The Journal of
Mathematics and Computer Science

Available online at
http://www.TJMCS.com

The Journal of Mathematics and Computer Science Vol .3 No.2 (2011) 165 - 175

Quasi-Permutation Representations for the Borel and Maximal Parabolic Subgroups of $SP(4,2^n)$

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Received: May 2011, Revised: June 2011
Online Publication: July 2011

Abstract

A square matrix over the complex field with non-negative integral trace is called a quasi-permutation matrix.Thus every permutation matrix over C is a quasi-permutation matrix . The minimal degree of a faithful representation of G by quasi-permutation matrices over the complex numbers is denoted by $c(G)$, and $r(G)$ denotes the minimal degree of a faithful rational valued complex character of G. In this paper $c(G)$ and $r(G)$ are calculated for the Borel or maximal parabolic subgroups of $SP(4,2^f)$.

Keywords: General linear group, Quasi-permutation.

1- Introduction

In 1963 Wong defined a quasi-permutation group of degree n to be a finite group G of automorphisms of an n-dimensional complex vector space such that every element of G has non-negative integral trace .The terminology drives from the fact that if

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2000 AMS Subject Classification:20C15

G is a finite group of permutations of a set Ω of size n, and we think of G as acting on the complex vector space with basis Ω, then the trace of an element $g \in G$ is equal to the number of points of Ω fixed by g.

Wong studied the extent to which some facts about permutation groups generalize to the quasi-permutation group situation.Then in 1994 Hartley with his colleague investigate further the analogy between permutation groups and quasi-permutation groups by studying the relation between the minimal degree of a faithful permutation representation of a given finite group G and the minimal degree of a faithful quasi-permutation representation . They also worked over the rational field and found some interesting results.(See [2],[8]) .

If F is a subfield of the complex numbers C, then a square matrix over F with non-negative integral trace is called a quasi-permutation matrix over F.Thus every permutation matrix over C is a quasi-permutation matrix. For a given finite group G, let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permutation matrices.

By a rational valued character we mean a character χ corresponding to a complex representation of G such that $\chi(g) \in Q$ for all $g \in G$. As the values of the character of a complex representation are algebraic numbers, a rational valued character is in fact integer valued. A quasi-permutation representation of G is then simply a complex representation of G whose character values are rational and non-negative. The module of such a representation will be called a quasi-permutation module. We will call a homomorphism from G to $GL(n,Q)$ a rational representation of G and its corresponding character will be called a rational character of G. Let $r(G)$ denote the minimal degree of a faithful rational valued character of G.

If $\varepsilon \in C$ is an algebraic number over Q, then the Galois group of $Q(\varepsilon)$ over Q is denoted by Γ.

Finding the above quantities have been carried out in some papers, for example in [3],[4] , [5] and [7] we found these for the groups $GL(2,q), SU(3,q^2), PSU(3,q^2)$, $SL(3,q), PSL(3,q)$ and $G_2(2^n)$ respectively.

In this paper we will calculate $c(G)$ and $r(G)$ for Borel or maximal parabolic subgroups of $SP(4,2^f)$.

2-Notation and preliminary results

Assume that E is a splitting field for G and that F is a subfield of E. If $\chi,\psi \in \mathrm{Irr}_E(G)$ we say that χ and ψ are Galois conjugate over F if $F(\chi) = F(\psi)$ and there exists $\sigma \in \mathrm{Gal}(F(\chi)/F)$ such that $\chi^\sigma = \psi$, where $F(\chi)$ denotes the field obtained by adding the values $\chi(g)$, for all $g \in G$, to F. It is clear that this defines an equivalence relation on $\mathrm{Irr}_E(G)$.

Let η_i for $0 \le i \le r$ be Galois conjugacy classes of irreducible complex characters of G. For $0 \le i \le r$ let φ_i be a representative of the class η_i, with $\varphi_o = 1_G$. Write

$\Psi_i = \sum_{\chi_i \in \eta_i} \chi_i$ and $K_i = ker\varphi_i$. We know that $K_i = ker\Psi_i$. For $I \subseteq \{0,1,2,\cdots,r\}$, put $K_I = \bigcap_{i \in I} K_i$. By definition of $r(G)$, $c(G)$ and using above notations we have:

$$r(G) = \min\{\xi(1) : \xi = \sum_{i=1}^{r} n_i \Psi_i, n_i \geq 0, K_I = 1 \; for \; I = \{i, i \neq 0, n_i > 0\}\}$$

$$c(G) = \min\{\xi(1) : \xi = \sum_{i=0}^{r} n_i \Psi_i, n_i \geq 0, K_I = 1 \text{ for } I = \{i, i \neq 0, n_i > 0\}\}$$

where $n_0 = -\min\{\xi(g) \mid g \in G\}$.

In [1] we defined $d(\chi), m(\chi)$ and $c(\chi)$ [See Definition 3.4]. Here we can redefine it as follows:

Definition 2.1.

Let χ be a complex charater of G, such that $\ker\chi = 1$ and $\chi = \chi_1 + \cdots + \chi_n$ for some $\chi_i \in Irr(G)$. Then define

(1) $d(\chi) = \sum_{i=1}^{n} |\Gamma_i(\chi_i)| \chi_i(1)$,

(2) $$m(\chi) = \begin{cases} 0 & \text{if } \chi = 1_G, \\ |\min\{\sum_{i=1}^{n} \sum_{\alpha \in \Gamma_i(\chi_i)} \chi_i^{\alpha}(g) : g \in G\}| & \text{otherwise}, \end{cases}$$

(3) $c(\chi) = \sum_{i=1}^{n} \sum_{\alpha \in \Gamma_i(\chi_i)} \chi_i^{\alpha} + m(\chi) 1_G$.

So

$$r(G) = \min\{d(\chi) : \ker\chi = 1\},$$

and

$$c(G) = \min\{c(\chi)(1) : \ker\chi = 1\}.$$

We can see all the following statements in [1].

Corollary 2.2.

Let $\chi \in Irr(G)$,then $\sum_{\alpha \in \Gamma(\chi)} \chi^{\alpha}$ is a rational valued character of G . Moreover $c(\chi)$ is a non-negative rational valued character of G and $c(\chi) = d(\chi) + m(\chi)$.

Lemma 2.3.

Let $\chi \in Irr(G), \chi \neq 1_G$. Then $c(\chi)(1) \geq d(\chi) + 1 \geq \chi(1) + 1$.

Lemma 2.4.

Let $\chi \in Irr(G)$. Then

(1) $c(\chi)(1) \geq d(\chi) \geq \chi(1)$;

(2) $c(\chi)(1) \leq 2d(\chi)$. Equality occurs if and only if $Z(\chi)/ker\chi$ is of even order.

3. Quasi-permutation representations

We begin with a brief summary of facts relevant to our treatment of the group .

Let K be the finite field with q elements, where $q = p^f$ and p is a prime number. Let $\overline{K}$ be the algebraic closure of K, and put

$$K_i = \{x \in \overline{K} \mid x^{q^i} = x\}.$$

Then K_i is the subfield of $\overline{K}$ with q^i elements , and $K_1 = K$. Let κ be a fixed generator of the multiplicative group K_4^* and put $\tau = \kappa^{q^2-1}, \theta = \kappa^{q^2+1}, \eta = \theta^{q-1}$ and $\gamma = \theta^{q+1}$. Then we have $<\theta> = K_2^*$ and $<\gamma> = K^*$. Choose a fixed isomorphism from the multiplicative group K_4^* into the multiplicative group of complex numbers, and let $\widetilde{\tau,\theta,\eta}$ and $\widetilde{\gamma}$ be the images of τ, θ, η and γ respectively under this isomorphism .

Let G be the $4-$dimensional symplectic group over K, that is ,

$$G = \{A \in GL(4, K) \mid {}^t AJA = J\},$$

where $J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$ and ${}^tA$ is the transposed matrix of A . For $t \in K$, define

$$x_a(t) = \begin{pmatrix} 1 & t & & \\ & 1 & & \\ & & 1 & -t \\ & & & 1 \end{pmatrix}, \quad x_b(t) = \begin{pmatrix} 1 & & & \\ & 1 & t & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$x_{a+b}(t) = \begin{pmatrix} 1 & & t & \\ & 1 & & t \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad x_{2a+b}(t) = \begin{pmatrix} 1 & & & t \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

and put $\Delta^+ = \{a, b, a+b, 2a+b\}$. Then for $r \in \Delta^+, \Xi_r = \{x_r(t) \mid t \in K\}$ is a subgroup of G isomorphism to the additive group of K , and we have the folloing commutator relations , where the commutator $x^{-1}y^{-1}xy$ is denoted by $[x, y]$:

$$[x_a(t), x_b(u)] = x_{a+b}(tu)x_{2a+b}(-t^2u),$$

$$[x_a(t), x_{a+b}(u)] = x_{2a+b}(2tu),$$

$$[x_r(t), x_s(u)] = 1, \quad \text{for all other pairs of } r, s \in \Delta^+.$$

Next, define $h(z_1,z_2)=\begin{pmatrix} z_1 & & & \\ & z_2 & & \\ & & z_2^{-1} & \\ & & & z_1^{-1} \end{pmatrix}$ for $z_i \in K_4^*$ and put

$U=\Xi_a\Xi_b\Xi_{a+b}\times\Xi_{2a+b}, \wp=\{h(z_1,z_2)\,|\,z_i\in K^*\}$ and $B=\wp U$. Then U is a Sylow p-subgroup of G, and B is the normalizer of U in G (called the Borel subgroup of G). Put $\omega_r = x_r(1)^t x_r(-1)x_r(1)$ for $r\in\Delta^+$. Especially,

$$\omega_a=\begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}, \omega_b=\begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix}.$$

Then G is generated by $B\cup\{\omega_a,\omega_b\}$. The maximal parabolic subgroups of G generated by $B\cup\{\omega_a,\}$ and $B\cup\{\omega_b,\}$ are denoted by P and Q respectively.

We know that every irreducible character of Borel subgroup B is the induced character of some linear character of a subgroup, that is, B is an $M-$group. The character table of B is given in Table (I) and the character tables of P and Q are given in Tables (II, III) of the Appendix of [6].

In the next theorem we shall determine $r(G)$ and $c(G)$ for a Borel subgroup of $SP(4,2^n)$.

Theorem 3.1.

Let G be a Borel subgroup of $SP(4,2^n)$, then

1) $r(B)=\begin{cases} 2mq(q-1) & \text{if} q\geq 4m+1, \\ \dfrac{q(q-1)^2}{2} & \text{otherwise} \end{cases}$

2) $c(B)=\begin{cases} 2mq^2 & \text{if} q\geq 4m+1, \\ \dfrac{q^2(q-1)}{2} & \text{otherwise} \end{cases}$ where $m=|\Gamma(\chi_4(k))|$.

Proof. By Definition 2.1, in order to calculate $r(G)$ and $c(G)$, we need to determine $d(\chi)$ and $c(\chi)(1)$ for all characters that are faithful or $\bigcap_\chi Ker\chi=1$.

Then by Corollary 2.2, Lemmas 2.3 and 2.4 and Table (I) of [6], for the Borel subgroup B we have:

$d(\chi_1)=|\Gamma(\chi_1(k,l))|\,\chi_1(k,l)(1)+|\Gamma(\theta_2(k))|\,\theta_2(k)(1)\geq \dfrac{q(q-1)^2}{2}+1$ and $c(\chi_1)(1)\geq 1+\dfrac{q^2(q-1)}{2}$,

$d(\chi_2) = |\Gamma(\chi_2(k))|\chi_2(k)(1) + |\Gamma(\theta_2(k))|\theta_2(k)(1) \geq \frac{(q-1)(q^2-q+2)}{2}$ and $c(\chi_2)(1) \geq \frac{q(q^2-q+2)}{2}$,

$d(\chi_3) = |\Gamma(\chi_3(k))|\chi_3(k)(1) + |\Gamma(\theta_2(k))|\theta_2(k)(1) \geq \frac{(q-1)(q^2-q+2)}{2}$ and $c(\chi_3)(1) \geq \frac{q(q^2-q+2)}{2}$,

$d(\chi_4) = |\Gamma(\chi_4(k))|\chi_4(k)(1) + |\Gamma(\theta_2(k))|\theta_2(k)(1) \geq \frac{q(q^2-1)}{2}$ and $c(\chi_4)(1) \geq \frac{q^2(q+1)}{2}$,

$d(\chi_5) = |\Gamma(\chi_5(k))|\chi_5(k)(1) + |\Gamma(\theta_2(k))|\theta_2(k)(1) \geq \frac{q(q^2-1)}{2}$ and $c(\chi_5)(1) \geq \frac{q^2(q+1)}{2}$,

$d(\chi_6) = |\Gamma(\theta_1)|\theta_1(1) + |\Gamma(\theta_2(k))|\theta_2(k)(1) \geq \frac{(q-1)^2(q+2)}{2}$ and $c(\chi_6)(1) \geq q^2(q-1)$,

$d(\chi_7) = |\Gamma(\chi_1(k,l))|\chi_1(k,l)(1) + |\Gamma(\theta_3(k))|\theta_3(k)(1) \geq \frac{q(q-1)^2}{2} + 1$ and $c(\chi_7)(1) \geq 1 + \frac{q^2(q-1)}{2}$,

$d(\chi_8) = |\Gamma(\chi_2(k))|\chi_2(k)(1) + |\Gamma(\theta_3(k))|\theta_3(k)(1) \geq \frac{(q-1)(q^2-q+2)}{2}$ and $c(\chi_8)(1) \geq \frac{q(q^2-q+2)}{2}$,

$d(\chi_9) = |\Gamma(\chi_3(k))|\chi_3(k)(1) + |\Gamma(\theta_3(k))|\theta_3(k)(1) \geq \frac{(q-1)(q^2-q+2)}{2}$ and $c(\chi_9)(1) \geq \frac{q(q^2-q+2)}{2}$,

$d(\chi_{10}) = |\Gamma(\chi_4(k))|\chi_4(k)(1) + |\Gamma(\theta_3(k))|\theta_3(k)(1) \geq \frac{q(q^2-1)}{2}$ and $c(\chi_{10})(1) \geq \frac{q^2(q+1)}{2}$,

$d(\chi_{11}) = |\Gamma(\chi_5(k))|\chi_5(k)(1) + |\Gamma(\theta_3(k))|\theta_3(k)(1) \geq \frac{q(q^2-1)}{2}$ and $c(\chi_{11})(1) \geq \frac{q^2(q+1)}{2}$,

$d(\chi_{12}) = |\Gamma(\theta_1)|\theta_1(1) + |\Gamma(\theta_3(k))|\theta_3(k)(1) \geq \frac{(q-1)^2(q+2)}{2}$ and $c(\chi_{12})(1) \geq q^2(q-1)$,

$d(\chi_{13}) = |\Gamma(\chi_5(k))|\chi_5(k)(1) + |\Gamma(\chi_4(k))|\chi_4(k)(1) \geq 2q(q-1)$ and $c(\chi_{13})(1) \geq 2q^2$,

$d(\theta_2(k)) = |\Gamma(\theta_2(k))|\theta_2(k)(1) = \frac{q(q-1)^2}{2}$ and $c(\theta_2(k))(1) = \frac{q^2(q-1)}{2}$,

$d(\theta_3(k)) = |\Gamma(\theta_3(k))|\theta_3(k)(1) = \frac{q(q-1)^2}{2}$ and $c(\theta_3(k))(1) = \frac{q^2(q-1)}{2}$,

An overall picture is provided by the Table(I):

Table (I)

χ	$d(\chi)$	$c(\chi)(1)$
χ_1	$\geq q(q-1)^2/2+1$	$\geq 1+q^2(q-1)/2$
χ_2	$\geq (q-1)(q^2-q+2)/2$	$\geq q(q^2-q+2)/2$
χ_3	$\geq (q-1)(q^2-q+2)/2$	$\geq q(q^2-q+2)/2$
χ_4	$\geq q(q^2-1)/2$	$\geq q^2(q+1)/2$
χ_5	$\geq q(q^2-1)/2$	$\geq q^2(q+1)/2$
χ_6	$\geq (q-1)^2(q+2)/2$	$\geq q^2(q-1)$

χ_7	$\geq q(q-1)^2/2+1$	$\geq 1+q^2(q-1)/2$
χ_8	$\geq (q-1)(q^2-q+2)/2$	$\geq q(q^2-q+2)/2$
χ_9	$\geq (q-1)(q^2-q+2)/2$	$\geq q(q^2-q+2)/2$
χ_{10}	$\geq q(q^2-1)/2$	$\geq q^2(q+1)/2$
χ_{11}	$\geq q(q^2-1)/2$	$\geq q^2(q+1)/2$
χ_{12}	$\geq (q-1)^2(q+2)/2$	$\geq q^2(q-1)$
χ_{13}	$\geq 2q(q-1)$	$\geq 2q^2$
$\theta_2(k)$	$q(q-1)^2/2$	$q^2(q-1)/2$
$\theta_3(k)$	$q(q-1)^2/2$	$q^2(q-1)/2$

Note that the characters $\theta_2(k)$ and $\theta_3(k)$ are rational , now let $|\Gamma(\chi_4(k))|=m$ where $\Gamma(\chi_4(k))=\Gamma(Q(\chi_4(k)):Q)=\Gamma(Q(\chi_5(k)):Q)$.

Now by above table and Definition 2.1 and Table (I) of [6], we have

$$\min\ \{d(\chi):Ker\chi=1\}=\begin{cases}2mq(q-1) & if q\geq 4m+1,\\ \frac{q(q-1)^2}{2} & otherwise,\end{cases}$$

$$\min\ \{c(\chi)(1):Ker\chi=1\}=\begin{cases}2mq^2 & \text{if} q\geq 4m+1,\\ \frac{q^2(q-1)}{2} & \text{otherwise.}\end{cases}$$

Hence $r(B)=\begin{cases}2mq(q-1) & \text{if} q\geq 4m+1,\\ \frac{q(q-1)^2}{2} & \text{otherwise,}\end{cases}$

and

$$c(B)=\begin{cases}2mq^2 & \text{if} q\geq 4m+1,\\ \frac{q^2(q-1)}{2} & \text{otherwise.W}\end{cases}$$

In the following theorem, we constructed the $r(G)$ and $c(G)$ of parabolic subgroup Q of $SP(4,2^n)$.

Theorem 3.2

Let G be a maximal parabolic subgroup P or Q of $SP(4,2^n)$, then

1) $r(G)=\frac{q(q-1)^2}{2}$

2) $c(G)=\frac{q^2(q-1)}{2}$.

Proof. Since the groups P and Q have similar proofs, we will prove only Q.In order to calculate $r(G)$ and $c(G)$, we need to determine $d(\chi)$ and $c(\chi)(1)$ for all characters that are faithful or $\bigcap_{\chi} Ker\chi = 1$.

Then by Corollary 2.2 , Lemmas 2.3,2.4 and Table (III) of [6], for the maximal parabolic subgroup Q we have :

$d(\chi_1) =|\Gamma(\chi_1^{'}(k))|\chi_1^{'}(k)(1)+|\Gamma(\chi_5^{'}(k))|\chi_5^{'}(k)(1) \geq q(q^2-1)+1$ and $c(\chi_1)(1) \geq q^3+q^2+1$,
$d(\chi_2) =|\Gamma(\chi_1^{'}(k))|\chi_1^{'}(k)(1)+|\Gamma(\chi_6^{'}(k))|\chi_6^{'}(k)(1) \geq q(q-1)^2+1$ and $c(\chi_2)(1) \geq q^3-q^2+1$,
$d(\chi_3) =|\Gamma(\chi_1^{'}(k))|\chi_1^{'}(k)(1)+|\Gamma(\theta_2^{'}(k))|\theta_2^{'}(k)(1) \geq \frac{q(q^2-1)}{2}+1$ and $c(\chi_3)(1) \geq \frac{(q^3+q^2+2)}{2}$,
$d(\chi_4) =|\Gamma(\chi_1^{'}(k))|\chi_1^{'}(k)(1)+|\Gamma(\theta_3^{'}(k))|\theta_3^{'}(k)(1) \geq \frac{q(q-1)^2}{2}+1$ and $c(\chi_4)(1) \geq \frac{(q^3-q^2+2)}{2}$,
$d(\chi_5) =|\Gamma(\chi_2^{'}(k))|\chi_2^{'}(k)(1)+|\Gamma(\chi_5^{'}(k))|\chi_5^{'}(k)(1) \geq q^3$ and $c(\chi_5)(1) \geq q^3+q^2+q+1$,
$d(\chi_6) =|\Gamma(\chi_2^{'}(k))|\chi_2^{'}(k)(1)+|\Gamma(\chi_6^{'}(k))|\chi_6^{'}(k)(1) \geq q(q^2-2q+2)$ and $c(\chi_6)(1) \geq q^3-q^2+q+1$,
$d(\chi_7) =|\Gamma(\chi_2^{'}(k))|\chi_2^{'}(k)(1)+|\Gamma(\theta_2^{'}(k))|\theta_2^{'}(k)(1) \geq \frac{q(q^2+1)}{2}$ and $c(\chi_7)(1) \geq \frac{(q^3+q^2+2q+2)}{2}$,
$d(\chi_8) =|\Gamma(\chi_2^{'}(k))|\chi_2^{'}(k)(1)+|\Gamma(\theta_3^{'}(k))|\theta_3^{'}(k)(1) \geq \frac{q(q^2-2q+3)}{2}$ and $c(\chi_8)(1) \geq \frac{(q^3-q^2+2q+2)}{2}$,
$d(\chi_9) =|\Gamma(\chi_3^{'}(k,l))|\chi_3^{'}(k,l)(1)+|\Gamma(\chi_5^{'}(k))|\chi_5^{'}(k)(1) \geq (q+1)(q^2-q+1)$ and $c(\chi_9)(1) \geq q^3+q^2+q+2$,
$d(\chi_{10}) =|\Gamma(\chi_3^{'}(k,l))|\chi_3^{'}(k,l)(1)+|\Gamma(\chi_6^{'}(k))|\chi_6^{'}(k)(1) \geq (q+1)+q(q-1)^2$ and $c(\chi_{10})(1) \geq q^3-q^2+q+2$,
$d(\chi_{11}) =|\Gamma(\chi_3^{'}(k,l))|\chi_3^{'}(k,l)(1)+|\Gamma(\theta_2^{'}(k))|\theta_2^{'}(k)(1) \geq (q+1)(q^2-q+2)/2$ and $c(\chi_{11})(1) \geq \frac{(q^3+q^2+2q+4)}{2}$,
$d(\chi_{12}) =|\Gamma(\chi_3^{'}(k,l))|\chi_3^{'}(k,l)(1)+|\Gamma(\theta_3^{'}(k))|\theta_3^{'}(k)(1) \geq \frac{(q+1)+q(q-1)^2}{2}$ and $c(\chi_{12})(1) \geq \frac{(q^3+q^2+2q+4)}{2}$,
$d(\chi_{13}) = \Gamma(\chi_4^{'}(k))|\chi_4^{'}(k)(1)+|\Gamma(\chi_5^{'}(k))|\chi_5^{'}(k)(1) \geq (q^2-1)(q+1)$ and $c(\chi_{13})(1) \geq q^2(q+2)$,
$d(\chi_{14}) =|\Gamma(\chi_4^{'}(k))|\chi_4^{'}(k)(1)+|\Gamma(\chi_6^{'}(k))|\chi_6^{'}(k)(1) \geq (q-1)(q^2+1)$ and $c(\chi_{14})(1) \geq q^3$,
$d(\chi_{15}) =|\Gamma(\chi_4^{'}(k))|\chi_4^{'}(k)(1)+|\Gamma(\theta_2^{'}(k))|\theta_2^{'}(k)(1) \geq (q+2)(q^2-1)/2$ and $c(\chi_{15})(1) \geq \frac{q^2(q+3)}{2}$
$d(\chi_{16}) = \Gamma(\chi_4^{'}(k))|\chi_4^{'}(k)(1)+|\Gamma(\theta_3^{'}(k))|\theta_3^{'}(k)(1) \geq \frac{(q-1)(q^2+q+2)}{2}$ and $c(\chi_{16})(1) \geq \frac{q^2(q+1)}{2}$
$d(\chi_{17}) =|\Gamma(\chi_7^{'}(k,l))|\chi_7^{'}(k,l)(1)+|\Gamma(\chi_5^{'}(k))|\chi_5^{'}(k)(1) \geq (q-1)(q^2+q+1)$ and $c(\chi_{17})(1) \geq q(q^2+q+1)$,
$d(\chi_{18}) =|\Gamma(\chi_7^{'}(k,l))|\chi_7^{'}(k,l)(1)+|\Gamma(\chi_6^{'}(k))|\chi_6^{'}(k)(1) \geq (q-1)(q^2-q+1)$ and $c(\chi_{18})(1) \geq q(q^2-q+1)$,
$d(\chi_{19}) =|\Gamma(\chi_7^{'}(k,l))|\chi_7^{'}(k,l)(1)+|\Gamma(\theta_2^{'}(k))|\theta_2^{'}(k)(1) \geq \frac{(q-1)(q^2+q+2)}{2}$ and $c(\chi_{19})(1) \geq \frac{q^2(q+3)}{2}$,
$d(\chi_{20}) =|\Gamma(\chi_7^{'}(k,l))|\chi_7^{'}(k,l)(1)+|\Gamma(\theta_3^{'}(k))|\theta_3^{'}(k)(1) \geq \frac{(q-1)(q^2-q+2)}{2}$ and $c(\chi_{20})(1) \geq \frac{q(q^2-q+2)}{2}$,
$d(\chi_{21}) =|\Gamma(\theta_1^{'})|\theta_1^{'}(1)+|\Gamma(\chi_5^{'}(k))|\chi_5^{'}(k)(1) \geq (2q-1)(q^2-1)$ and $c(\chi_{21})(1) \geq 2(q^3+q^2-1)$,
$d(\chi_{22}) =|\Gamma(\theta_1^{'})|\theta_1^{'}(1)+|\Gamma(\chi_6^{'}(k))|\chi_6^{'}(k)(1) \geq (2q+1)(q-1)^2$ and $c(\chi_{22})(1) \geq 2q^2(q-1)$,

$d(\chi_{23}) = |\Gamma(\theta_1^{'})|\theta_1^{'}(1) + |\Gamma(\theta_2^{'}(k))|\theta_2^{'}(k)(1) \geq \frac{(3q-2)(q^2-1)}{2}$ and $c(\chi_{23})(1) \geq \frac{q^2(3q-1)}{2}$,

$d(\chi_{24}) = |\Gamma((\theta_1^{'})|\theta_1^{'}(1) + |\Gamma(\theta_3^{'}(k))|\theta_3^{'}(k)(1) \geq \frac{(3q+2)(q-1)^2}{2}$ and $c(\chi_{24})(1) \geq \frac{3q^2(q-1)}{2}$,

$d(\chi_5^{'}(k)) = |\Gamma(\chi_5^{'}(k))|\chi_5^{'}(k)(1) \geq q(q^2-1)$ and $c(\chi_5^{'}(k))(1) \geq q^2(q+1)$,

$d(\chi_6^{'}(k)) = |\Gamma(\chi_6^{'}(k))|\chi_6^{'}(k)(1) \geq q(q-1)^2$ and $c(\chi_6^{'}(k))(1) \geq q^2(q-1)$,

$d(\theta_2^{'}(k)) = |\Gamma(\theta_2^{'}(k))|\theta_2^{'}(k)(1) = \frac{q(q^2-1)}{2}$ and $c(\theta_2^{'}(k))(1) = \frac{q^2(q+1)}{2}$,

$d(\theta_3^{'}(k)) = |\Gamma(\theta_3^{'}(k))|\theta_3^{'}(k)(1) = \frac{q(q-1)^2}{2}$ and $c(\theta_3^{'}(k))(1) = \frac{q^2(q-1)}{2}$.

The values are set out in the following table :

Table (II)

χ	$d(\chi)$	$c(\chi)(1)$
χ_1	$\geq q(q^2-1)+1$	$\geq q^3+q^2+1$
χ_2	$\geq q(q-1)^2+1$	$\geq q^3-q^2+1$
χ_3	$\geq q(q^2-1)/2+1$	$\geq (q^3+q^2+2)/2$
χ_4	$\geq q(q-1)^2/2+1$	$\geq (q^3-q^2+2)/2$
χ_5	$\geq q^3$	$\geq q^3+q^2+q+1$
χ_6	$\geq q(q^2-2q+2)$	$\geq q^3-q^2+q+1$
χ_7	$\geq q(q^2+1)/2$	$\geq (q^3+q^2+2q+2)/2$
χ_8	$\geq q(q^2-2q+3)/2$	$\geq (q^3-q^2+2q+2)/2$
χ_9	$\geq (q+1)(q^2-q+1)$	$\geq q^3+q^2+q+2$
χ_{10}	$\geq (q+1)+q(q-1)^2$	$\geq q^3-q^2+q+2$
χ_{11}	$\geq (q+1)(q^2-q+2)/2$	$\geq (q^3+q^2+2q+4)/2$
χ_{12}	$\geq (q+1)+q(q-1)^2/2$	$\geq (q^3-q^2+2q+4)/2$
χ_{13}	$\geq (q+1)(q^2-1)$	$\geq q^2(q+2)$
χ_{14}	$\geq (q-1)(q^2+1)$	$\geq q^3$
χ_{15}	$\geq (q+2)(q^2-1)/2$	$\geq q^2(q+3)/2$
χ_{16}	$\geq (q-1)(q^2+q+2)/2$	$\geq q^2(q+1)/2$
χ_{17}	$\geq (q-1)(q^2+q+1)$	$\geq q(q^2+q+1)$
χ_{18}	$\geq (q-1)(q^2-q+1)$	$\geq q(q^2-q+1)$
χ_{19}	$\geq (q-1)(q^2+q+2)/2$	$\geq q^2(q+3)/2$
χ_{20}	$\geq (q-1)(q^2-q+2)/2$	$\geq q(q^2-q+2)/2$
χ_{21}	$\geq (2q-1)(q^2-1)$	$\geq 2(q^3+q^2-1)$

χ_{22}	$\geq (2q+1)(q-1)^2$	$\geq 2q^2(q-1)$
χ_{23}	$\geq (3q-2)(q^2-1)/2$	$\geq q^2(3q-1)/2$
χ_{24}	$\geq (3q+2)(q-1)^2/2$	$\geq 3q^2(q-1)/2$
$\chi_5^{'}(k)$	$\geq q(q^2-1)$	$\geq q^2(q+1)$
$\chi_6^{'}(k)$	$\geq q(q-1)^2$	$\geq q^2(q-1)$
$\theta_2^{'}(k)$	$q(q^2-1)/2$	$q^2(q+1)/2$
$\theta_3^{'}(k)$	$q(q-1)^2/2$	$q^2(q-1)/2$

Now by Table (II) and Definition 2.1 ,we have

min $\{d(\chi) : Ker\chi = 1\} = \dfrac{q(q-1)^2}{2}$ and

min $\{c(\chi)(1) : Ker\chi = 1\} = \dfrac{q^2(q-1)}{2}$.

Hence $r(G) = \dfrac{q(q-1)^2}{2}$, $c(G) = \dfrac{q^2(q-1)}{2}$, and the result follows.

References

[1] H. Behravesh , *Quasi-permutation representations of p-groups of class 2* , J.London Math. Soc. (2)55 , 251-26 (1997).

[2] J.M. Burns , B. Goldsmith , B. Hartley and R. Sandling , *on quasi-permutation representations of finite groups*, Glasgow Math, J. 36 , 301-308 (1994).

[3] M.R. Darafsheh , M. Ghorbany , A. Daneshkhah and H. Behravesh, *Quasi-permutation representation of the group* $GL(2,q)$, Journal of Algebra 243,142-167(2001).

[4] M.R. Darafsheh and M.Ghorbany, *Quasi-permutation representations of the groups* $SU(3,q^2)$ *and* $PSU(3,q^2)$, Southest Asian Bulletin of Mathemetics (2002)26 ,395-406.

[5] M.R. Darafsheh and M.Ghorbany, *Quasi-permutation representations of the groups* $SL(3,q)$ *and* $PSL(3,q)$, Iranian Journal of Science and Technology, Tra.A, Vol. 26, No. A1 (2002)145-154.

[6] H. Enomoto , *The characters of the finite symplectic group* $SP(4,q), q = 2^f$, Osaka J. Math. 9(1972), 75-94.

[7] M.R. Darafsheh and M.Ghorbany, *Special representations of the group* $G_2(2^n)$ *with minimal degrees*, Southest Asian Bulletin of Mathemetics (2006)30: 663-670.

[8] W.J. Wong , *Linear groups analogous to permutation groups* , J. Austral. Math. Soc (Sec. A) 3 , 180-184 ,(1963).