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Multiplicity results for a Kirchhoff-type doubly eigenvalue boundary value problem

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Abstract

This paper is concerned with the existence of at least three weak solutions to a class of Kirchhoff-type doubly eigenvalue boundary value problem. The technical approach is mainly based on a very recent three critical points theorem due to B. Ricceri [On a three critical points theorem revisited, Nonlinear Anal., 70 (2009) 3084-3089.]

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1 Introduction

Consider the following Kirchhoff-type problem

$$\begin{cases} -K(\int_a^b |u'(x)|^2 dx)u'' = \lambda f(x,u) + \mu g(x,u), \\ u(a) = u(b) = 0 \end{cases} \tag{1}$$

where $K : [0, +\infty[\to R$ is a continuous function, $f, g : [a, b] \times R \to R$ are two Carathéodory functions and $\lambda,\ \mu > 0$.

Our approach is studying problem (1) relies on the following three critical points theorem (see also [16] for an earlier version as well as [3,11] for related results).

Theorem A. [Ricceri, 15] Let X be a reflexive real Banach space, $I \subseteq R$ an interval, $\Phi : X \longrightarrow R$ a sequentially weakly lower semicontinuous C^1 functional bounded on each bounded subset of X whose derivative admits a continuous inverse on X^* and $J : X \longrightarrow R$ a C^1 functional with compact derivative.
Assume that

$$\lim_{||x||\to+\infty}(\Phi(x) + \lambda J(x)) = +\infty$$

for all $\lambda \in I$, and that there exists $\rho \in R$ such that

$$\sup_{\lambda\in I}\inf_{x\in X}(\Phi(x) + \lambda(J(x) + \rho)) < \inf_{x\in X}\sup_{\lambda\in I}(\Phi(x) + \lambda(J(x) + \rho)).$$

Then, there exist a non-empty open set interval $A \subseteq I$ and a positive real number q with the following property: for every $\lambda \in A$ and every C^1 functional $\Psi : X \longrightarrow R$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$\Phi'(u) + \lambda J'(u) + \mu\Psi'(u) = 0$$

has at least three solutions in X whose norms are less than q.

In the subsequent proofs, we also use the next result precisely to verify the minimax inequality in Theorem A.

Proposition B. [Bonanno, 3] Let X be a non-empty set and $\Phi,\ J$ two real functions on X. Assume that $\Phi(x) \geq 0$ for every $x \in X$ and there exists $u_0 \in X$ such that $\Phi(u_0) = J(u_0) = 0$. Further, assume that exist $u_1 \in X$, $r > 0$ such that
(κ_1) $\Phi(u_1) > r$
(κ_2) $\sup_{\Phi(x)<r}(-J(x)) < r\frac{-J(u_1)}{\Phi(u_1)}$.
Then, for every $\nu > 1$ and for every $\rho \in R$ satisfying

$$\sup_{\Phi(x)<r}(-J(x)) + \frac{r\frac{-J(u_1)}{\Phi(u_1)} - \sup_{\Phi(x)<r}(-J(x))}{\nu} < \rho < r\frac{-J(u_1)}{\Phi(u_1)},$$

one has

$$\sup_{\lambda\in R}\inf_{x\in X}(\Phi(x) + \lambda(J(x) + \rho)) < \inf_{x\in X}\sup_{\lambda\in[0,\sigma]}(\Phi(x) + \lambda(J(x) + \rho))$$

where

$$\sigma = \frac{\nu r}{r\frac{-J(u_1)}{\Phi(u_1)} - \sup_{\Phi(x)<r}(-J(x))}.$$

Problems of Kirchhoff-type have been widely investigated, and among the papers, we refer to the papers [2,6,7,10,12,13,19] and references therein.

Recently, B. Ricceri in an interesting paper [17] established the existence of at least three weak solutions to a class of Kirchhoff-type doubly eigenvalue boundary value problem using Theorem 2 of [14].

The purpose of this paper is to establish the existence of a non-empty open set interval $A \subseteq I$ and a positive real number q with the following property: for each $\lambda \in A$ and for each Carathéodory function $g : [a, b] \times R \to R$ such that $\sup_{|\xi|\leq s} |g(., \xi)| \in L^1(a, b)$ for all $s > 0$, there is $\delta > 0$ such that, for each $\mu \in [0, \delta]$, problem (1) admits at least three weak solutions in $W_0^{1,2}(a, b)$ whose norms are less than q.

For a thorough account on the existence of at least solutions to some second order differential equations with Dirichlet boundary value condition by Ricceri's three critical points theorem [16], we refer to [1,4,5,8,9].

For basic notations and definitions we refer to [18].

2 Preliminaries

We say that u is a weak solution to (1) if $u \in W_0^{1,2}(a, b)$ and

$$K(\int_a^b |u'(x)|^2 dx) \int_a^b u'(x)v'(x)dx - \int_a^b (\lambda f(x, u(x)) + \mu g(x, u(x)))v(x) = 0$$

for every $v \in W_0^{1,2}(a, b)$.

In the sequel, X will denote the Sobolev space $W_0^{1,2}(a, b)$ equipped with the norm

$$|| u || = \left(\int_a^b |u'(x)|^2 dx \right)^{1/2}.$$

Let $K : [0, +\infty[\to R$ be a continuous function such that there exists a positive number m with $K(t) \geq m$ for all $t \geq 0$ and let $f : [a, b] \times R \to R$ be a Carathéodory function such that $\sup_{|\xi|\leq s} |f(., \xi)| \in L^1(a, b)$ for all $s > 0$.

Corresponding to K and f we introduce the functions $\tilde{K} : [0, +\infty[\to R$ and $F : [a, b] \times R \to R$ as follows

$$\tilde{K}(t) = \int_0^t K(s)ds \text{ for all } t \geq 0 \tag{2}$$

and

$$F(x, t) = \int_0^t f(x, s)ds \quad \text{for all } (x, t) \in [a, b] \times R. \tag{3}$$

3 Results

Our main results fully depend on the following technical lemma:

Lemma 1. Assume that there exist positive constants c, d, α and β with $\beta - \alpha < b - a$ such that

(i) $\tilde{K}(d^2(\frac{\alpha+\beta}{\alpha\beta})) > \frac{4mc^2}{b-a}$,
(ii) $F(x,t) \geq 0$ for each $(x,t) \in ([a, a+\alpha] \cup [b-\beta, b]) \times [0, d]$,
(iii) $\int_a^b \sup_{t\in[-c,c]} F(x,t)dx < \frac{4mc^2}{b-a} \frac{\int_{a+\alpha}^{b-\beta} F(x,d)dx}{\tilde{K}(d^2(\frac{\alpha+\beta}{\alpha\beta}))}$.
Then, there exist $r > 0$ and $w \in X$ such that $\tilde{K}(||w||^2) > 2r$ and

$$\int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x,t)dx < 2r\frac{\int_a^b F(x,w(x))dx}{\tilde{K}(||w||^2)}.$$

Proof: We put

$$w(x) = \begin{cases} \frac{d}{\alpha}(x-a) & \text{if } a \leq x < a+\alpha, \\ d & \text{if } a+\alpha \leq x \leq b-\beta, \\ \frac{d}{\beta}(b-x) & \text{if } b-\beta < x \leq b \end{cases} \tag{4}$$

and $r = \frac{2mc^2}{b-a}$. It is easy to see that $w \in X$ and, in particular, one has

$$||w||^2 = d^2(\frac{\alpha+\beta}{\alpha\beta}).$$

Hence, from (i) we have $\tilde{K}(||w||^2) > 2r$. Since $0 \leq w(x) \leq d$ for each $x \in [a,b]$, condition (ii) ensures that

$$\int_a^{a+\alpha} F(x,w(x))dx + \int_{b-\beta}^b F(x,w(x))dx \geq 0.$$

Moreover, owing to our assumptions, we obtain

$$\int_a^b \sup_{t\in[-\sqrt{\frac{r(b-a)}{2m}},\sqrt{\frac{r(b-a)}{2m}}]} F(x,t)dx < \frac{4mc^2}{b-a}\frac{\int_{a+\alpha}^{b-\beta} F(x,d)dx}{\tilde{K}(d^2(\frac{\alpha+\beta}{\alpha\beta}))} \leq 2r\frac{\int_a^b F(x,w(x))dx}{\tilde{K}(||w||^2)}.$$

So, the proof is complete. □

Now, we state our main result.

Theorem 1. Assume that there exist positive constants c, d, α and β with $\beta - \alpha < b - a$ such that (i), (ii) and (iii) in Lemma 1 hold. Furthermore, suppose that

(iv) $\frac{(b-a)^2}{2m} \limsup_{|t|\to+\infty} \frac{F(x,t)}{t^2} < \frac{1}{\theta}$ for almost every $x \in [a,b]$ and for all $t \in R$, and for some θ satisfying

$$\theta > \frac{\frac{2mc^2}{b-a}}{\frac{4mc^2}{b-a}\frac{\int_{a+\alpha}^{b-\beta} F(x,d)dx}{\tilde{K}(d^2(\frac{\alpha+\beta}{\alpha\beta}))} - \int_a^b \sup_{t\in[-c,c]} F(x,t)dx};$$

(v) there exists a continuous function $h : [0, +\infty[\to R$ such that

$$h(tK(t^2)) = t$$

for all $t \geq 0$.

Then, there exist a non-empty open interval $A \subseteq]0, \theta]$ and a number $q > 0$ with the following property: for each $\lambda \in A$ and for an arbitrary Carathéodory function $g : [a, b] \times R \to R$ such that $\sup_{|\xi| \leq s} |g(., \xi)| \in L^1(a, b)$ for all $s > 0$, there is $\delta > 0$ such that, whenever $\mu \in [0, \delta]$, problem (1) admits at least three weak solutions in X whose norms are less than q.

Remark 1. Other candidates for test function w in (4) can be considered to other versions of the statement.

Remark 2. Even if $K(t) = 1$ for all $t \geq 0$ and $\mu = 0$ in (1), we obtain the extension of previous works, specially when $f(x, t) = f(t)$ for every $(x, t) \in [a, b] \times R$, $[a, b] = [0, 1]$ and $\alpha = \beta = \frac{1}{4}$, Theorem 2 reduced to Theorem 2 in [4]. When $\alpha = \beta = \frac{b-a}{4}$, Theorem 2 becomes to Theorem 2 in [5].

Here, we want to point out a remarkable particular situation of Theorem 1.

Corollary 1. Assume that there exist positive constants c, d, p_1, p_2, α and β with $\beta - \alpha < b - a$ such that (ii) in Lemma 1 holds, and
(j) $p_1 d^2 (\frac{\alpha+\beta}{\alpha\beta}) + \frac{p_2}{2} d^4 (\frac{\alpha+\beta}{\alpha\beta})^2 > \frac{4p_1c^2}{b-a}$,
(jj) $\int_a^b \sup_{t\in[-c,c]} F(x, t)dx < \frac{4p_1c^2}{b-a} \frac{\int_{a+\alpha}^{b-\beta} F(x,d)dx}{p_1 d^2(\frac{\alpha+\beta}{\alpha\beta}) + \frac{p_2}{2} d^4 (\frac{\alpha+\beta}{\alpha\beta})^2}$,
(jjj) $\frac{(b-a)^2}{2p_1} \limsup_{|t|\to+\infty} \frac{F(x,t)}{t^2} < \frac{1}{\theta}$ for almost every $x \in [a, b]$ and for all $t \in R$, and for some θ satisfying

$$\theta > \frac{\frac{2p_1c^2}{b-a}}{\frac{4p_1c^2}{b-a} \frac{\int_{a+\alpha}^{b-\beta} F(x,d)dx}{p_1 d^2(\frac{\alpha+\beta}{\alpha\beta}) + \frac{p_2}{2} d^4 (\frac{\alpha+\beta}{\alpha\beta})^2} - \int_a^b \sup_{t\in[-c,c]} F(x, t)dx}.$$

Then, there exist a non-empty open interval $A \subseteq]0, \theta]$ and a number $q > 0$ with the following property: for each $\lambda \in A$ and for an arbitrary Carathéodory function $g : [a, b] \times R \to R$ such that $\sup_{|\xi| \leq s} |g(., \xi)| \in L^1(a, b)$ for all $s > 0$, there is $\delta > 0$ such that, whenever $\mu \in [0, \delta]$, problem

$$\begin{cases} -(p_1 + p_2 \int_a^b |u'(x)|^2 dx)u'' = \lambda f(x, u) + \mu g(x, u), \\ u(a) = u(b) = 0 \end{cases} \tag{5}$$

admits at least three weak solutions in X whose norms are less than q.

Proof: For fixed $p_1,\ p_2 > 0$, set $K(t) = p_1 + p_2 t$ for all $t \geq 0$. Bearing in mind that $m = p_1$, from (j), (jj) and (jjj) we find (i), (iii) and (iv) respectively. In particular, we note that from the setting of K, there exists a continuous function $h : [0, +\infty[\to R$ such that

$$h(tK(t^2)) = t$$

for all $t \geq 0$, because the function K is non-decreasing in $[0, +\infty[$, with $K(0) > 0$, then $t \to tK(t^2)$ $(t \geq 0)$ is increasing and onto $[0, +\infty[$ (see [17, Remark 4]). So, Assumption (jj) is satisfied. Hence, Theorem 1 yields the conclusion. □

Finally, we conclude this section by giving an example to illustrate our results.

Example 1. Consider the problem

$$\begin{cases} -(1 + \int_0^1 |u'(x)|^2 dx)u'' = \lambda(e^{-u}u^{12}(13 - u) + 1) + \mu g(x, u), \\ u(0) = u(1) = 0 \end{cases} \tag{6}$$

where $g : [0, 1] \times R \to R$ is a fixed L^1-Carathéodory function and $\lambda, \mu > 0$. Choose $K(t) = 1 + t$ for all $t \geq 0$ and $f(x, t) = f(t) = e^{-t}t^{12}(13 - t) + 1$ for every $t \in [0, 1]$. Assumptions (j) and (jj) are satisfied by choosing, for instance $d = 2,\ c = 1, [a, b] = [0, 1]$ and $\alpha = \beta = \frac{1}{4}$. In particular, since $\limsup_{|t| \to +\infty} \frac{F(x,t)}{t^2} = 0$, we see that the assumption (jjj) is fulfilled. So, Corollary 1 is applicable to the problem (6) for every $\theta > \frac{272}{2^{12}e^{-2} - 136e^{-1} - 135}$.

4 Proof of Theorem 1

For each $u \in X$, we put

$$\Phi(u) = \frac{1}{2}\tilde{K}(||u||^2)$$

and

$$J(u) = -\int_\Omega F(x, u(x))dx$$

where $\tilde{K}$ and F are given in (2) and (3), respectively. Clearly, Φ is a sequentially weakly lower semicontinuous C^1 functional as well as is bounded on each bounded subset of X, and J is a C^1 functional with compact derivative. In particular, one has

$$\Phi'(u)(v) = K(\int_a^b |u'(x)|^2 dx) \int_a^b u'(x)v'(x)dx$$

and

$$J'(u)(v) = -\int_\Omega f(x, u(x))v(x)dx$$

for every $v \in X$. We claim that Φ' admits a continuous inverse on X (we identity X to X^*). To this end, we need to find a continuous operator $T : X \to X$ such that $T(\Phi'(u)) = u$ for all $u \in X$.
Let $T : X \to X$ be the operator defined by

$$T(v) = \begin{cases} \frac{h(||v||)}{||v||}v & \text{if } v \neq 0 \\ 0 & \text{if } v = 0, \end{cases}$$

where h is defined in the statement. Since, h is continuous and $h(0)=0$, we have that the operator T is continuous in X. So, for every $u \in X$, taking into account that $\inf_{t\geq 0} K(t) \geq m > 0$, using (v) we obtain

$$T(\Phi'(u)) = T(K(||u||^2)u) = \frac{h(K(||u||^2)||u||)}{K(||u||^2)||u||}K(||u||^2)u = \frac{||u||}{K(||u||^2)||u||}K(||u||^2)u = u.$$

Hence, our claim is proved. Moreover, since $m \leq K(s)$ for all $s \in [0, +\infty[$, we have

$$\Phi(u) \geq \frac{m}{2}||u||^2 \quad \text{for all } u \in X. \tag{7}$$

Further, thanks to (iv), there exist two constants $\gamma,\ \tau \in R$ with $0 < \gamma < \frac{1}{\theta}$ such that

$$\frac{(b-a)^2}{2m}F(x,t) \leq \gamma t^2 + \tau \text{ for a.e. } x \in (a,b) \text{ and all } t \in R.$$

Fix $u \in X$. Then

$$F(x,u(x)) \leq \frac{2m}{(b-a)^2}(\gamma |u(x)|^2 + \tau) \quad \text{for all } x \in (a,b). \tag{8}$$

Then, for any fixed $\lambda \in]0,\theta]$, taking into account that

$$\max_{x\in[a,b]} |u(x)| \leq \frac{(b-a)^{\frac{1}{2}}}{2}||u||, \tag{9}$$

from (7) and (8), we obtain

$$\begin{aligned}
\Phi(u) + \lambda J(u) &= \frac{1}{2}\tilde{K}(||u||^2) - \lambda \int_a^b F(x,u(x))dx \\
&\geq \frac{m}{2}||u||^2 - \frac{2\theta m}{(b-a)^2}\left(\gamma \int_a^b |u(x)|^2 + \tau(b-a)\right) \\
&\geq \frac{m}{2}||u||^2 - \frac{2\theta m}{(b-a)^2}\left(\gamma \frac{(b-a)^2}{4}||u||^2 + \tau(b-a)\right) \\
&= \frac{m}{2}(1-\gamma\theta)||u||^2 - \frac{2\theta\tau m}{b-a},
\end{aligned}$$

and so

$$\lim_{||u||\to+\infty} (\Phi(u) + \lambda J(u)) = +\infty.$$

Now, we claim that there exist $r > 0$ and $w \in X$ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} (-J(u)) < r\frac{-J(w)}{\Phi(w)}.$$

Moreover, taking into account that (9), from (7) we have

$$\Phi^{-1}(]-\infty, r]) \subseteq \left\{ u \in X; |u(x)| \leq \sqrt{\frac{r(b-a)}{2m}} \text{ for all } x \in [a,b] \right\},$$

and it follows that

$$\sup_{u \in \Phi^{-1}(]-\infty, r[)} (-J(u)) \leq \int_a^b \sup_{t \in [-\sqrt{\frac{r(b-a)}{2m}}, \sqrt{\frac{r(b-a)}{2m}}]} F(x,t)dx.$$

Now, thanks to Lemma 1, there exist $r > 0$ and $w \in X$ such that $\tilde{K}(||w||^2) > 2r$ and

$$\int_a^b \sup_{t \in [-\sqrt{\frac{r(b-a)}{2m}}, \sqrt{\frac{r(b-a)}{2m}}]} F(x,t)dx < 2r \frac{\int_a^b F(x, w(x))dx}{\tilde{K}(||w||^2)}.$$

So, $\Phi(w) > r$ and

$$\sup_{u \in \Phi^{-1}(]-\infty, r])} (-J(u)) < r \frac{-J(w)}{\Phi(w)}.$$

Hence, due to the choice of θ, taking $u_0 = 0$ and $u_1 = w$, by Proposition B, for a suitable constant ρ, we obtain

$$\sup_{\lambda \in R} \inf_{u \in X} (\Phi(u) + \lambda J(u) + \rho\lambda) < \inf_{u \in X} \sup_{\lambda \in [0,\theta]} (\Phi(u) + \lambda J(u) + \rho\lambda).$$

For any fixed L^1-Carathéodory function $g : [a,b] \times R \to R$, set

$$\Psi(u) = -\int_a^b \int_0^{u(x)} g(x,s)dsdx.$$

It is well known that Ψ is a continuously differentiable functional whose differential $\Psi'(u) \in X^*$, at $u \in X$ is given by

$$\Psi'(u)(v) = -\int_a^b g(x, u(x))v(x)dx \text{ for every } v \in X,$$

such that $\Psi' : X \to X^*$ is a compact operator. Now, all the assumptions of Theorem A, are satisfied. Hence, applying Theorem A, taking into account that the critical points of the functional $\Phi + \lambda J + \mu\Psi$ are exactly the weak solutions of the problem (1), we have the conclusion. □

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