1 \\
&\int_D K(x_1,\xi)u(\xi,t)d\xi = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t}\Big\{\alpha_1(x_1)\overset{\sim}{u}(x_1,\gamma_1(x_1),\lambda) \\
&+\alpha_2(x_1,\overset{\sim}{u}(x_1,\gamma_2(x_1),\lambda) + \int_{a_1}^{b_1}[K_1(x_1,\xi_1)\overset{\sim}{u}(\xi_1,\gamma_1(\xi_1),\lambda) + K_2(x_1,\xi_1)\overset{\sim}{u}(\xi_1,\gamma_2(\xi_1),\lambda) \\
&\int_D K(x_1,\xi)\overset{\sim}{u}(\xi,\lambda)d\xi\Big\}d\lambda = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t}\overset{\sim}{\alpha}(x_1,\lambda)d\lambda = \alpha(x_1,t)
\end{aligned}
$$

i.e. relation (23) satisfies in boundary condition (2) ([12]).

Finaly by substituting relation (23) into (3) we obtain

$$
u(x,0) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\overset{\sim}{u}(x,\lambda)d\lambda. \tag{24}
$$

From fundamental solution (11) and asymptotic relation (22) we observe that: If we change the Laplace line with the limits of right half side circles, with the help of Jordan Lemma ([14],[15]) it is obvious that relation (24) vanishes. We may therefore write down the following theorem for the mixed and complex problem (1-3).

**Theorem 5:** Let conditions in theorem 4 hold, solution to complex problem (1-3) exists. It is unique and it is in the form (23).

# 7 Related Open Problems

(a) Verifying boundary value problem (6) and (8) where boundary $\Gamma$ of region $D$, is not Lyapunov curve.

$$
\big(\gamma_k(x_1) \in C^{(1)}([a_1,b_1]) \ , \ \ k = 1,2\big)
$$

(b) Verifying problem (6) and (8) where $\Gamma$ is Lyapunov, but $D$ is not a connected region.

(c) Finding the soultion for the following problem:

$$\frac{\partial u(x)}{\partial x_3} = \frac{\partial u(x)}{\partial x_2} + i\frac{\partial u(x)}{\partial x_1} \qquad , \qquad x = (x_1, x_2, x_3) \in D \subset I\!R^3$$

with boundary condition

$$\sum_{j=1}^{2} \left[ \alpha_j(x')u(x', \gamma_j(x)) + \int_S K_j(x', \xi')u(\xi', \gamma_j(\xi'))d\xi' \right] + \int_D K(x', \xi)u(\xi)d\xi = \alpha(x')$$

where $x' = (x_1, x_2) \in S \subset I\!R^2$ and

$$\Gamma_k \; : \; x_3 = \gamma_k(x') \qquad , \qquad k = 1, 2 \qquad x' \in S \;\; , \;\; i = \sqrt{-1}$$

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