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CHARACTERIZATION THE DELETABLE SET OF VERTICES IN THE ($p-3$)- REGULAR GRAPHS

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Abstract

In this paper we characterized the (p - 3)- regular graphs which have a 3−deletable and a 4−deletable set of vertices.

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1. Introduction

The roots of our study in deletable set of vertices are in the problem of reducibility of graphs. The concept of reducibility is well studied for some classes of lattices by Bordalo and Monjardet [1996]. In fact they proved that the class of pseudo complemented lattices as well as the class of semimodular lattices is reducible. Kharat and Waphare [2001] identified some classes of posets which are reducible. Further, they have introduced a concept of reducibility number for posets. Akram and Waphare [2008] introduced analogous concepts in graphs. In fact they defined the deletable vertex or the deletable set of vertices and the reducible class of graphs as follows.

Definition 1.1: Let $\mathcal{G}$ be a class of graphs satisfying some property P. A vertex (edge) v is called $deletable$ with respect to $\mathcal{G}$ if $G - v \in \mathcal{G}$. In general, a set S of vertices (edges) is called $deletable$ with respect to $\mathcal{G}$ if $G - S \in \mathcal{G}$. Generally, if $|S| = k$ then we say that S is a $k - deletable$ set.

Definition 1.2: Let $\mathcal{G}$ be a class of graphs satisfying certain property P. The class $\mathcal{G}$ is called *vertex (edge) reducible* if for any $G \in \mathcal{G}$ either G is the trivial graph (null graph) or it contains a vertex (edge) v such that $G - v \in \mathcal{G}$.

We use the concept of dominating set as given in Slater [1995].

Definition 1.3: A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called *dominating set* if every vertex $v \in V$ is either an element of S or is adjacent to an element of S.

For the undefined concepts and terminology we refer the reader to Wilson [1978],Clark [1991], Harary [1969], West [1999] and Tutte [1984].

We need the following lemma in Akram [2008].

Lemma 1.4: *Let R be an r-regular graph with p vertices. Suppose $U = \{u_1, u_2, \cdots, u_k\}$ is a deletable set of vertices with respect to the class of regular graphs $\mathcal{R}$. Then the following statements are true.*

1. $r - d_{[U]}(u_i) \le p - k, i = 1,2, \cdots k$ *, where* $[U]$ *denotes the vertex induced subgraph induced by* U*.*
2. $\frac{rk-2m}{p-k} = r - j$ *where* $m = |E([U])|$ *and the* j *=the degree of every vertex in* $R - U$ *. In particular,* $p - k$ *divides* $rk - 2m$ *for some* $0 \le m \le \frac{rk}{2}$*.*
3. $r - j \le k$*, where* j *is the degree of every vertex in* $R - U$*.*

2. Characterization the deletable set of vertices

In this section we characterized the $(p - 3) -$ regular graphs which contain a $3 - deletable$ and a $4 - deletable$ set of vertices.

Proposition 2.1: *Let G be a $(p-3)-$ regular graph on p vertices.Then G contains a $3 - deletable$ set and a $4 - deletable$ set if and only if $G \cong C_5, G \cong K_{3,3}, G \cong$ one of the eight graphs in Figure 1 or $G \cong N_3 + [(P_1 \cup P_1) + H]$ for some $(p-10)-$ regular graph H on $(p-7)$ vertices.*

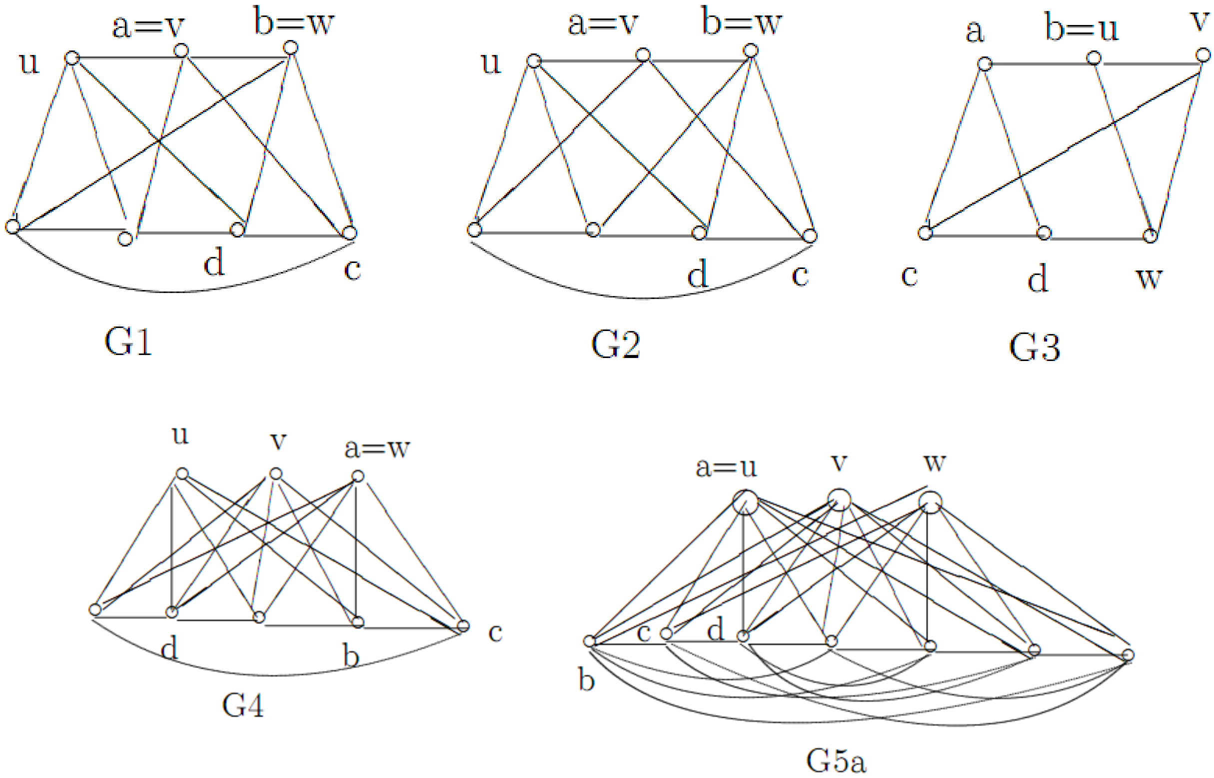
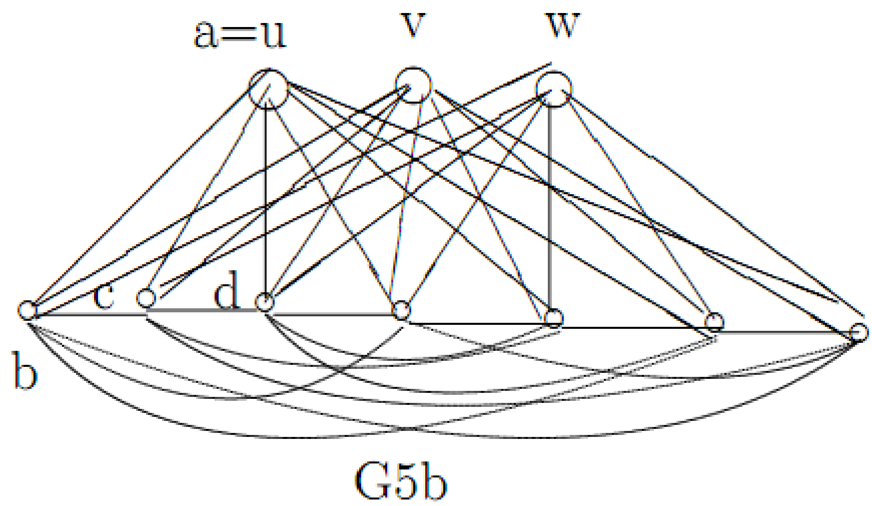


Figure 1(continued)



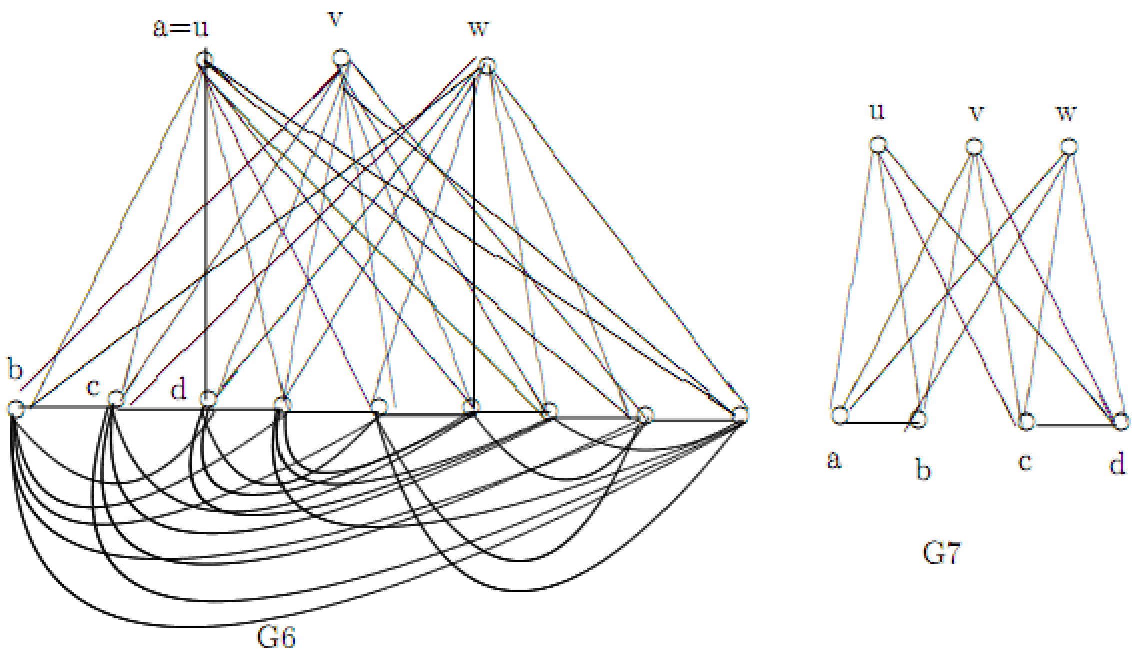


Figure 1

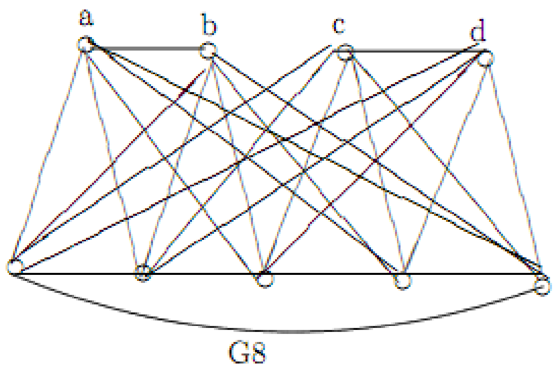


Figure 2

Proof: Suppose that $A = \{u, v, w\}$ and $B = \{a, b, c, d\}$ are two deletable sets in G. By Lemma 1.4, we have $\frac{3(p-3)-2i}{p-3}$ and $\frac{4(p-3)-2j}{p-4}$ are integers for some $i = 0,1,2,3$ and $j = 0,1,\cdots,6$. Therefore $(p-3)$ divides 2i and $(p-4)$ divides $4(p-3)-2j$.

For $i = 1$, we have $p = 5$ and $G \cong C_5$.

For $i = 2$, we have $p = 5$ or 7. If p = 5 , then $G \cong C_5$. Suppose $p = 7$, then $r = 7 - 3 = 4$. In this case $[A]$ is a path and $G - A$ is a 4-cycle. The only 4-regular graphs are G_1 and G_2 as shown in Figure 1.

For $i = 3$, we have $p = 5, 6$ or 9 and $[A] \cong C_3$. As there is no 2-regular graph on 5 vertices containing a triangle the case $p = 5$ is impossible. If $p = 6$, then $G \cong G_3$ as shown in Figure 1. If $p = 9$, then $G - A$ is a 4-regular graph on 6 vertices, since $\frac{4(p-3)-2j}{p-4} = \frac{24-2j}{5}$ is an integer. We have $D \cong [B] \cong P_2 \cup N_1$ or $P_1 \cup P_1$. By Lemma 1.4 (1) the first case is impossible. The only possible graph is G_8 as shown in Figure 2. In G_8 we do not have a set of three vertices forming a triangle and which is deletable. Hence this case is impossible.

Lastly we consider $i = 0$. Suppose $j = 0$. As $\frac{4(p-3)}{(p-4)}$ is an integer, $p = 5, 6\ or\ 8$ and the corresponding quotient $p = \frac{4(p-3)}{p-4}$, is 8,6 or 5 respectively, which is impossible by Lemma 1.4(3).

Suppose $j = 1$. As $\frac{2(2p-7)}{p-4} = \frac{[2(p-4)+2(p-3)]}{p-4}$ is an integer, $p = 5\ or\ 6$ and the quotient $\frac{2(2p-7)}{p-4}$ is 6 or 5 respectively, which is impossible by Lemma 1.4(3).

Suppose $j = 3$. As $\frac{4(p-3)-6}{p-4} = \frac{2(2p-6-3)}{p-4} = \frac{2(p-4)+2(p-5)}{p-4}$ is an integer, $p = 5\ or\ 6$ and the corresponding quotient $\frac{4(p-3)-6}{p-4}$ is 2 or 3 respectively. There is no 2-regular graph on 5 vertices having 3 non-adjacent vertices. $p = 6$ and in this case $G \cong K_{3,3}$.

Suppose $j = 4$. Then $\frac{4(p-3)-8}{p-4} = \frac{4(p-5)}{p-4}$ is an integer. Hence $p = 5, 6\ or\ 8$. As above $p = 5$ is impossible. For $p = 6$, we must have $G \cong K_{3,3}$.

Let $p = 8$. In this case there is a unique 5-regular graph containing 3-non-adjacent vertices, namely, $G_4 \cong N_3 + C_5$ as shown in Figure 1.

Suppose $j = 5$. Then $\frac{4(p-3)-10}{p-4} = \frac{2(2p-6-5)}{p-4} = \frac{2(p-4+p-7)}{p-4}$ is an integer. Hence $\frac{2(p-7)}{p-4} = \frac{2(p-4)-6}{p-4}$ is an integer, which implies that $\frac{6}{p-4}$ is an integer. Hence $p = 5, 6, 7\ or\ 10$.

The case $p = 5$ is impossible by Lemma 1.4. For $p = 6, r = 3$. We cannot have both $N_3, N_1 + P_2$ as induced subgraphs in a 3-regular graph on 6 vertices. Thus this case is impossible.

For $p = 7$, we have $G \cong N_3 + (P_1 \cup P_1) \cong G_7$ as shown in Figure 1.

For $p = 10$, we must have $|\{a, b, c, d\} \cap \{u, v, w\}| = 1$. The common vertex must have degree 3 in $[\{a, b, c, d\}]$. Then it is easy to see that $G \cong G_{5a}\ or\ G \cong G_{5b}$.

Suppose $j = 6$. Then $\frac{4(p-3)-12}{p-4} = \frac{4(p-3-3)}{p-4} = \frac{4(p-6)}{p-4} = \frac{4(p-4-2)}{p-4} = \frac{4(p-4)-8}{p-4}$ is an integer. This implies that $\frac{8}{p-4}$ is an integer. Hence $p = 5, 6, 8\ or\ 12$.

It can be observed that $p = 5\ or\ 6$ is impossible, since we cannot have both K_4, N_3 as induced subgraphs in a regular graph with $p = 5$, or 6.

Suppose p = 8. In this case also we can see that $|A \cap B| \neq 0, 2, 3$. The intersection being a singleton is also impossible as a vertex in A which is not in B will have three neighbors in $\{a, b, c, d\}$ and that is not possible since any vertex not in B should have precisely two neighbors in B. Hence there is no graph with $p = 8$, having a 3-deletable set as well as a 4-deletable set.

Now consider the case $p = 12$. As the quotient $\frac{4(p-3)-2j}{p-4} = 3$, each vertex not in B has precisely three neighbors in B. Also we have $|A \cap B| = 1$, as $G - [A \cup B]$ is a 4-regular graph on 6 vertices, and there is a unique 4-regular graph on 6 vertices. Thus we get that there is a unique graph $G \cong G_6$ in Figure 1 on 12 vertices with 3-deletable and 4-deletable subsets.

It only remains to consider the case $i = 0$ and $j = 2$. In this case $\frac{3(p-3)}{p-3} = 3$ and $\frac{4(p-3)-4}{p-4} = 4$. Therefore u, v, w are non-adjacent and are joined to all the remaining vertices. Again $[B]$ has two edges and each vertex is joined to all the remaining vertices. Note that if a vertex is isolated in $[B]$ then its degree in G is at most $p - 4$ which is impossible.
Hence $[\{a, b, c, d\}] \cong P_1 \cup P_1$. Now it can be observed that $A \cap B = \emptyset$. Therefore, we get that either $p = 7$ or $p \geq 10$. If $p = 7$ then $G \cong N_3 + [P_1 \cup P_1] \cong G_7$, and if $p \geq 10$, then we must have $G \cong N_3 + [(P_1 \cup P_1) + H]$, where H is a $(p - 10)$ -regular graph on $(p - 7)$ vertices.

Conversely, if $G \cong N_3 + [(P_1 \cup P_1) + H]$, $C_5, K_{3,3}$ *or* $G \cong$ one of the eight graphs in Figure 1, then clearly G contains a 3-deletable set as well as a 4-deletable set (see Figure 1).

Corollary 2.2: *There is no 6-regular graph on 9 vertices which contains a 3-deletable subset as well as a 4-deletable subset.*

Proposition 2.3: *There is no 9-regular graph on 30 vertices having a 3-deletable set and a 4-deletable set.*
Proof: Let G be a 9-regular graph on 30 vertices. Suppose $A = \{u, v, w\}$ and $B = \{a, b, c, d\}$ are deletable subsets in G. By Lemma 1.4, we have $\frac{3(9)-2i}{30-3}, \frac{4(9)-2j}{30-4}$ are integers for some $i = 0,1,2,3$ and $j = 0,1, \cdots ,6$. We have $i = 0$, $j = 5$ and each of the corresponding quotient is 1. Therefore, $[A] \cong N_3, [B] \cong N_1 + P_3$, A is an independent dominating set, B is a dominating set, $N(a) - B, N(b) - B$, $N(c) - B$ and $N(d) - B$ are mutually disjoint and $N(u)$, $N(v)$ and $N(w)$ are also mutually disjoint. Clearly, exactly two of B are in one of $N[u]$, $N[v]$, $N[w]$, say $a, b \in N[u]$. If a, b are both different from u, then u is a common neighbor for a, b , which is impossible. If u is one of a, b then one of v, w is not adjacent to any of a, b, c, d which is impossible.

Proposition 2.4: *There is no 11-regular graph on 36 vertices having a 3-deletable set and a 4-deletable set.*
Proof: Let G be an 11-regular graph on 36 vertices.
Suppose $A = \{u, v, w\}$ and $B = \{a, b, c, d\}$ are deletable subsets in G. By Lemma 1.4, $\frac{3(11)-2i}{36-3}, \frac{4(11)-2j}{36-4}$ are integers for some $i = 0,1,2,3$ and $j = 0, \cdots ,6$. We have $i = 0$ and $j = 6$.
Therefore $[A] \cong N_3$, and A is dominating and $N(u), N(v), N(w)$ are mutually disjoint.
Similarly, $[B] \cong K_4$, and B is dominating and $N(a) - B, N(b) - B$, $N(c) - B$, $N(d) - B$ are mutually disjoint. If $A \cap B = \emptyset$, then two of a, b, c, d are in one of $N(u), N(v), N(w)$, which is impossible. $|A \cap B| \neq 2$ or 3, as $[A] \cong N_3$ and $[B] \cong K_4$. If $|A \cap B| = 1$, say $a = u$, then $b, c, d \in N(u)$. Hence each of $v, w \notin N(a, b, c, d)$, which is impossible.

Proposition 2.5: *There is no 9-regular graph on 28 vertices having a 3-deletable set and a 4-deletable set.*

Proof: Let G be a 9-regular graph on 28 vertices. Suppose $A = \{u, v, w\}$ and $B = \{a, b, c, d\}$ are deletable subsets in G. By Lemma 1.4, we have $\frac{3(9)-2i}{28-3} = \frac{4(9)-2j}{28-4}$ are integers for some $i = 0,1,2,3$ and $j = 0, \cdots, 6$. We have $i = 1$and $j = 6$. Therefore $[A] \cong P_1 \cup N_1$, say u is adjacent to v. A is dominating and $N(u)$, $N(v)$, $N(w)$ are mutually disjoint. Similarly, $[B] \cong K_4$, B is dominating and $N(a) - B, N(b) - B$, $N(c) - B$, $N(d) - B$ are mutually disjoint. If $A \cap B = \emptyset$, then two of a, b, c, d are in one of $N(u), N(v)$, $N(w)$, which is impossible. If $|A \cap B| = 2 \; or \; 3$, then we cannot get $[B] \cong K_4$. If $|A \cap B| = 1$, with $a = u$, then $b, c, d \in N(u) - v$ and $w \notin N(a, b, c, d)$, which is impossible. Similarly, we arrive at a contradiction when $|A \cap B| = 1$, with $a = v \; or \; a = w$.

Proposition 2.6: *There is no 7-regular graph on 18 vertices having a 3-deletable set and a 4-deletable set.*

Proof: Let G be an 7-regular graph on 18 vertices. Suppose $A = \{u, v, w\}$ and $B = \{a, b, c, d\}$ are deletable subsets in G. By Lemma 1.4, $\frac{3(7)-2i}{18-3} = \frac{4(7)-2j}{18-4}$ are integers for some $i = 0,1,2,3$ and $j = 0, \cdots, 6$. We have $i = 3$ and $j = 0$. Therefore $[A] \cong C_3$, A is dominating and $N(u) - A$, $N(v) - A$, $N(w) - A$ are mutually disjoint. Similarly, $[B] \cong N_4$, B is dominating and each vertex in $V(G) - B$ is adjacent to exactly two from $\{a, b, c, d\}$. If $A \cap B = \emptyset$, then two of $\{a, b, c, d\}$ are in one of $N(u), N(v)$, $N(w)$, then we have either one from $\{u, v, w\} \notin N(a, b, c, d)$ or two of $\{u, v, w\}$ are adjacent by only one from a, b, c, d, which is impossible. $|A \cap B| \neq 2 \; or \; 3$, as $[B] \cong N_4$ and $[A] \cong C_3$ If $|A \cap B| = 1$, then either 3 or 4 vertices from $\{a, b, c, d\}$ have a common neighbor, which is impossible.

Proposition 2.7: *There is no 10-regular graph on 18 vertices having a 3-deletable set and a 4-deletable set*.

Proof: Let G be an 10-regular graph on 18 vertices. Suppose $A = \{u, v, w\}$ and $B = \{a, b, c, d\}$ are deletable subsets in G. By Lemma 1.4, $\frac{3(10)-2i}{18-3} = \frac{4(10)-2j}{18-4}$ are integers for some $i = 0,1,2,3$ and $j = 0, \cdots, 6$. We have $i = 0$ and $j = 6$. Therefore $[A] \cong N_3$, A is dominating and every vertex in $V(G) - A$, is adjacent to two from $\{u, v, w\}$. Similarly, $[B] \cong K_4$, B is dominating and each vertex in $V(G) - B$ is adjacent to two from $\{a, b, c, d\}$. If $A \cap B = \emptyset$, then it is clear we cannot get $[B] \cong K_4$ in $V(G) - \{u, v, w\}$ such that each vertex of $\{u, v, w\}$ is adjacent by two from $\{a, b, c, d\}$. If $A \cap B = 2 \; or \; 3$, it is clear that we cannot get $[B] \cong K_4$. Then we have $|A \cap B| = 1$, say $u = a$. Then $b, c, d \in N(u)$. It is clear that one from $\{v, w\}$ is adjacent to only one from $\{a, b, c, d\}$, which is impossible.

Proposition 2.8: *There is no 18-regular graph on 28 vertices having a 3-deletable set and a 4-deletable set.*

Proof: Let G be an 18-regular graph on 28 vertices.
Suppose $A = \{u, v, w\}$ and $B = \{a, b, c, d\}$ are deletable subsets in G. By Lemma 1.4, $\frac{3(18)-2i}{28-3} = \frac{4(18)-2j}{28-4}$ are integers for some $i = 0,1,2,3$ and $j = 0, \cdots, 6$. We have $i = 2$ and $j = 0$. Therefore $[A] \cong P_2$, A is dominating and each vertex in $V(G) - A$, is adjacent to two from

$\{u,v,w\}$. Similarly, $[B] \cong N_4$, B is dominating and each vertex in $V(G)-B$ is adjacent to three from $\{a,b,c,d\}$. If $A \cap B = \emptyset$, then one from $\{u,v,w\}$is adjacent to only two from $\{a,b,c,d\}$ which is impossible. If $|A \cap B| = 2$ *or* 3, then we cannot get $[B] \cong K_4$. If $|A \cap B| = 1$, then $\{a,b,c,d\}$ have a common neighbor, which is impossible.

Proposition 2.9: *There is no 24-regular graph on 36 vertices having a 3-deletable set and a 4-deletable set.*
Proof: Let G be an 24-regular graph on 36 vertices.
Suppose $A = \{u,v,w\}$ and $B = \{a,b,c,d\}$ are deletable subsets in G. By Lemma 1.4, $\frac{3(24)-2i}{36-3} = \frac{4(24)-2j}{36-4}$ are integers for some $i = 0,1,2,3$ and $j = 0,\cdots,6$. We have $i = 3$ and $j = 0$. Therefore $[A] \cong C_3$, A is dominating and each vertex in $V(G)-A$, is adjacent to two from $\{u,v,w\}$. Similarly, $[B] \cong N_4$, B is dominating and each vertex in $V(G)-B$ is adjacent to three vertices from $\{a,b,c,d\}$. If $|A \cap B| = \emptyset$;, then one from $\{u,v,w\}$ is adjacent to only two from $\{a,b,c,d\}$, which is impossible. If $|A \cap B| = 2$ *or* 3, then we cannot get $[B] \cong N_4$. If $|A \cap B| = 1$, then a,b,c,d have a common neighbor, which is impossible.

Proposition 2.10: *There is no 20-regular graph on 30 vertices having a 3-deletable set and a 4-deletable set.*
Proof: Let G be an 20-regular graph on 30 vertices.
Suppose $A = \{u,v,w\}$ and $B = \{a,b,c,d\}$ are deletable subsets in G. By Lemma 1.4, $\frac{3(20)-2i}{30-3} = \frac{4(20)-2j}{30-4}$ are integers for some $i = 0,1,2,3$ and $j = 0,\cdots,6$. We have $i = 3$ and $j = 1$. Therefore $[A] \cong C_3$, A is dominating and each vertex in $V(G)-A$, is adjacent to two from $\{u,v,w\}$. Similarly, $[B] \cong P_1 \cup N_2$, B is dominating and each vertex in $V(G)-B$ is adjacent to three from $\{a,b,c,d\}$. If $|A \cap B| = \emptyset$, then there is a vertex from $\{u,v,w\}$ which is adjacent to only two from a,b,c,d, which is impossible. If $|A \cap B| = 2$ *or* 3, then we cannot get $[B] \cong P_1 \cup N_2$. If $|A \cap B| = 1$, then we have one from $\{u,v,w\}$ is a common neighbor for $\{a,b,c,d\}$, which is impossible.

Proposition 2.11: *There is no 16-regular graph on 24 vertices having a 3-deletable set and a 4-deletable set*.
Proof: Suppose $A = \{u,v,w\}$ and $B = \{a,b,c,d\}$ are deletable subsets in G. By Lemma 1.4, $\frac{3(16)-2i}{24-3} = \frac{4(16)-2j}{24-4}$ are integers for some $i = 0,1,2,3$ and $j = 0,\cdots,6$. We have $i = 3$ and $j = 2$. Therefore $[A] \cong C_3$, A is dominating and each vertex in $V(G)-A$ is adjacent to two from $\{u,v,w\}$. Similarly, $[B] \cong P_1 \cup P_1$ *or* $P_2 \cup N_1$, B is dominating and each vertex in $V(G)-A$ is adjacent to three from $\{a,b,c,d\}$.
Let $[B] \cong P_1 \cup P_1$. if $|A \cap B| = \emptyset$, then there is a vertex from $\{u,v,w\}$ which is adjacent to only two from $\{a,b,c,d\}$, which is impossible. If $|A \cap B| = 2$ *or* 3, then we cannot get $[B] \cong P_1 \cup P_1$. If $|A \cap B| = 1$ then a,b,c,d have a common neighbor, which is impossible.
Let $[B] \cong P_2 \cup N_1$, then by the same arguments as above, we have a contradiction.

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