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d_2-coloring of a Graph

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Abstract

A subset S of V is called an i-set$(i \geq 2)$ if no two vertices in S have the distance i. The 2-set number $\alpha_2(G)$ of a graph is the maximum cardinality among all 2-sets of G. A d_2-*coloring* of a graph is an assignment of colors to its vertices so that no two vertices have the distance two get the same color. The d_2-*chromatic number* $\chi_{d_2}(G)$ of a graph G is the minimum number of d_2-colors need to G. In this paper, we initiate a study of these two new parameters.

1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph without loops and multiple edges . The order and size of G are denoted by p and q respectively. For graph theoretical terms we refer Chartrand and Lesniak [1].

A subset S of V is an *independent* set if no two vertices in S are adjacent. An assignment of colors to the vertices of a graph so that no two adjacent vertices get the same color is called a *coloring* of the graph and an assignment of colors to the vertices of a graph so that no two vertices have the distance 1 or 2 get the same color is called a *distant 2-coloring* of G. This motivates to

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define two new parameters as follows:

A subset S of V is called an i-set($i \geq 2$) if no two vertices in S have the distance i. The 2-set number $\alpha_2(G)$ of a graph is the maximum cardinality among all 2-sets of G. A *d_2-coloring* of a graph is an assignment of colors to its vertices so that no two vertices have the distance two get the same color. The *d_2-chromatic number* $\chi_{d_2}(G)$ of a graph G is the minimum number of d_2-colors need to G.

The corona of two graphs G_1 and G_2 is the graph $G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2. Any undefined term in this paper may be found in Chartrand and Lesniak [1]. In this paper, we initiate a study of these two new parameters.

2 2-set number of a Graph

Definition 2.1. A subset S of V is called an i-set($i \geq 2$) of G if no two vertices in S have the distance i. The 2-set number $\alpha_2(G)$ of a graph is the maximum cardinality among all 2-sets of G.

Remark 2.2. From the above definition, if $i = 1$, then S is an independent set of G.

Proposition 2.3. $\alpha_2(K_p) = p$.

Proof. In K_p, no two vertices have distance two and hence $\alpha_2(K_p) = p$. □

Proposition 2.4. $\alpha_2(P_p) = \begin{cases} \frac{p}{2} + 1 & \textit{if } p \equiv 2(\textit{mod } 4) \\ \lceil \frac{p}{2} \rceil & \textit{otherwise} \end{cases}$.

Proof. Consider $P_p : v_1 - v_2 - \cdots - v_p$.
Case 1. $p \equiv 2(\text{mod } 4)$
Let $S = \{v_1, v_2, v_5, v_6, \ldots, v_{p-5}, v_{p-4}, v_{p-1}, v_p\}$. Then no two vertices in S have distance two and is a maximum 2-set of P_p. Thus $\alpha_2(P_p) = \frac{p}{2} + 1$.
Case 2.
If $p \equiv 0(\text{mod } 4)$, then $S = \{v_1, v_2, v_5, v_6, \ldots, v_{p-7}, v_{p-6}, v_{p-3}, v_{p-2}\}$ is a maximum 2-set. If $p \equiv 1(\text{mod } 4)$, then $S = \{v_1, v_2, v_5, \ldots, v_{p-4}, v_{p-3}, v_p\}$ is a maximum 2-set. If $p \equiv 3(\text{mod } 4)$, then $S = \{v_1, v_2, v_5, v_6, \ldots, v_{p-6}, v_{p-5}, v_{p-2}, v_{p-1}\}$ is a maximum 2-set. In this case, we get $\alpha_2(P_p) = \lceil \frac{p}{2} \rceil$. □

One can prove the following Proposition in analogous to the above.

Proposition 2.5. $\alpha_2(C_p) = \begin{cases} \frac{p}{2} - 1 & if \ p \equiv 2(mod\ 4) \\ \lfloor \frac{p}{2} \rfloor & otherwise \qquad , p > 3 \end{cases}$

Proposition 2.6. $\alpha_2(K_{m,n}) = 2.$

Proof. For any edge $e = uv$, whose end vertices form a maximum 2-set of $K_{m,n}$ and hence $\alpha_2(K_{m,n}) = 2$. □

Lemma 2.7. *Let G be a graph with p vertices . Then $\alpha_2(G) = p$ if and only if G is complete or a union of complete components.*

Proof. Suppose $\alpha_2(G) = p$. Then no two vertices in G have distance two. If $d(u, v) = k$, $2 < k < \infty$, for some $u, v \in V(G)$, then $d(w, u) = 2$ for some $w \in V(G)$, a contradiction. If $d(u, v) = 1$, for all $u, v \in V(G)$, then G is complete. If $d(u, v) = \infty$, for some $u, v \in V(G)$, then $G = \cup_{i=1}^{m} G_i$, where G_i are connected components of G. Since $\alpha_2(G) = p$, $\alpha_2(G_i) = |V(G_i)|$ for all $1 \leq i \leq m$ and so each G_i is complete.
Converse is obvious. □

The following Proposition is immediate from the definition.

Proposition 2.8. *For a connected graph G, $\alpha_2(G) \geq \omega(G)$.*

Theorem 2.9. *Let G be a connected graph with $p > 2$ vertices. Then $\alpha_2(G) = p - 1$ if and only if $\omega(G) = p - 1$.*

Proof. Suppose $\alpha_2(G) = p-1$. Let S be a maximum 2-set of G with $|S| = p-1$. Then no two vertices in S have distance two. If $d(u, v) = k > 2$ for some $u, v \in S$, then $d(u, w) = 2$, for some $w \in S$, a contradiction. Thus any two vertices in S have distance one, the subgraph induced by S is complete and hence $\omega(G) = p-1$. Conversely, suppose $\omega(G) = p-1$. Then G is not complete and by Lemma 2.7, $\alpha_2(G) \leq p - 1$. If $\alpha_2(G) < p - 1$, then $\omega(G) < p - 1$, a contradiction. Hence $\alpha_2(G) = p - 1$. □

Proposition 2.10. *For any connected graph G with p vertices, $\alpha_2(K_m \circ G) = m\alpha_2(G)$.*

Proof. In $K_m \circ G$, for every $i \neq j$, $d(u, v) = 3$, for every u in i^{th} copy of G, v in j^{th} copy of G and hence $\alpha_2(K_m \circ G) = m\alpha_2(G)$. □

In view of Proposition 2.10, we have following corollary.

Corollary 2.11. *$\alpha_2(K_m \circ G) = m$ if and only if $G \cong K_1$*

Lemma 2.12. *Let G be a tree with p vertices. Then $1 \leq \alpha_2(G) \leq p + 1 - \Delta$.*

Proof. Obviously $\alpha_2(G) \geq 1$. Let $S \subset V(G)$ with $d(u, v) = 2$ for all $u, v \in S$. Then $|S| = \Delta$ and so maximum 2-set contains at most $p - (\Delta - 1)$ vertices. Thus $\alpha_2(G) \leq p + 1 - \Delta$. □

Theorem 2.13. *Let G be a connected bipartite graph with (V_1, V_2)-partition of $V(G)$, $|V_1| > 1$ and $|V_1| \geq |V_2|$. Then,*
(i) $\alpha_2(\overline{G}) = |V_1| + |V_2|$ if and only if G is complete.
(ii) V_1 is a maximum 2-set of $\overline{G}$ if and only if G is not complete.

Proof. (i) Suppose $\alpha_2(\overline{G}) = |V_1| + |V_2|$. Since G is connected and by Lemma 2.7, $\overline{G}$ is union of two complete components and hence G is complete. Converse is obvious.
(ii) Suppose V_1 is a maximum 2-set of $\overline{G}$. Then for every $v \in V_2$, $d_{\overline{G}}(v, u) = 2$, for some $u \in V_1$ and so at least one shortest path $u' - w' - v'$ is in $\overline{G}$, $u', w' \in V_1$, $v' \in V_2$. Thus $w'v' \in E(\overline{G})$ and so $w'v' \notin E(G)$. Hence G is not complete. Conversely, suppose G is not complete. Then for each $x \in V_1$, $y \in V_2$, there exists $x' \in V_2$, $y' \in V_1$ such that $d_{\overline{G}}(x, x') = 2$ and $d_{\overline{G}}(y, y') = 2$ and so V_1 is a maximum 2-set of G. Hence $\alpha_2(\overline{G}) = |V_1|$. □

Lemma 2.14. *For any connected graphs G and H, $\alpha_2(G+H) = \omega(G)+\omega(H)$.*

Proof. If G and H are complete, then $G + H$ is complete and by Lemma 2.7, $\alpha_2(G + H) = \omega(G) + \omega(H)$.
Suppose G is not complete. Then $G + H$ is not complete, $\text{diam}(G + H) = 2$ and by Proposition 2.8, $\alpha_2(G + H) \geq \omega(G + H)$. Suppose $\alpha_2(G + H) = k > \omega(G + H)$. Then $G + H$ contains K_k, $k > \omega(G + H)$, a contradiction. Hence $\alpha_2(G + H) = \omega(G + H) = \omega(G) + \omega(H)$. □

In view of Lemma 2.14, we have following corollary.

Corollary 2.15. *$\alpha_2(W_p) = 3$, $p > 4$.*

Theorem 2.16. *Let G and H be connected with $|V(G)| > 1$ and $|V(H)| > 1$. Then $\alpha_2(G + H) = 4$ if and only if G and H are triangle free graphs.*

Proof. Suppose $\alpha_2(G+H) = 4$. Then by Lemma 2.14, $\omega(G) + \omega(H) = 4$ and so $\omega(G) = \omega(H) = 2$. Thus G and H are triangle free graphs. Conversely, suppose G and H are triangle free graphs, then $\omega(G) = \omega(H) = 2$ and by Lemma 2.14, $\alpha_2(G+H) = 4$. □

Theorem 2.17. *Let G be a connected graph with $diam(G) = 2$. Then $\alpha_2(G) = 2$ if and only if G is triangle free graph.*

Proof. Suppose $\alpha_2(G) = 2$. If G contains K_3, then by Proposition 2.8, $\alpha_2(G) \geq 3$, a contradiction. Conversely, suppose G is triangle free graph. If $\alpha_2(G) = k > 2$, then by hypothesis, G contains K_k, a contradiction. □

In view of Theorems 2.16, 2.17, we have following corollary.

Corollary 2.18. *Let G and H be connected graphs with diameter is 2. If G and H are triangle free graphs then $\alpha_2(G+H) = \alpha_2(G) + \alpha_2(H)$.*

Theorem 2.19. *Let G be a any tree. Then $\alpha_2(G) = 2$ if and only if G is a star or bistar.*

Proof. Suppose $\alpha_2(G) = 2$. If G is not a star, then $diam(G) = k \geq 3$. Suppose $diam(G) = k > 3$. Then G has a path of length k and so $\alpha_2(G) \geq \alpha_2(P_{k+1})$. By Proposition 2.4, this is not possible. Thus diam$(G) = 3$ and so G is a bistar. If G is not a bistar, then diam$(G) \neq 3$ and so $diam(G) = 1$ or 2. Hence G is a star. Converse is obvious. □

3 d_2-chromatic number of a Graph

Definition 3.1. A *d_2-coloring* of a graph is an assignment of colors to its vertices so that no two vertices have the distance two get the same color. The *d_2-chromatic number* $\chi_{d_2}(G)$ of a graph G is the minimum number of d_2-colors need to G. A graph G is k d_2-colorable(resp. d_2-chromatic) if $\chi_{d_2}(G) \leq k$(resp. $\chi_{d_2}(G) = k$).

Proposition 3.2. $\chi_{d_2}(K_n) = \chi_{d_2}(\overline{K_n}) = 1$.

Proof. Since $d(u,v) = 1$ for all $u, v \in V(K_n)$, every vertex in K_n assigned the same color and hence $\chi_{d_2}(K_n) = 1$. Since $d(u,v) = \infty$ for all $u, v \in V(\overline{K_n})$, $\chi_{d_2}(\overline{K_n}) = 1$. □

Proposition 3.3. $\chi_{d_2}(P_p) = 2$ *for* $p > 2$.

Proof. Consider $P_p : v_1 - v_2 - \cdots - v_p$.
Case 1. $p \equiv 0(\text{mod } 4)$
Let $\Omega_1 = \{v_1, v_2, v_5, v_6, \ldots, v_{p-3}, v_{p-2}\}$ and $\Omega_2 = \{v_3, v_4, v_7, v_8, , \ldots, v_{p-1}, v_p\}$. Then Ω_1 and Ω_2 are 2-sets and every vertex in Ω_1 make a distance two to at least one vertex in Ω_2 and vice versa. Hence we assign one color to Ω_1 and another one color to Ω_2 and so $\chi_{d_2}(P_p) = 2$.
Case 2. $p \equiv 1(\text{mod } 4)$
Let $\Omega_1 = \{v_1, v_2, v_5, v_6, \ldots, v_{p-4}, v_{p-3}, v_p\}$ and $\Omega_2 = \{v_3, v_4, v_7, v_8, , \ldots, v_{p-2}, v_{p-1}\}$. Then Ω_1 and Ω_2 are 2-sets and every vertex in Ω_1 make a distance two to at least one vertex in Ω_2 and vice versa. Hence we assign one color to Ω_1 and another one color to Ω_2 and so $\chi_{d_2}(P_p) = 2$.
Case 3. $p \equiv 2(\text{mod } 4)$
Let $\Omega_1 = \{v_1, v_2, v_5, v_6, \ldots, v_{p-1}, v_p\}$ and $\Omega_2 = \{v_3, v_4, v_7, v_8, , \ldots, v_{p-3}, v_{p-2}\}$. Then Ω_1 and Ω_2 are 2-sets and every vertex in Ω_1 make a distance two to at least one vertex in Ω_2 and vice versa. Hence we assign one color to Ω_1 and another one color to Ω_2 and so $\chi_{d_2}(P_p) = 2$.
Case 4. $p \equiv 3(\text{mod } 4)$
Let $\Omega_1 = \{v_1, v_2, v_5, v_6, \ldots, v_{p-2}, v_{p-1}\}$ and $\Omega_2 = \{v_3, v_4, v_7, v_8, , \ldots, v_{p-4}, v_{p-3}, v_p\}$. Then Ω_1 and Ω_2 are 2-sets and every vertex in Ω_1 make a distance two to at least one vertex in Ω_2 and vice versa. Hence we assign one color to Ω_1 and another one color to Ω_2 and so $\chi_{d_2}(P_p) = 2$. □

Proposition 3.4. $\chi_{d_2}(C_p) = \begin{cases} 2 & if \ \ p \equiv \ 0 \ (mod \ 4) \\ 3 & \quad otherwise \qquad , p > 3 \end{cases}$

Proof. Consider $C_p : v_1 - v_2 - \cdots - v_p - v_1$.
Case 1. $p \equiv 0(\text{mod } 4)$
Let $\Omega_1 = \{v_1, v_2, v_5, v_6, \ldots, v_{p-3}, v_{p-2}\}$ and $\Omega_2 = \{v_3, v_4, v_7, v_8, , \ldots, v_{p-1}, v_p\}$. Then Ω_1 and Ω_2 are 2-sets and every vertex in Ω_1 make a distance two to at least one vertex in Ω_2 and vice versa. Hence we assign one color to Ω_1 and another one color to Ω_2 and so $\chi_{d_2}(C_p) = 2$.
Case 2. $p \equiv 1, 2, 3(\text{mod } 4)$
We consider first p_1-vertices, where $p_1 \equiv 0(\text{mod } 4)$, as in case 1, assigning two colors to p_1-vertices of C_p. Remaining there are $k(= 1 \ or \ 2 \ or \ 3)$vertices.
If $k = 1$ then $p = p_1 + 1$ and $d(v_{p_1+1}, v_{p_1-1}) = d(v_{p_1+1}, v_2) = 2$, where $v_2 \in \Omega_1$

and $v_{p_1-1} \in \Omega_2$. So assign a new color to v_{p_1+1}.
If $k = 2$ then $p = p_1 + 2$ and $d(v_{p_1+2}, v_2) = d(v_{p_1+2}, v_{p_1}) = d(v_{p_1+1}, v_1) = d(v_{p_1+1}, v_{p_1-1}) = 2$,where $v_1, v_2 \in \Omega_1$ and $v_{p_1}, v_{p_1-1} \in \Omega_2$. So assign a new color to both v_{p_1+1} and v_{p_1+2}.
If $k = 3$ then $p = p_1+3$ and $d(v_{p_1+2}, v_1) = d(v_{p_1+2}, v_{p_1}) = 2$, where $v_1 \in \Omega_1$ and $v_{p_1} \in \Omega_2$. So assign a new color to v_{p_1+2}. Also color of vertex set Ω_1 assign to v_{p_1+1} and color of vertex set Ω_2 assign to v_{p_1+3} because $\Omega_1 \cup \{v_{p_1+1}\}$, $\Omega_2 \cup \{v_{p_1+3}\}$ are 2-sets and $d(v_{p_1+1}, v_{p_1+3}) = 2$.
Hence in this cases, $\chi_{d_2}(C_p) = 3$. □

Proposition 3.5. $\chi_{d_2}(K_{m,n}) = m$ *for* $m \geq n$.

Proof. Let $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$ be the partition of $V(K_{m,n})$. Clearly $d(u_i, u_j) = 2 = d(v_k, v_l)$ for every $i \neq j, k \neq l$. So we assign m-colors to vertices in V_1 with each vertex has different color. Since for every $u \in V_1, d(u, v) = 1$ for every $v \in V_2$, same set of colors of vertices in V_1 use to vertices in V_2.Hence $\chi_{d_2} = m$. □

Remark 3.6. In general, $\chi_{d_2}(K_{n_1,n_2,\ldots,n_t}) = max\{n_1, n_2, \ldots, n_t\}$.

Proposition 3.7. *For* $p \geq 5$, $\chi_{d_2}(W_p) = \begin{cases} \frac{p-1}{2} & \textit{if } p \textit{ is odd} \\ \frac{p}{2} & \textit{if } p \textit{ is even} \end{cases}$

Proof. Note that $W_p = C_{p-1} + K_1$. Let $V(C_{p-1}) = \{v_1, v_2, \ldots, v_{p-1}\}$ and $V(K_1) = v_p$.
Consider $C_{p-1} = v_1 - v_2 - \cdots - v_{p-1} - v_1$
Case 1. p is odd
Consider the maximum edge independent set $S = \{e_i : e_i = (v_i, v_{i+1}) \in E(W_p)$ for $i = 1, 3, 5, \ldots, p-2\}$ and $|S| = \beta'(W_p)$. Then $V(C_p) = \bigcup_i \{v_i, v_{i+1}\}$. Since $d(u, v) = 2$,for some $u \in \{v_i, v_{i+1}\}$ and $v \in \{v_j, v_{j+1}\}$,for every $i \neq j$,there are β' colors assigning to $V(C_p)$ with end vertices of each edge in S receive the same color. Also $d(v_p, v_k) = 1, k = 1$ to $p-1$,center vertex v_p receives any one color from β' colors. Hence $\chi_{d_2}(W_p) = \beta'(W_p) = \frac{(p-1)}{2}$, if p is odd.
Case 2. p is even
Consider the maximum edge independent set $S' = \{e_i : e_i = (v_i, v_{i+1}) \in E(W_p)$ for $i = 1, 3, 5, \ldots, p-1\}$ and also $V(W_p) = \bigcup_i \{v_i, v_{i+1}\}$. As in case 1, there are β' colors assigned to $V(W_p)$. Hence $\chi_{d_2}(W_p) = \beta'(W_p) = \frac{p}{2}$,if p is even. □

Proposition 3.8. *A connected graph G is one d_2-colorable if and only if G is complete.*

Proof. Suppose G is one d_2-colorable. Then no two vertices have distance two in G. Since G is connected,any two vertices must have distance one. Hence G is complete. Converse follows from Proposition 3.2. □

Proposition 3.9. *Let G be a connected graph with $p > 1$ vertices. Then $\chi_{d_2}(G) = p - 1$ if and only if G is a star graph*

Proof. Suppose $\chi_{d_2}(G) = p - 1$. Let $S \subset V(G)$ with $d(u, v) = 2$. By hypothesis, $|S| = p - 1$ and S is an independent set of G. Since G is connected, there exists a vertex $w \in V - S$ such that $d(w, u) = 1$ for all $u \in S$. Thus G is a star graph. Converse is obvious. □

Theorem 3.10. *For any connected graph G with $p > 1$ vertices, $1 \leq \chi_{d_2}(G) \leq p - 1$.*

Proof. Obviously $\chi_{d_2}(G) \geq 1$. Let S be a subset of $V(G)$ such that any two vertices in S have distance two. Then S can have at most $p - 1$ vertices and so $\chi_{d_2}(G) \leq p - 1$. □

Remark 3.11. Let G be a connected graph with diameter two. Then $\chi_{d_2} \geq 2$

Theorem 3.12. *If G has a perfect matching, then $\chi_{d_2}(G) \leq \beta'$.*

Proof. Let $M = \{e_i : e_i = (v_i, v_{i+1}) \in E(G), i = 1 \text{ to } \beta'\}$ be a maximum matching of G. Then $|M| = \beta'(G)$ and $V(G) = \bigcup_i \{v_i, v_{i+1}\}$. There are β' colors assigning to the vertices of G with end vertices of each edge in M receive the same color because $d(u, v) = 2$ for some $u \in \{v_i, v_{i+1}\}$ and $v \in \{v_j, v_{j+1}\}$, for every $i \neq j$. This is a maximum possible d_2-coloring of G. Hence $\chi_{d_2}(G) \leq \beta'$. □

Theorem 3.13. *Let G be a bipartite graph. Then $\chi_{d_2} \geq \Delta$.*

Proof. Let (V_1, V_2) be a partition of $V(G)$ and let $v \in V_1$ with $deg(v) = \Delta$. Clearly the subgraph induces by $N[v]$ in G is a star graph $K_{1,\Delta}$ and so $\chi_{d_2}(G) \geq \chi_{d_2}(K_{1,\Delta}) = \Delta$ in G. Hence $\chi_{d_2} \geq \Delta$. □

Theorem 3.14. *If G is a connected bipartite graph, then $\overline{G}$ is two d_2-colorable. But converse is not true.*

Proof. Let (V_1, V_2) be a partition of $V(G)$. Clearly V_1 and V_2 induces a complete subgraphs of $\overline{G}$. If G is complete, then $\chi_{d_2}(\overline{G}) = 1$. If G is not complete, for every $u \in V_1, d_{\overline{G}}(u, v) = 2$ for some $v \in V_2$ and also for every $v \in V_2, d_{\overline{G}}(v, u) = 2$ for some $u \in V_1$. Hence assigning one color to all the vertices in V_1 and another color to all the vertices in V_2. Hence $\chi_{d_2}(\overline{G}) = 2$. Thus $\overline{G}$ is two d_2-colorable.
Conversely, consider the graph given in the Figure 1. Clearly $\chi_{d_2}(\overline{G}) = 2$ but G is not bipartite. □

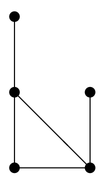

Figure 1

Remark 3.15. In general, if G is connected k-partite then $\overline{G}$ is k d_2-colorable.

Theorem 3.16. *If G is a tree, then $\chi_{d_2} = \Delta$*

Proof. Since G is bipartite and by Theorem 3.13, $\chi_2 \geq \Delta$. Suppose $\chi_{d_2} = k > \Delta$. Then $V(G)$ contains S such that $|S| = k$ and $d(u, v) = 2$ for all $u, v \in S$. This is not possible because $V(G)$ contains S' such that $d(u', v') = 2$ for all $u', v' \in S'$ and $|S'| = \Delta$. Hence $\chi_{d_2} = \Delta$. □

Corollary 3.17. *If G is connected and $\chi_{d_2} > \Delta$, then G contains cycle.*

Proof. Suppose G contains no cycle. Then G is a tree and by Theorem 3.16, $\chi_{d_2} = \Delta$, a contradiction. □

Theorem 3.18. *Let G be a graph. Then $\chi_{d_2} + \overline{\chi}_{d_2} = p$ and $\chi_{d_2}\overline{\chi}_{d_2} = p - 1$ if and only if $G = K_{1,p-1}$ or $\overline{K}_{1,p-1}$, $p > 1$.*

Proof. Assume that $\chi_{d_2}+\overline{\chi}_{d_2} = p$ and $\chi_{d_2}\overline{\chi}_{d_2} = p-1$, then ${\chi_{d_2}}^2 - p\,\chi_{d_2} + p - 1 = 0$ and $\overline{\chi}^2_{d_2} - p\,\overline{\chi}_{d_2} + p - 1 = 0$. Solving these equations, we get $\chi_{d_2} = p - 1,\ 1$ and $\overline{\chi}_{d_2} = p - 1,\ 1$. By hypothesis, $\chi_{d_2} = p - 1$ and $\overline{\chi}_{d_2} = 1$ or $\chi_{d_2} = 1$ and $\overline{\chi}_{d_2} = p - 1$. By Proposition 3.9, $G = K_{1,p-1}$ or $\overline{K}_{1,p-1}$. Conversely, suppose $G = K_{1,p-1}$ or $\overline{K}_{1,p-1}$, $p > 1$. If $G = K_{1,p-1}$, then $\chi_{d_2} = p - 1$ and $\overline{\chi}_{d_2} = 1$ and hence $\chi_{d_2} + \overline{\chi}_{d_2} = p$ and $\chi_{d_2}\overline{\chi}_{d_2} = p - 1$. If $G = \overline{K}_{1,p-1}$, then $\chi_{d_2} = 1$ and $\overline{\chi}_{d_2} = p - 1$ and hence $\chi_{d_2} + \overline{\chi}_{d_2} = p$ and $\chi_{d_2}\overline{\chi}_{d_2} = p - 1$. □

Corollary 3.19. *Let G be a graph with p vertices. If $p = p_1 + 1$, where p_1 is prime and $\chi_d\overline{\chi}_{d_2} = p_1$, then $G = K_{1,p_1}$ or $G = \overline{K}_{1,p_1}$*

Proof. Since $\chi_d\overline{\chi}_{d_2} = p_1$, $\chi_{d_2} = p_1$ and $\overline{\chi}_{d_2} = 1$ or $\chi_{d_2} = 1$ and $\overline{\chi}_{d_2} = p_1$. If $\chi_{d_2} = p_1$ and $\overline{\chi}_{d_2} = 1$, then $\chi_{d_2} + \overline{\chi}_{d_2} = p$ and $\chi_{d_2}\overline{\chi}_{d_2} = p_1$. By Theorem 3.18, $G = K_{1,p_1}$. Similarly, if $\chi_{d_2} = 1$ and $\overline{\chi}_{d_2} = p_1$, then $G = \overline{K}_{1,p_1}$. □

Theorem 3.20. *For any graph G with p vertices, $2 \leq \alpha_2(G) + \chi_{d_2}(G) \leq p+1$.*

Proof. Obviously $\alpha_2(G) + \chi_{d_2}(G) \geq 2$. Let $S \subset V(G)$ such that S contains exactly one vertex from each color class of G. Then $|S| = \chi_{d_2}(G)$ and $d(u, v) = 2$ for all $u, v \in S$. Thus any maximum 2-set contains at most $p - (|S| - 1)$ vertices and so $\alpha_2(G) \leq p - \chi_{d_2}(G) + 1$. □

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