The Journal of
Mathematics and Computer Science

Available online at
http://www.TJMCS.com

The Journal of Mathematics and Computer Science Vol .3 No.2 (2011) 117 - 125

Multiple solutions for a two-point boundary value problem depending on two parameters

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Received: April 2011, Revised: June 2011
Online Publication: July 2011

Abstract

In this paper we deal with the existence of at least three weak solutions for a two-point boundary value problem with Neumann boundary condition. The approach is based on variational methods and critical point theory.

Keywords- Three solutions; Critical point; Multiplicity results; Neumann problem.

AMS subject classification: 35J20; 34A15.

1 Introduction

In this work, based on a three critical points theorem due to Ricceri [9], we are interested in ensuring the existence of at least three solutions for the following Neumann problem

$$\begin{cases} -(|u'|u')' + u = \lambda f(x,u) + \mu g(x,u), \\ u'(0) = u'(1) = 0 \end{cases} \tag{1}$$

where $\lambda,\ \mu > 0$, $f : [0,1] \times R \to R$ is a continuous function and $g : [0,1] \times R \to R$ is an L^1-Caratéodory function.

Precisely, we deal with the existence of an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q, such that, for each $\lambda \in \Lambda$, the problem (1) admits at least three weak solutions whose norms in $W^{1,3}([0,1])$ are less than q.

We say that u is a weak solution to the problem (1) if $u \in W^{1,3}([0,1])$ and

$$\int_0^1 |u'(x)|u'(x)v'(x)dx + \int_0^1 u(x)v(x)dx - \lambda \int_0^1 f(x,u(x))v(x)dx - \mu \int_0^1 g(x,u(x))v(x)dx = 0$$

for all $v \in W^{1,3}([0,1])$.

Problem of the above type with Dirichlet and Neumann boundary conditions were widely in these latest years and we refer to [1-8] and the reference therein for more details.

In [4], using variational methods, the author ensures the existence of at least three weak solutions in $W_0^{1,2}([0,1])$ for the problem

$$\begin{cases} u'' + \lambda f(u) = 0, \\ u(0) = u(1) = 0 \end{cases} \tag{2}$$

where $\lambda > 0$ and $f : R \to R$ is a continuous function, while in their interesting paper [3], R.I. Avery and J. Henderson studied the problem (2) (independent of λ, in that case), where $f : R \to R$ is a continuous function and λ is a real parameter, by using multiple fixed-point theorem to obtain three symmetric positive solutions under growth conditions on f.

Also, M. Ramaswamy and R. Shivaji recently in [8] established the existence of three positive solutions for classes of nondecreasing, p-sublinear functions f belonging to $C^1([0,\infty))$ for a p-Laplacian version of [3], i.e., the problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

where $p > 1$, $\lambda > 0$ is a parameter and Ω is a bounded domain in R^N; $N \geq 2$ with $\partial\Omega$ of class C^2 and connected.

In [6] the authors obtained the existence of an open interval $\Lambda \subseteq [0, \infty[$ such that for each $\lambda \in \Lambda$, problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda f(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial v} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\lambda \in]0, +\infty[$, $\Omega \subset R^N$ is non-empty bounded open set with a boundary of class C^1, $a \in L^\infty(\Omega)$, with $\text{essinf}_\Omega a > 0$, $f : \Omega \times R \to R$ a function, $p \geq 2$ and v is the outer unit normal to $\partial\Omega$, and in [2] the authors proved the existence of an open interval $]\lambda',\ \lambda''[$ for each λ of problem (4) depending on λ admit at least three solutions.

In [1], we obtained the existence of an interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$ problem

$$\begin{cases} \Delta_p u + \lambda f(x,u) = a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5}$$

admits at least three weak solutions in whose norms in $W_0^{1,p}(\Omega)$ are less than q, where $\Omega \subset R^N (N \geq 2)$ is an non-empty bounded open set with a smooth boundary $\partial\Omega$, $p > N$, $\lambda > 0$, $f : \Omega \times R \to R$ is a continuous function and positive weight function $a(x) \in C(\overline{\Omega})$.

In the present paper, our approach is based on a three critical points theorem proved in [9], recalled below for the reader's convenience (Theorem 2.1), on a technical lemma (Lemma 2.3) that allows us to apply it.

Theorem 2.4 which is our main result, ensures the existence of an open interval $\Lambda \subseteq [0, \infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the problem (1) admits at least three weak solutions whose norms in $W^{1,3}([0, 1])$ are less than q.

As the consequences of Theorem 2.4, we obtain Corollary 2.5 and Theorem 2.6.

Corollary 2.5 ensures the existence of three weak solutions for the problem:

$$\begin{cases} -u'' + u = \lambda h_1(x) h_2(u) + \mu g(x,u), \\ u'(0) = u'(1) = 0 \end{cases} \tag{6}$$

where $h_1 : [0, 1] \to R$ and $h_2 : R \to R$ are two continuous functions such that $h_1(x) \geq 0$ in $[0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$ and $h_2(t) \geq 0$ in $[0, \frac{d}{4}]$ for positive constant $d \in R$ which will be taken later.

Finally, Theorem 2.6 ensures the existence of three weak solutions for the problem:

$$\begin{cases} -u'' + u = \lambda f(u) + \mu g(u), \\ u'(0) = u'(1) = 0 \end{cases} \tag{7}$$

where $f, g : R \to R$ are two continuous functions, as Example 2.7 shows.

The aim of the present paper is to extend the main result of [4] in the case Neumann boundary condition.

2 Main results

First we here recall for the reader's convenience the three critical points theorem of [9](see [11] for the related results) and Proposition 3.1 of [10]:

Theorem 2.1. Let X be a reflexive real Banach space, $I \subseteq R$ an interval, $\Phi : X \longrightarrow R$ a sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of X, whose derivative admits a continuous inverse on X^* and $J : X \longrightarrow R$ a C^1 functional with compact derivative.

Assume that

$$\lim_{||x|| \to +\infty} (\Phi(x) + \lambda J(x)) = +\infty$$

for all $\lambda \in I$, and that there exists $\rho \in R$ such that

$$\sup_{\lambda \in I} \inf_{x \in X} (\Phi(x) + \lambda(J(x) + \rho)) < \inf_{x \in X} \sup_{\lambda \in I} (\Phi(x) + \lambda(J(x) + \rho)).$$

Then, there exist a non-empty open set interval $A \subseteq I$ and a positive real number q with the following property: for every $\lambda \in A$ and every C^1 functional $\Psi : X \longrightarrow R$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$\Phi'(u) + \lambda J'(u) + \mu \Psi'(u) = 0$$

has at least three solutions in X whose norms are less than q.

Proposition 2.2. Let X be a non-empty set and Φ, J two real function on X. Assume that there are $r > 0$ and $x_0, x_1 \in X$ such that

$$\Phi(x_0) = J(x_0) = 0,\ \Phi(x_1) > r,$$

$$\sup_{x \in \Phi^{-1}(]-\infty, r])} (-J(x)) < r \frac{-J(x_1)}{\Phi(x_1)}.$$

Then, for each ρ satisfying

$$\sup_{x \in \Phi^{-1}(]-\infty, r])} (-J(x)) < \rho < r \frac{-J(x_1)}{\Phi(x_1)},$$

one has

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) + \lambda(\rho + J(x))) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\rho + J(x))).$$

Here and in the sequel, X will denote the Sobolev space $W^{1,3}([0,1])$ with the norm

$$|| u ||= \left(\int_0^1 |u'(x)|^3 dx + \int_0^1 |u(x)|^3 dx \right)^{1/3},$$

and put

$$F(x,t) = \int_0^t f(x, \xi) d\xi$$

for each $(x,t) \in [0,1] \times R$.

Our main results fully depend on the following lemma:

Lemma 2.3. Assume that there exist two positive constants c and d with $\sqrt[3]{113} d > \sqrt[3]{28}\ c$ such that

(*i*) $F(x,t) \geq 0$ for each $(x,t) \in ([0, \frac{1}{4}] \cup [\frac{3}{4}, 1]) \times [0, \frac{d}{4}]$,
(*ii*) $\frac{113}{c^3} \max_{(x,t) \in [0,1] \times [-c,c]} F(x,t) < \frac{28}{d^3} \int_{\frac{1}{4}}^{\frac{3}{4}} F(x, \frac{d}{4}) dx$.

Then, there exist $r > 0$ and $w \in X$ such that $||w||^3 > 3r$ and

$$\max_{(x,t)\in[0,1]\times[-\sqrt[3]{12r},\ \sqrt[3]{12r}]} F(x,t) < 3r\frac{\int_0^1 F(x,w(x))dx}{||w||^3}.$$

Proof: We put

$$w(x) = \begin{cases} 4dx^2, & 0 \le x \le \frac{1}{4} \\ \frac{d}{4}, & \frac{1}{4} \le x \le \frac{3}{4} \\ 4d(1-x)^2, & \frac{3}{4} \le x \le 1 \end{cases}$$

and $r = 3c^3$. It is easy to see that $w \in X$ and, in particular, one has

$$||w||^3 = \frac{113}{112}d^3.$$

Hence, taking into account that $\sqrt[3]{113}\ d > \sqrt[3]{28}\ c$, one has

$$3r < ||w||^3.$$

Since $0 \le w(x) \le \frac{d}{4}$ for each $x \in [0,1]$, condition (i) ensures that

$$\int_0^{\frac{1}{4}} F(x,w(x))dx + \int_{\frac{3}{4}}^1 F(x,w(x))dx \ge 0. \qquad (8)$$

Moreover, from (ii) and (8), we have

$$\max_{(x,t)\in[0,1]\times[-\sqrt[3]{12r},\ \sqrt[3]{12r}]} F(x,t) < \frac{28}{113}(\frac{c}{d})^3 \int_{\frac{1}{4}}^{\frac{3}{4}} F(x,\frac{d}{4})dx \le 3r\frac{\int_0^1 F(x,w(x))dx}{||w||^3}.$$

Namely,

$$\max_{(x,t)\in[0,1]\times[-\sqrt[3]{12r},\ \sqrt[3]{12r}]} F(x,t) < 3r\frac{\int_0^1 F(x,w(x))dx}{||w||^3}.$$

So, the proof is complete. □

Now, we state our main results:

Theorem 2.4. Assume that there exist three positive constants c, d, s with $\sqrt[3]{113}d > \sqrt[3]{28}\ c$, $s < 3$ and a positive function $a \in L^1$ such that

(i) $F(x,t) \ge 0$ for each $(x,t) \in ([0,\frac{1}{4}] \cup [\frac{3}{4},1]) \times [0,\frac{d}{4}]$,
(ii) $\frac{113}{c^3} \max_{(x,t)\in[0,1]\times[-c,c]} F(x,t) < \frac{28}{d^3} \int_{\frac{1}{4}}^{\frac{3}{4}} F(x,\frac{d}{4})dx$,
(iii) $F(x,t) \le a(x)(1+|t|^s)$ almost everywhere in $[0,1]$ and for each $t \in R$.

Then, there exist a non-empty open set $A \subseteq]0,+\infty[$ and a real number $q > 0$ with the following property: for every $\lambda \in A$ and for an arbitrary L^1-Carathéodory function $g : [0,1] \times R \to R$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the problem (1) has at least three weak solutions in X whose norms are less than q.

Proof: For each $u \in X$, we put

$$\Phi(u) = \frac{||u||^3}{3},$$

$$J(u) = -\int_0^1 F(x,u(x))dx.$$

Of course, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semi continuous functional whose Gâteaux derivative admits a continuous inverse on X^* and J is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. In particular, for all $u, v \in X$ one has

$$\Phi'(u)(v) = \int_0^1 |u'(x)|u'(x)v'(x)dx + \int_0^1 u(x)v(x)dx,$$

$$J'(u)(v) = -\int_0^1 f(x,u(x))v(x)dx.$$

Furthermore, thanks to (iii), for each $\lambda > 0$, one has that

$$\lim_{||u||\to+\infty}(\Phi(u) + \lambda J(u)) = +\infty.$$

We claim that there exist $r > 0$ and $w \in X$ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-J(u)) < r\frac{(-J(w))}{\Phi(w)}.$$

However, taking into account that for every $u \in X$, one has

$$\max_{x\in[0,1]}|u(x)| \le \sqrt[3]{4}||u||,$$

it follows that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-J(u)) = \sup_{||u||^3\le 3r}\int_0^1 F(x,u(x))dx \le \max_{(x,t)\in[0,1]\times[-\sqrt[3]{12r},\ \sqrt[3]{12r}]} F(x,t).$$

Now, thanks to Lemma 2.3, there exist $r > 0$ and $w \in X$ such that $||w||^3 > 3r$ and

$$\max_{(x,t)\in[0,1]\times[-\sqrt[3]{12r},\ \sqrt[3]{12r}]} F(x,t) < 3r\frac{\int_0^1 F(x,w(x))dx}{||w||^3},$$

so, our claim is true. Fix ρ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-J(u)) < \rho < r\frac{(-J(w))}{\Phi(w)},$$

and with $x_0 = 0$ and $x_1 = w$, from Proposition 2.2, we obtain the minimax inequality

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda J(u) + \rho\lambda) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda J(u) + \rho\lambda).$$

For any fixed L^1-Carathéodory function $g : [0, 1] \times R \to R$, set

$$\Psi(u) = -\int_0^1 \left(\int_0^{u(t)} g(t,\xi)d\xi\right) dt.$$

It is well known that Ψ is a continuously differentiable functional whose differential $\Psi'(u) \in X^*$, at $u \in X$ is given by

$$\Psi'(u)v = -\int_0^1 g(t, u(x))v(x)dx \quad \text{for every } v \in X,$$

such that $\Psi' : X \to X^*$ is a compact operator. Now, all the assumptions of Theorem 2.1, are satisfied. Now, applying Theorem 2.1, taking into account that the critical points of the functional $\Phi + \lambda J + \mu\Psi$ are exactly the weak solutions of the problem (1), we have the conclusion. Hence, our conclusion follows from Theorem 2.1. □

Let $h_1 \in C([0, 1])$ and $h_2 \in C(R)$ be two functions. Put

$$H(t) = \int_0^t h_2(\xi)d\xi$$

for all $t \in R$. We have the following consequence of Theorem 2.4:

Corollary 2.5. Assume that there exist three positive constants c, d, s with $\sqrt[3]{113}d > \sqrt[3]{28}\, c$, $s < 3$ and a positive function $b \in L^1$ such that

(j) $h_1(x) \geq 0$ for each $x \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$ and $h_2(t) \geq 0$ for each $t \in [0, \frac{d}{4}]$,
(jj) $\frac{113}{c^3}\max_{(x,t)\in[0,1]\times[-c,c]} h_1(x)H(t) < \frac{28}{d^3}H(\frac{d}{4})\int_{\frac{1}{4}}^{\frac{3}{4}} h_1(x)dx$,
(jjj) $H(t) \leq b(x)(1 + |t|^s)$ almost everywhere in $[0, 1]$ and for each $t \in R$.

Then, there exist a non-empty open set $A \subseteq]0, +\infty[$ and a real number $q > 0$ with the following property: for every $\lambda \in A$ and for an arbitrary L^1-Carathéodory function $g : [0, 1] \times R \to R$, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the problem (6) has at least three weak solutions in X whose norms are less than q.

Proof: Put $f(x, t) = h_1(x)h_2(t)$ for each $(x, t) \in [0, 1] \times R$, and note that

$$\max_{(x,t)\in\overline{\Omega}\times[-c,c]} F(x,t) = \max_{(x,t)\in[0,1]\times[-c,c]} h_1(x)H(t),$$

choosing $b(x) = \frac{a(x)}{h_1(x)}$ for almost every $x \in [0,1]$, it is easy to verify that all the assumptions of Theorem 2.4 are fulfilled. So, the proof is complete. □

We now want to point out a consequence of Theorem 2.4 when the function f does not depend on x:

Theorem 2.6. Let $f : R \to R$ be a continuous function. Put $F(t) = \int_0^t f(\xi)d\xi$ for each $t \in R$ and assume that there exist four positive constants c, d, s and η with $\sqrt[3]{113}d > \sqrt[3]{28}\, c$ and $s < 3$ such that

(i') $F(t) \geq 0$ for each $t \in [0, \frac{d}{4}]$,
(ii') $113\frac{\max_{t\in[-c,c]} F(t)}{c^3} < 28\frac{F(\frac{d}{4})}{d^3}$,
(iii') $F(t) \leq \eta(1 + |t|^s)$ for each $t \in R$.

Then, there exist a non-empty open set $A \subseteq]0, +\infty[$ and a real number $q > 0$ with the following property: for every $\lambda \in A$ and for an arbitrary continuous function $g : R \to R$, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the problem (7) has at least three weak solutions in X whose norms are less than q.

Finally, we illustrate the results of Theorem 2.6:

Example 2.7. Consider the problem

$$\begin{cases} -(|u'|u')' + u = \lambda(2u^{19}e^{-u^2}(10 - u^2)) + \mu g(u), \\ u'(0) = u'(1) = 0. \end{cases} \tag{9}$$

By choosing, for instance $c = 1,\ d = 8,\ s = 2$ and η sufficiently large, all the assumptions of Theorem 2.6 are satisfied. So , there exist a non-empty open set $A \subseteq]0, +\infty[$ and a real number $q > 0$ with the following property: for every $\lambda \in A$ and for an arbitrary continuous function $g : R \to R$, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the problem (9) has at least three weak solutions in $W^{1,3}([0, 1])$ whose norms are less than q.

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