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On positive weak solutions for some nonlinear elliptic boundary value problems involving the p-Laplacian

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Abstract

This study concerns the existence of positive weak solutions to boundary value problems of the form

$$\begin{cases} -\Delta_p u = g(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where Δ_p is the so-called p-Laplacian operator i.e. $\Delta_p z = div\,(|\nabla z|^{p-2}\nabla z)$, $p > 1$, Ω is a smooth bounded domain in R^N $(N \geq 2)$ with $\partial\Omega$ of class C^2, and connected, and $g(x,0) < 0$ for some $x \in \Omega$ (semipositone problems). By using the method of sub-super solutions we prove the existence of the positive weak solution to special types of $g(x,u)$.

Keywords: Positive weak solutions, p-Laplacian, sub-super solution
AMS Subject Classification: 35J65

1 Introduction

In this paper we consider the existence of positive weak solution to boundary value problems of the form

$$\begin{cases} -\Delta_p u = g(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Δ_p is the so-called p-Laplacian operator i.e. $\Delta_p z = div\,(|\nabla z|^{p-2}\nabla z)$, $p > 1$, Ω is a smooth bounded domain in R^N $(N \geq 2)$ with $\partial\Omega$ of class C^2, and connected, and $g(x,0) < 0$ for some $x \in \Omega$ (semipositone problems). In particular, we first study the case when $g(x,u) = a(x)\,u^{p-1} - b(x)\,u^{q-1} - ch(x)$, where $q > p$ and $a(x), b(x)$ are $C^1(\bar{\Omega})$ functions that $a(x)$ is allowed to be negative near the boundary of Ω, and $b(x) > b_0 > 0$ for $x \in \Omega$. Here $h : \overline{\Omega} \longrightarrow R$ is a $C^1(\bar{\Omega})$ function satisfying $h(x) \geq 0$ for $x \in \Omega$, $h(x) \not\equiv 0$, and $\max_{x\in\bar{\Omega}} h(x) = 1$. We prove that there exists a $c_0 = c_0(\Omega, a, b) > 0$ such that for $0 < c < c_0$ there exists a positive solution.

Problems involving the "p-Laplacian" arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mapping (see [10]) as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids.

The above equation arises in the studies of population biology of one species with u representing the concentration of the species or the population density, and $ch(x)$ representing the rate of harvesting (see [7]).

In the earlier paper [1] we consider the problem (1) with $p = 2$. The purpose of this paper is to extend this study to the p-Laplacian case. The case when $p = 2$ (the Laplacian operator), $a(x), b(x)$ are positive constants throughout $\bar{\Omega}$, has been studied in [7]. Also recently in [8] the authors extend this study to the p-Laplacian case. In [3] the authors studied the case when $c = 0$ (non-harvesting case), $b(x) \equiv 1$ for $\bar{\Omega}$ and $a(x)$ is apositive function throughout $\bar{\Omega}$. However the $c > 0$ case is a semipositone problem $(g(x,0) < 0)$ and studying positive solutions in this case is significantly harder. Here we consider the challenging semipositone case $c > 0$. Semipositone problems have been of great interest during the past two decades, and continue to pose mathematically difficult problems in the study of positive solutions (see [2, 5]).

We next study the case when $g(x,u) = \lambda\, m(x)\, f(u)$, where the weight m satisfying $m \in C(\Omega)$ and $m(x) \geq m_0 > 0$ for $x \in \Omega$, $f \in C^1[0,\rho)$ is a nondecreasing function for some $\rho > 0$ such that $f(0) < 0$ and there exist $\alpha \in (0,\rho)$ such that $f(t)(t-\alpha) \geq 0$ for $t \in [0,\rho]$.

See [5] where positive solution is obtained for large λ when $m(x) \equiv 1$ for $x \in \Omega$ and f is p-sublinear at infinity. We are interested in the existence of a positive solution in a range of λ without assuming any condition on f at infinity. Our approach is based on the method of sub-super solutions, see [3, 9].

2 Existence results

Let $W_0^{1,s} = W_0^{1,s}(\Omega), s > 1$, denote the usual Sobolev space. We give the definition of weak solution and sub-super solution of (1).

Definition 2.1. We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution to (1) if for any $v \in W_0^{1,p}$ with $v \geq 0$ we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u \,.\nabla v \, dx \; = \; \int_\Omega g(x,u) \, v \, dx.$$

However in this paper, we in fact study the existence of $C^1(\overline{\Omega})$ solutions that strictly positive in Ω.

Definition 2.2. We say that $\psi \in W_0^{1,p}(\Omega)$ is a subsolution to (1) if

$$\int_\Omega |\nabla \psi|^{p-2} \nabla \psi \,.\nabla v \, dx \; \leq \; \int_\Omega g(x,\psi) \, v \, dx,$$

hold for all $v \in W_0^{1,p}$ with $v \geq 0$.

Definition 2.3. We say that $z \in W_0^{1,p}(\Omega)$ is a supersolution to (1) if

$$\int_\Omega |\nabla z|^{p-2} \nabla z \,.\nabla v \, dx \; \geq \; \int_\Omega g(x,z) \, v \, dx,$$

hold for all $v \in W_0^{1,p}$ with $v \geq 0$.

Now if there exists sub and super solutions ψ and z respectively such that $0 \leq \psi \leq z$ for $x \in \Omega$, then (1) has a positive solution $u \in W_0^{1,p}(\Omega)$ such that $\psi \leq u \leq z$ (see [3, 4]). We shall obtain the existence of positive weak solution to problem (1) by constructing a positive subsolution ψ and supersolution z.

To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} -\Delta_p \phi = \lambda \, |\phi|^{p-2} \, \phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

Let $\phi_1 \in C^1(\overline{\Omega})$ be the eigenfunction corresponding to the first eigenvalue λ_1 of (2) such that $\phi_1(x) > 0$ in Ω, and $||\phi_1||_\infty = 1$. It can be shown that $\frac{\partial \phi_1}{\partial n} < 0$ on $\partial\Omega$ and hence, depending on Ω, there exist positive constants k, η, μ such that

$$\lambda_1 \, \phi_1^p - |\nabla \phi_1|^p \; \leq -k, \qquad x \in \bar{\Omega}_\eta, \tag{3}$$

$$\phi_1 \geq \mu, \qquad x \in \Omega_0 = \Omega \setminus \bar{\Omega}_\eta, \tag{4}$$

with $\bar{\Omega}_\eta = \{x \in \Omega \mid d(x, \partial\Omega) \leq \eta\}$. Further assume that there exists a constants $a_0, a_1 > 0$ such that $a(x) \geq -a_0$ in $\bar{\Omega}_\eta$ and $a(x) \geq a_1$ in $\Omega_0 = \Omega \setminus \bar{\Omega}_\eta$.

We will also consider the unique solution, $\zeta \in C^1(\overline{\Omega})$, of the boundary value problem

$$\begin{cases} -\Delta_p \zeta = 1, & x \in \Omega, \\ \zeta = 0, & x \in \partial\Omega, \end{cases}$$

to discuss our existence result. It is known that $\zeta > 0$ in Ω and $\frac{\partial \zeta}{\partial n} < 0$ on $\partial\Omega$.

First we obtain the existence of positive weak solution of (1) in the case when $g(x,u) = a(x)\, u^{p-1} - b(x)\, u^{q-1} - ch(x)$.

Theorem 2.4. Suppose that $a_0 < k\,(p/(p-1))^{p-1}$ and $\lambda_1\,(p/(p-1))^{p-1} < a_1$. Then there exists $c_0 = c_0(\Omega, a_0, a_1, b) > 0$ such that if $0 < c < c_0$ then the problem (1) has a positive solution u.

Proof. To obtain the existence of positive weak solution to problem (1), we constructing a positive subsolution ψ and supersolution z. We shall verify that $\psi = \delta\, \phi_1^{p/(p-1)}$ is a subsolution of (1), where $\delta > 0$ is small and specified later (note that $||\psi||_\infty \leq \delta$). Let the test function $w \in W_0^{1,p}$ with $w \leq 0$. A calculation shows that

$$\begin{aligned} \int_\Omega |\nabla \psi|^{p-2}\, \nabla \psi\, .\nabla w &= \delta^{p-1}\,(p/p-1)^{p-1} \int_\Omega \phi_1\, |\nabla \phi_1|^{p-2}\, \nabla \phi_1\, .\nabla w\, dx \\ &= \delta^{p-1}\,(p/p-1)^{p-1} \{ \int_\Omega |\nabla \phi_1|^{p-2}\, \nabla \phi_1\, \nabla(\phi_1\, w)\, dx - \int_\Omega |\nabla \phi_1|^p\, w\, dx \} \\ &= \delta^{p-1}\,(p/p-1)^{p-1} \int_\Omega (\lambda_1\, \phi_1^p - |\nabla \phi_1|^p)\, w\, dx. \end{aligned} \tag{5}$$

Thus ψ is a subsolution if

$$\delta^{p-1}\,(p/p-1)^{p-1} \int_\Omega (\lambda_1\, \phi_1^p - |\nabla \phi_1|^p)\, w\, dx \;\leq\; \int_\Omega (a(x)\, \psi^{p-1} - b(x)\, \psi^{q-1} - ch(x))\, w\, dx.$$

Now $\lambda_1\, \phi_1^p - |\nabla \phi_1|^p \;\leq\; -k$ in $\bar{\Omega}_\eta$, and therefore

$$\begin{aligned} \delta^{p-1}\,(p/p-1)^{p-1}\,(\lambda_1\, \phi_1^p - |\nabla \phi_1|^p) &\leq -k\, \delta^{p-1}\,(p/p-1)^{p-1} \\ &\leq -a_0\, \delta^{p-1} - ||b||_\infty\, \delta^{q-1} - c, \end{aligned}$$

if

$$\delta < \theta_1 = \Big(\frac{k\,(p/p-1)^{p-1} - a_0}{||b||_\infty} \Big)^{1/q-p},$$

$$c \leq \hat{c}(\delta) = \delta^{p-1}(k\,(p/p-1)^{p-1} - a_0 - ||b||_\infty\, \delta^{q-p}).$$

Clearly $\hat{c}(\delta) > 0$.

Furthermore, we note that $\phi_1 \geq \mu > 0$ in $\Omega_0 = \Omega \setminus \bar{\Omega}_\eta$, and therefore

$$\begin{aligned} \delta^{p-1}\,(p/p-1)^{p-1}\,(\lambda_1\,\phi_1^p - |\nabla\phi_1|^p) &\leq \lambda_1\,\delta^{p-1}\,(p/p-1)^{p-1} \\ &\leq a_1\,\delta^{p-1}\,\phi_1^p - ||b||_\infty\,\delta^{q-1} - c, \end{aligned}$$

if

$$\delta < \theta_2 = (\frac{(a_1 - (p/p-1)^{p-1}\,\lambda_1)\,\mu^p}{||b||_\infty})^{1/q-p},$$

$$c \leq \bar{c}(\delta) = \delta^{p-1}((a_0 - (p/p-1)^{p-1}\,\lambda_1)\,\mu^p - ||b||_\infty\,\delta^{q-p}).$$

Clearly $\bar{c}(\delta) > 0$. Choose $\theta = \min\{\theta_1, \theta_2\}$ and $\delta = \theta/2$. Then simplifying, both $\hat{c}$ and $\bar{c}$ are greater than $(\frac{\theta}{2})^{q-1}\,(2^{q-p}\,||b||_\infty - ||b||_\infty)$. Hence if $c \leq (\frac{\theta}{2})^{q-1}\,(2^{q-p}\,||b||_\infty - ||b||_\infty) = c_0(\Omega, a_0, a_1, b)$ then ψ is a subsolution.

Next, we construct a supersolution z of (1). We denote $z = N\zeta(x)$, where the constant $N > 0$ is large and to be chosen later. We shall verify that z is a supersolution of (1). To this end, let Let $w(x) \in W_0^{1,p}(\Omega)$ with $w \geq 0$. Then we have

$$\begin{aligned} \int_\Omega |\nabla z|^{p-2}\nabla z.\nabla w\,dx &= N^{p-1}\int_\Omega |\nabla\zeta|^{p-2}\nabla\zeta.\nabla w\,dx \\ &= N^{p-1}\int_\Omega w\,dx. \end{aligned}$$

Thus z is a supersolution if

$$N^{p-1}\int_\Omega w\,dx \geq \int_\Omega (a(x)\,z^{p-1} - b(x)\,z^{q-1} - ch(x))\,w\,dx,$$

and therefore if $N \geq N_0^{1/(p-1)}$ where $N_0 = \sup_{[0,(||a||_\infty/b_0)^{1/q-p}]}(\,||a||_\infty\,v^{p-1} - b_0\,v^{q-1})$, we have

$$\int_\Omega |\nabla z|^{p-2}\,\nabla z\,.\nabla w\,dx \geq \int_\Omega (a(x)\,z^{p-1} - b(x)\,z^{q-1} - ch(x))\,w\,dx,$$

and hence z is supersolution of (1). Since $\zeta > 0$ and $\partial\zeta/\partial n < 0$ on $\partial\Omega$, we can choose N large enough so that $\psi \leq z$ is also satisfied. Hence Theorem 2.4 is proven. □

Now, we obtain the existence of positive weak solution of (1) in the case when $g(x,u) = \lambda\,m(x)\,f(u)$. Assume that there exist positive constants $r_1, r_2 \in (\alpha, \rho]$ satisfying:

(H.1) $\frac{r_2}{r_1} \geq \max\{\lambda_1^{1/p-1}\,(\frac{p\,||\zeta||_\infty\,\mu^{p/1-p}}{p-1}),\ \frac{p\,||\zeta||_\infty}{p-1}\,(\frac{\lambda_1\,||m||_\infty\,f(r_1)}{m_0\,\mu^p\,f(r_2)})^{1/p-1}\}$,

(H.2) $k\, f(r_1) > \lambda_1\, |f(0)|$.

Theorem 2.5. Let $(H.1), (H.2)$ hold. Then there exist $\lambda_* < \tilde{\lambda}$ such that (1) has a positive solution for $\lambda \in [\lambda_*, \tilde{\lambda}]$.

Proof. Let λ_1, ϕ_1, be as before. We now construct our positive subsolution. Let $\psi = r_1\, \mu^{p/1-p}\, \phi_1^{p/(p-1)}$. Let the test function $w(x) \in W_0^{1,p}(\Omega)$ with $w \geq 0$. Using a calculation similar to the one in the proof of Theorem 2.4, we have

$$\int_\Omega |\nabla\psi|^{p-2}\, \nabla\psi\,.\nabla w = (\frac{p}{p-1}\, r_1\, \mu^{p/1-p})^{p-1} \int_\Omega (\lambda_1\, \phi_1^p - |\nabla\phi_1|^p)\, w\, dx.$$

Thus ψ is a subsolution if

$$(\frac{p}{p-1}\, r_1\, \mu^{p/1-p})^{p-1} \int_\Omega (\lambda_1\, \phi_1^p - |\nabla\phi_1|^p)\, w\, dx \,\leq\, \lambda \int_\Omega m(x)\, f(\psi)\, w\, dx,$$

Now $\lambda_1\, \phi_1^p - |\nabla\phi_1|^p \,\leq\, -k$ in $\bar{\Omega}_\eta$, and therefore

$$\begin{aligned}(\frac{p}{p-1}\, r_1\, \mu^{p/1-p})^{p-1}\, (\lambda_1\, \phi_1^p - |\nabla\phi_1|^p) \;&\leq\; -k\, (\frac{p}{p-1}\, r_1\, \mu^{p/1-p})^{p-1}\\ &\leq\; \lambda\, m(x)\, f(\psi),\end{aligned}$$

if

$$\lambda \leq \hat{\lambda} = \frac{k\, (\frac{p}{p-1}\, r_1\, \mu^{p/1-p})^{p-1}}{m_0\, |f(0)|}.$$

Furthermore, we note that $\phi_1 \geq \mu > 0$ in $\Omega_0 = \Omega \setminus \bar{\Omega}_\eta$, and therefore

$$\psi = r_1\, \mu^{p/1-p}\, \phi_1^{p/(p-1)} \geq r_1\, \mu^{p/1-p}\, \mu^{p/(p-1)} = r_1,$$

thus $f(\psi) \geq f(r_1)$. Hence if

$$\lambda \,\geq\, \lambda_* = \frac{\lambda_1\, (\frac{p}{p-1}\, r_1\, \mu^{p/1-p})^{p-1}}{m_0\, f(r_1)},$$

we have

$$\begin{aligned}(\frac{p}{p-1}\, r_1\, \mu^{p/1-p})^{p-1}\, (\lambda_1\, \phi_1^p - |\nabla\phi_1|^p) \;&\leq\; \lambda_1\, (\frac{p}{p-1}\, r_1\, \mu^{p/1-p})^{p-1}\\ &\leq\; \lambda\, m_0\, f(r_1)\\ &\leq\; \lambda\, m(x)\, f(\psi).\end{aligned}$$

We get $\lambda_* < \hat{\lambda}$ by using $(H.2)$. Therefore if $\lambda_* \leq \lambda \leq \hat{\lambda}$, then ψ is subsolution.

Next, we construct a supersolution z of (1) such that $z \geq \psi$. We denote $z = \frac{r_2}{||\zeta||_\infty} \zeta(x)$. We shall verify that z is a super solution of (1). To this end, let Let $w(x) \in W_0^{1,p}(\Omega)$ with $w \geq 0$. Then we have

$$\int_\Omega |\nabla z|^{p-2} \nabla z . \nabla w \, dx = (\frac{r_2}{||\zeta||_\infty})^{p-1} \int_\Omega w \, dx. \tag{6}$$

Thus z is a super solution if

$$(\frac{r_2}{||\zeta||_\infty})^{p-1} \int_\Omega w \, dx \geq \lambda \int_\Omega m(x) \, f(z) \, w \, dx.$$

But $f(z) \leq f(r_2)$ and hence z is a super solution if

$$\lambda \leq \bar{\lambda} = \frac{(r_2/||\zeta||_\infty)^{p-1}}{||m||_\infty \, |f(r_2)}.$$

We easily see that $\lambda_* < \bar{\lambda}$, by using $(H.1)$. Finally, using $(5),(6)$ and the weak comparoson principle [4], we see that $\psi \leq z$ in Ω when $(H.1)$ is satisfied. Therefore (1) has a positive solution for $\lambda \in [\lambda_*, \tilde{\lambda}]$, where $\tilde{\lambda} = \min\{\hat{\lambda}, \bar{\lambda}\}$. This completes the proof of Theorem 2.5. □

Remark 2.6. Theorem 2.5 holds no matter what the growth condition of f is, for large u. Namely, f could satisfy p-superlinear or p-linear growth condition at infinity.

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