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On Lorentzian α-Sasakian manifolds

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Abstract

We study Ricci-semi symmetric, ϕ-Ricci semisymmetric and ϕ-symmetric Lorentzian α-Sasakian manifolds. Also, we study a Lorentzia α-Sasakian manifold satisfies $S(X,\xi).R = 0$.

keywords: Ricci semisymmetric Lorentzia α-Sasakian manifold, ϕ-Ricci symmetric Lorentzian α-Sasakian manifold, ϕ-symmetric Lorentzian α-Sasakian manifold.

1 Introduction

The notion of local symmetry of Rimannian manifolds have been weakened by many authors in several ways to the different extent. As a weaker version of local symmetry, Takahashi [6], introduced the notion of locally ϕ-symmetry on sasakian manifolds. In respect of contact Geometry, the notion of ϕ-symmetry was introduced and studied by Boeckx, Buecken and Vanhecke [2], with several examples. In [3], De studied the notion of ϕ-symmetry with several examples for Kenmotsu manifolds. In 1977, Adati and Matsumoto defined Para-sasakian manifold and special Para-Sasakian manifolds [4], which are special classes of an almost para contact manifold introduced by sato [5].

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2 Preliminaries

A differentiable manifold M of dimension n is called a Lorentzian α-Sasakian manifold if it admits a $(1,1)$ tensor filed ϕ, a contravariant vector field ξ, a covariant vector field η and Lorentzian metric g which satisfy [4, 7]

$$\phi^2 = I + \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$\phi\xi = 0, \ \eta(\phi X) = 0, \tag{2.4}$$

$$g(X,\xi) = \eta(X), \tag{2.5}$$

for all $X, Y \in TM$. From the above relations it follows that a Lorentzian α-Sasakian manifold satisfies

$$\nabla_X \xi = -\alpha\phi X \tag{2.6}$$

$$(\nabla_X \eta)Y = -\alpha g(X,Y), \tag{2.7}$$

$$(\nabla_X \phi)Y = \alpha g(X,Y)\xi - \alpha\eta(Y)X, \tag{2.8}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Also, a Lorentzian α-Sasakian manifold M is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{2.9}$$

for any vector fields X, Y where a, b are functions on M.
Further, on such an From the above relations it follows that a Lorentzian α-Sasakian manifold satisfiesthe following relations hold[7]

$$R(X,Y)\xi = \alpha^2(\eta(Y)X + \eta(X)Y), \tag{2.10}$$

$$R(\xi, X)Y = \alpha^2(g(X,Y)\xi + \eta(Y)X), \tag{2.11}$$

$$R(\xi, X)\xi = \alpha^2(X + \eta(X)\xi), \tag{2.12}$$

$$S(X,\xi) = (n-1)\alpha^2\eta(X), \tag{2.13}$$

$$Q\xi = (n-1)\alpha^2\xi, \tag{2.14}$$

$$S(\xi,\xi) = -(n-1)\alpha^2, \tag{2.15}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\alpha^2\eta(X)\eta(Y), \tag{2.16}$$

for any vector fields X, Y, Z, where $R(X,Y)Z$ is the curvature tensor, and S is the Ricci tensor.

Definition 2.1 *An n-dimensional Lorentzian α-Sasakian manifold is said to be an Einstein manifold if its Ricci tensor satisfies the condition*

$$S(X,Y) = \lambda g(X,Y), \tag{2.17}$$

where λ is a constant.

Definition 2.2 *A Lorentzian α-Sasakian manifold is said to be Ricci-semi symmetric if its Ricci tensor satisfies the condition*

$$R(X,Y).S=0, \tag{2.18}$$

for any vector fields X, Y.

3 Main Results

In this section, we prove the following theorems:

Theorem 3.1 *Let M be an n-dimensional Lorentzian α-Sasakian manifold. If M is Ricci semisymmetric then it is an η-Einstein manifold.*

Proof. Suppose that M is Ricci semisymmetric then in view of (2.18) we have

$$R(X,Y).S=0,$$

this implies that

$$S(R(X,Y)U,V)+S(U,R(X,Y)V)=0. \tag{3.1}$$

Putting $X=\xi$ in (3.1) we get

$$S(R(\xi,Y)U,V)+S(U,R(\xi,Y)V)=0. \tag{3.2}$$

Using (2.11) in (3.2) we get

$$S(\alpha^2(g(Y,U)\xi+\eta(U)Y),V)+S(U,\alpha^2(g(Y,V)\xi+\eta(V)Y))=0,$$

which implies

$$\begin{aligned}0=&\alpha^2 g(Y,U)S(\xi,V)+\alpha^2\eta(U)S(Y,V)\\&+\alpha^2 g(Y,V)S(U,\xi)+\alpha^2\eta(V)S(U,Y),\end{aligned} \tag{3.3}$$

Putting $U=\xi$ in (3.3) and using (2.2), (2.5) and (2.13) we obtain

$$S(Y,V)=-(n-1)\alpha^2 g(Y,V)+2(n-1)\alpha^2\eta(Y)\eta(V).$$

Therefore, in view of (2.9), M is an η-Einstein manifold. This completes the proof of the theorem.

Definition 3.2 *A Lorentzian α-Sasakian manifold M is said to be ϕ-Ricci symmetric if the Ricci operator satisfies*

$$\phi^2((\nabla_X Q)(Y))=0,$$

for all vector fields X and Y on M and $S(X,Y)=g(QX,Y)$ [4].

If X and Y are orthogonal to ξ, then manifold is said to be locally ϕ-Ricci symmetric.

Theorem 3.3 *An n-dimensional Lorentzian α-Sasakian manifold is ϕ-Ricci symmetric if and only if manifold is an Einstein manifold.*

Proof. Suppose that the manifold is ϕ-Ricci symmetric then in view of Definition 3.2 we have

$$\phi^2((\nabla_X Q)(Y))=0.$$

Using (2.1) in above equation we obtain

$$(\nabla_X Q)(Y)+\eta((\nabla_X Q)(Y))\xi=0. \tag{3.4}$$

Taking inner product of (3.4) with Z we get

$$g((\nabla_X Q)(Y),Z)+\eta((\nabla_X Q)(Y))\eta(Z)=0,$$

which implies

$$g(\nabla_X Q(Y) - Q(\nabla_X Y), Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0,$$

which on simplifying gives

$$g(\nabla_X Q(Y), Z) - S(\nabla_X Y, Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0. \tag{3.5}$$

Replacing Y by ξ in (3.5) we get

$$g(\nabla_X Q(\xi), Z) - S(\nabla_X \xi, Z) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0. \tag{3.6}$$

Using (2.4), (2.13) and (2.14) in (3.6) we obtain

$$-(n-1)\alpha^3 g(\phi X, Z) + \alpha S(\phi X, Z) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0. \tag{3.7}$$

Replacing Z by ϕZ in (3.7) we get

$$S(\phi X, \phi Z) = (n-1)\alpha^2 g(\phi X, \phi Z). \tag{3.8}$$

Using (2.3) and (2.16) in (3.8) we obtain

$$S(X, Z) = (n-1)\alpha^2 g(X, Z).$$

Therefore, the manifold is an Einstein manifold.

Next, suppose that the manifold is an Einstein manifold. Then in view of (2.17) we have $S(X,Y) = \lambda g(X,Y)$, wher $S(X,Y) = g(QX,Y)$ and λ is constant. Hence $QX = \lambda X$. Therefore, we obtain $\phi^2((\nabla_X Q)(Y)) = 0$. This completes the proof.

Theorem 3.4 *An n-dimensional $(n > 3)$, Lorentzian α-Sasakian manifold satisfying the condition $S(X,\xi).R = 0$ is an η-Einstein manifold.*

Proof. Since $S(X,\xi).R = 0$ we have

$$(S(X,\xi).R)(U,V)Z = 0,$$

which implies

$$\begin{aligned} 0 &= ((X \wedge_S \xi).R)(U,V)Z \\ &= (X \wedge_S \xi)R(U,V)Z + R((X \wedge_S \xi)U,V)Z \\ &+ R(U,(X \wedge_S \xi)V)Z + R(U,V)(X \wedge_S \xi)Z, \end{aligned} \tag{3.9}$$

where endomorphism $X \wedge_S Y$ is defined by

$$(X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y. \tag{3.10}$$

Using (3.10) in (3.9) we get by virtue of (2.13)

$$\begin{aligned} 0 = (n-1)\alpha^2[&\eta(R(U,V)Z)X + \eta(U)R(X,V)Z \\ &+ \eta(V)R(U,X)Z + \eta(Z)R(U,V)X] \\ &- S(X,R(U,V)Z)\xi - S(X,U)R(\xi,V)Z \\ &- S(X,V)R(U,\xi)Z - S(X,Z)R(U,V)\xi, \end{aligned}$$

taking the inner product with ξ we obtain

$$\begin{aligned} 0 = (n-1)\alpha^2[&\eta(R(U,V)Z)\eta(X) + \eta(U)\eta(R(X,V)Z) \\ &+ \eta(V)\eta(R(U,X)Z) + \eta(Z)\eta(R(U,V)X)] \\ &+ S(X,R(U,V)Z) - S(X,U)\eta(R(\xi,V)Z) \\ &- S(X,V)\eta(R(U,\xi)Z) - S(X,Z)\eta(R(U,V)\xi). \end{aligned}$$

Putting $U = Z = \xi$ in the above equation an using (2.10)-(2.13) we get

$$0 = (n-1)\alpha^2[-2\alpha^2\eta(V)\eta(X) + \alpha^2 g(V,X) - \alpha^2\eta(V)\eta(X)]$$

$$+(n-1)\alpha^4\eta(V)\eta(X)+\alpha^2 S(X,V),$$

with simplify of the last equation we have

$$S(X,V)=-(n-1)\alpha^2 g(X,V)+2(n-1)\alpha^2\eta(X)\eta(V).$$

Therefore, in view of (2.9) manifold is an η-Einstein manifold. The proof is complete.

Definition 3.5 *A Lorentzian α-Sasakian manifold M is said to be ϕ-symmetric if*

$$\phi^2((\nabla_W R)(X,Y)Z)=0,$$

for all vector fields X, Y, Z, W on M [6].

Theorem 3.6 *A ϕ-symmetric Lorentzian α-Sasakian manifold is an η-Einstein manifold.*

Proof. If manifold is ϕ-symmetric then in view of Definition 3.5 we have

$$\phi^2((\nabla_W R)(X,Y)Z)=0,$$

by virtue of (2.1) we get

$$(\nabla_W R)(X,Y)Z+\eta((\nabla_W R)(X,Y)Z)\xi=0,$$

taking inner product with U, we obtain

$$g((\nabla_W R)(X,Y)Z,U)+\eta((\nabla_W R)(X,Y)Z)g(\xi,U)=0. \quad (3.11)$$

Let $\{e_i\}$, $i=1,2,...,n$, be an orthonormal basis of tangent space at any point of the manifold. Then by putting $X=U=e_i$ in (3.11) and taking summation over i, $1\le i\le n$, we have

$$(\nabla_W S)(Y,Z)+\sum_{i=1}^{n}\eta((\nabla_W R)(e_i,Y)Z)g(\xi,e_i)=0.$$

Replacing $Z=\xi$ in the above equation, we obtain

$$(\nabla_W S)(Y,\xi)+\sum_{i=1}^{n}\eta((\nabla_W R)(e_i,Y)\xi)g(\xi,e_i)=0. \quad (3.12)$$

The second term of (3.12), takes the form

$$\eta((\nabla_W R)(e_i,Y)\xi)=g(\nabla_W R(e_i,Y)\xi,\xi)-g(R(\nabla_W e_i,Y)\xi,\xi)$$
$$-g(R(e_i,\nabla_W Y)\xi,\xi)-g(R(e_i,Y)\nabla_W\xi,\xi),$$

with simplify of the above equation we have

$$\eta((\nabla_W R)(e_i,Y)\xi)=0. \quad (3.13)$$

The equations (3.12) and (3.13) imply that

$$(\nabla_W S)(Y,\xi)=0,$$

which gives

$$\nabla_W(S(Y,\xi))-S(\nabla_W Y,\xi)-S(Y,\nabla_W\xi)=0,$$

in view of (2.6) and (2.6) we obtain

$$(n-1)\alpha^2\nabla_W\eta(Y)-(n-1)\alpha^2\eta(\nabla_W Y)+\alpha S(Y,\phi W)=0. \quad (3.14)$$

Replacing Y by ϕY in (3.14) we get

$$S(\phi Y,\phi W)=(n-1)\alpha g((\nabla_W\phi)Y,\xi). \quad (3.15)$$

Using (2.2), (2.8) and (2.16) in the above equation we have

$$S(Y,W) = -(n-1)\alpha^2 g(W,Y) - 2(n-1)\alpha^2 \eta(Y)\eta(W).$$

This implies that manifold is an η-Einstein.

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