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Nonexistence of result for some p-Laplacian Systems

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Abstract

We study the nonexistence of positive solutions for the system

$$\begin{cases} -\Delta_p u = \lambda f(v), & x \in \Omega \\ -\Delta_p v = \mu g(u), & x \in \Omega \\ u = 0 = v, & x \in \partial\Omega \end{cases}$$

where Δ_p denotes the p-Laplacian operator defined by $\Delta_p z = div(|\nabla z|^{p-2}\nabla z)$ for $p > 1$ and Ω is a smooth bounded domain in R^N $(N \geq 1)$, with smooth boundary $\partial\Omega$, and λ, μ are positive parameters. Let $f, g : [0, \infty) \to R$ be continuous and we assume that there exist positive numbers K_i and M_i; $i = 1;2$ such that $f(v) \leq k_1 v^{p-1} - M_1$ for all $v \geq 0$; and $g(u) \leq k_2 u^{p-1} - M_2$ for all $u \geq 0$; We establish the nonexistence of positive solutions when $\lambda\mu$ is large.

1. Introduction

In this work we first consider a non-existence result for positive solutions in $C^1(\Omega)$ to the following reaction-diffusion system

$$\begin{cases} -\Delta_p u = av^{p-1} - bv^{\gamma-1} - c, & x \in \Omega, \\ -\Delta_p v = au^{p-1} - bu^{\gamma-1} - c, & x \in \Omega \\ u = 0 = v, & x \in \partial\Omega \end{cases} \tag{1}$$

where Δ_p denotes the p-Laplacian operator defined by $\Delta_p z = div(|\nabla z|^{p-2}\nabla z)$; $p > 1$, $\gamma(> p)$; a, b and c are positive constants, Ω is a smooth bounded domain in R^N $(N \geq 1)$ $\gamma(> p)$; a, b and c are positive constants, $\partial\Omega$ is a smooth bounded domain in R^N $(N \geq 1)$ with smooth boundary.

We first show that if $a \leq \lambda_1$;where λ_1 is the first eigenvalue of $-\Delta_p$ with Dirichlet boundary conditions, (1) has no positive solutions. Next we consider the system

$$\begin{cases} -\Delta_p u = \lambda f(v), \ x \in \Omega \\ -\Delta_p v = \mu g(u), \ x \in \Omega \\ u = 0 = v \ , \quad x \in \partial\Omega \end{cases} \tag{2}$$

where Ω is a smooth bounded domain in R^N $(N \geq 1)$; $\partial\Omega$ is its smooth boundary and λ , μ are positive parameters. Let $f, g : [0, \infty) \to R$ be continuous and assume that there exist positive numbers K_i and M_i , $i = 1;2$ such that

$$f(v) \leq k_1 v^{p-1} - M_1 \quad (v \geq 0) \tag{3}$$

and

$$g(u) \leq k_2 u^{p-1} - M_2 \quad (u \geq 0) \tag{4}$$

We discuss a nonexistence result when $\lambda\mu$ is small . In [7] , discussed (2) when there exist positive numbers K_i and M_i , $i = 1;2$ such that $f(v) \geq k_1 v^{p-1} - M_1$ $(v \geq 0)$ and $g(u) \geq k_2 u^{p-1} - M_2$ $(u \geq 0)$

Definition 1.1. A pair of nonnegative functions (u, v) in $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ are called a weak solution of (2) if they satisfies

$$\int_\Omega |\nabla u|^{p-2} \nabla u . \nabla w dx = \int_\Omega [\lambda f(v)] w dx$$

and

$$\int_\Omega |\nabla v|^{p-2} \nabla v . \nabla w dx = \int_\Omega [\mu g(u)] w dx$$

for all test function $w \in W_0^{1,p}(\Omega)$.

In the case when $p = 2$, system (2) studied by Dalmasso [4]. For existence results of positive solutions for (2) see [2, 3, 5]. For corresponding results in the single equations case, see [1] for (1) and [6] for (2).

2. Non-existence results

In this section we state our main results. To prove the non-existence results we use estimates on the first eigenvalue of $-\Delta_p$ with Dirichlet boundary conditions.

Theorem 2.1. Let q be such that $\frac{1}{p}+\frac{1}{q}=1$.Then for $a \le \lambda_1$, (1) has no positive solution.

Proof. Suppose not, i.e., assume that there exist a positive solution (u, v) of (1) , Since for any $w \in C_0^\infty(\Omega)$ we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla w dx = \int_\Omega \left[av^{p-1} - bv^{\gamma-1} - c\right]w dx$$

it follows that

$$\int_\Omega |\nabla u|^p\ dx = \int_\Omega \left[av^{p-1} - bv^{\gamma-1} - c\right]u dx$$

and since $b > 0$, we get

$$\int_\Omega |\nabla u|^p\ dx \le \int_\Omega \left[av^{p-1} - c\right]u dx \qquad (5)$$

But

$$\int_\Omega |\nabla u|^p\ dx \ge \lambda_1 \int_\Omega |u|^p\ dx = \lambda_1 \int_\Omega u^p dx \qquad (6)$$

since $\lambda_1 = \inf_{z \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla z|^p dx}{\int_\Omega |z|^p dx}$ is the first eigenvalue of $-\Delta_p$ with Dirichlet boundary conditions.

Combining (5) and (6), we obtain

$$\lambda_1 \int_\Omega u^p dx \le a\int_\Omega uv^{p-1} dx - c\int_\Omega u dx \qquad (7)$$

Similarly, we obtain

$$\lambda_1 \int_\Omega v^p dx \le a\int_\Omega vu^{p-1} dx - c\int_\Omega v dx \qquad (8)$$

But recall that $ab \le \frac{a^P}{p}+\frac{b^q}{q}$ if $\frac{1}{p}+\frac{1}{q}=1$. Taking $a = u$, $b = v^{p-1}$ and $a = v$, $b = u^{p-1}$, we see that $uv^{p-1} \le \frac{u^p}{p}+\frac{v^p}{q}$ and $vu^{p-1} \le \frac{v^p}{p}+\frac{u^p}{q}$ respectively .Thus adding (7) and(8), we get

$$\lambda_1\int_\Omega u^p dx+\lambda_1\int_\Omega v^p dx$$
$$\leq a\int_\Omega uv^{p-1}dx+a\int_\Omega vu^{p-1}dx-c\int_\Omega (u+v)dx$$
$$< a\int_\Omega uv^{p-1}dx+a\int vu^{p-1}dx$$
$$\leq a\int_\Omega\left[\frac{u^p}{p}+\frac{v^p}{q}\right]dx+a\int_\Omega\left[\frac{v^p}{p}+\frac{u^p}{q}\right]dx$$
$$=a\left(\frac{1}{p}+\frac{1}{q}\right)\int_\Omega u^p dx+a\left(\frac{1}{p}+\frac{1}{q}\right)\int_\Omega v^p dx$$
$$=a\int_\Omega\left[u^p+v^p\right]dx.$$

This implies
$$(\lambda_1-a)\int_\Omega\left[u^p+v^p\right]dx<0$$
which is contradiction if $a\leq\lambda_1$. Thus (1) has no positive solution for $a\leq\lambda_1$. □

Now we consider the system (2) and we prove:

Theorem 2.2. Let (3)–(4) hold. Then the system (2) has no positive solutions if
$$\lambda\mu<\frac{\lambda_1^2}{k_1k_2}$$

Proof. Suppose $u>0$ and $v>0$ be $C^1(\bar{\Omega})$ functions such that (u,v) is a solution of (2). We prove our theorem by arriving at a contradiction. Multiplying the first equation in (2) by a positive eigenfunction say ϕ_1 corresponding to λ_1, we obtain
$$-\int_\Omega \Delta_P u\phi_1 dx=\int_\Omega \lambda f(v)\phi_1 dx$$
and hence using (3); we get
$$-\int_\Omega \Delta_P u\phi_1 dx\leq\int_\Omega \lambda(k_1v^{P-1}-M_1)\phi_1 dx$$

That is
$$\int_\Omega u^{P-1}\lambda_1\phi_1 dx\leq\int_\Omega \lambda(k_1v^{P-1}-M_1)\phi_1 dx \qquad (9)$$
Similarly using the second equation in (2) and (4) we obtain
$$\int_\Omega v^{P-1}\lambda_1\phi_1 dx\leq\int_\Omega \mu(k_2u^{P-1}-M_2)\phi_2 dx \qquad (10)$$
Combining (9) and (10) we obtain

$$\int_{\Omega}\left[\lambda_1-(\lambda\mu)\frac{k_1k_2}{\lambda_1}\right]v^{P-1}\phi_1 dx \le \int_{\Omega}\left[\mu(k_2u^{P-1}-M_2)-(\lambda\mu)\frac{k_1k_2}{\lambda_1}v^{P-1}\right]\phi_1 dx$$

$$\le \int_{\Omega}\mu\left[\lambda\frac{k_2m_1}{\lambda_1}+M_2\right]\phi_1 dx.$$

This clearly require $\lambda\mu \ge \frac{\lambda_1^2}{k_1k_2}$. Hence , we get the result. □

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