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DYNAMICAL SYSTEMS ON FINSLER MODULES

M. HASSANI

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ABSTRACT. In this paper we investigate the generalized derivations and show that if E be a simple full Finsler A-module and let $\delta : D(\delta) \subseteq E \to E$ be a d-derivation.Then either δ is closable or both of the sets $\{x \pm \delta(x) : x \in E\}\}$ are dense in $E \bigoplus E$. We also describe dynamical systems on a full Finsler module E over C^*- algebra A as a one -parameter group.

1. INTRODUCTION

Hilbert C^*- modules are significant keys in theory of operator algebras, operator K-theory, theory of operator spaces so on (see [4]) Recall that a (left) Hilbert C^*- module over a C^*-algebra A is a left A-module E equipped with A-inner product $<,>$ which is a A-linear in the first and conjugate linear in the second variable such that E is Banach space with the norm $\|x\| = \| < x, x > \|^{\frac{1}{2}}$.

Finsler modules over C^*-algebras are generalization of Hilbert C^*-modules .Let A_+ be the positive cone of a C^*- algebra A. Suppose that E is complex linear space which is a left A-module (and $\lambda(ax) = (\lambda a)x = a(\lambda x)$ where $\lambda \in C, a \in A$ and $x \in E$)equipped with a map $\rho_A : E \to A_+$ such that (i) The map $\|.\|_E : x \to \|\rho_A(x)\|$ is a Banach space norm on E, and (ii) $\rho(ax)^2 = a\rho(x)^2a^*$ for all $a \in A$ and $x \in E$. Then E is called a Finsler A-module .

This definition is introduced in the works of N.C.Phillips and N.Weaver [6]. A Finsler A-module is said to be full if the linear span $\{\rho_A(x)^2 : x \in E\}$ denoted by $< \rho_A(E) >$ is dense in A. For example , if E is a (full) Hilbert C^*-module over A then E together with $\rho_A(x) =< x, x >$ is a (full)Finsler module.

In this paper ,we investigate the generalized derivations. This notion first appeared in the context of operator algebra [7].

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In the sequel, as main result, we describe dynamical system on a full A-Finsler module E as a one - parameter group of unitaries on E. The reader is referred to [6],[8] for more details on Finsler modules and to [9] for more information in C^*-dynamical systems.

2. Preliminaries.

Definition 2.1. Let E and F be Finsler modules over C^*-algebra A and B respectively and $\varphi : A \to B$ be a $*$- homomorphism of C^*-algebras. A linear operator $\psi : E \to F$ is said to be a φ-homomorphism of Finsler modules if the following conditions are satisfied:

$(i)\psi(ax) = \varphi(a)\psi(x)$

$(ii)\rho_B(\psi(x)) = \varphi(\rho_A(x))$

where $x \in E$ and $a \in A$. Recall that ψ is said to be module map if it satisfies in condition (i) .If E, F and G are Finsler modules over C^*-algebras A, B and C resp;$\varphi_1 : A \to B$, and $\varphi_2 : B \to C$ are $*$-homomorphism of C^*-algebras, and $\psi_1 : E \to F$ and $\psi_2 : F \to G$ are φ_1-homomorphism and φ_2-homomorphism of Finsler modules resp.,then it is straightforward to show that $\psi_2\psi_1 : E \to G$ is a $\varphi_2\varphi_1$-homomorphism of Finsler modules.

Definition 2.2. Let A and B be C^*- algebras , E and F be Finsler modules over C^* algebras A and B respectively. A linear operator $\psi : E \to F$ is said to be a unitary operator if there exists an injective homomorphism of C^* algebra $\varphi : A \to B$ such that ψ is a surjective φ homomorphism.The following useful theorem which can be found in [1].

Theorem 2.3. *Let A and B be C^*- algebras , E and F be Finsler modules over C^* algebras A and B respectively .If $\psi : E \to F$ is a unitary operator of Finsler modules, then ψ is isometry .Also if F is a full Finsler module over B , then φ is a $*$ - isomorphism of C^* -algebras .*

Remark 2.4. Fullness condition can not be dropped in above theorem .For example :

Example 2.5. Let $B = C[0, 1], A = E = \{f \in B, f(0) = 0\}$ and $F = \{f \in B, f(1) = 0\}$ .Then E is a full Finsler A-module with respect to the norm Finsler $\rho_A(f) = |f|$ and F is a Finsler B-module with respect to the norm Finsler $\rho_B(f) = |f|$ which is not full. Let $\psi : E \to F$ with $\psi(f)(t) = f(1 - t)$ for all $t \in [0, 1]$ and $\varphi : A \to B$ with $\varphi(f) = \psi(f)$.It is clear to show that ψ is a bijective bounded operator and $\psi(af) = \varphi(a)\psi(f)$ and $\rho_B(\psi(f)) = \varphi(\rho_A(f))$ for all $a \in A$ and $f \in E$. But φ is not $*$-isomorphism, since it is not surjective. We denote by $U(E)$ the group of all unitary operators of E onto E. We end this section whit the following lemma which can be founded in [2].

Lemma 2.6. *Let E be a full Finsler A-module and $a \in A$. Then $ax = 0$ for all $x \in E$ iff $a = 0$.*

3. GENERALIZED DERIVATION

Definition 3.1. Let E be full Finsler $\mathcal{A}$-module . A linear map $\delta : D(\delta) \subseteq E \to E$ where $D(\delta)$ is a dense subspace of E is called a generalized derivation if there exists a mapping $d : D(d) \to A$ where $D(d)$ is a dense subalgebra of A such that $D(\delta)$ is an algebraic left $D(d)$-module, and $\delta(ax) = a\delta(x) + d(a)x$ for all $x \in D(\delta)$ and all $a \in D(d)$.

In this case d must be derivation since for any $a, b \in D(d)$ and $x \in D(\delta)$ we have

$$\delta(abx) = ab\delta(x) + d(ab)x$$

on the other hand,

$\delta(abx) = \delta(a(bx)) = a\delta(bx) + d(a)bx = ab\delta(x) + ad(b)x + d(a)bx$

whence

$$(d(ab) - (ad(b) + d(a)b))x = 0$$

for all $x \in D(\delta)$. Thus by lemma [2.6] we obtain $d(ab) = ad(b) + d(a)b$ since $D(\delta)$ is dense in E.

Similarly we can show that d is linear so $d : D(d) \subseteq A \to A$ is a derivation. We call δ a d-derivation

Theorem 3.2. *Let E be a simple full Finsler A-module in the sense that is has no trivial left A-module and let $\delta : D(\delta) \subseteq E \to E$ be a d-derivation.Then either δ is closable or both of the sets $\{x \pm \delta(x) : x \in E\}\}$ are dense in $E \bigoplus E$*

Proof. let $S(\delta)$ be the separating space of δ that is

$$S(\delta) = \{x \in E, \exists x_n \subseteq D(\delta), x_n \to 0, \delta(x_n) \to x\}.$$

Then $S(\delta)$ is a closed subspace of E. Let $a \in A, \in S(\delta)$. Thus there exists a sequence $\{x_n\} \subseteq D(\delta)$ such that $x_n \to 0$ and $\delta(x_n) \to x$, so we have $ax_n \to 0$ and $\delta(ax_n) = a\delta(x_n) + d(a)x_n \to ax$

Hence $ax \in S(\delta)$.Thus $S(\delta)$ is a left submodule of E By the hypothesis $S(\delta) = \{0\}$ or $S(\delta) = E$ .If $S(\delta) = \{0\}$ then δ is closable. If $S(\delta) = E$ then rang of δ is dense. Hence both of the sets $\{x \pm \delta(x) : x \in E\}$ are dense in $E \bigoplus E$. □

4. DYNAMICAL SYSTEMS

Definition 4.1. Let E be a full Finsler A-module. A map α from the real line $\mathbb{R}$ to $U(E)$ which maps t to α_t is said to be a one - parameter group of unitaries if

$(i)\alpha_0 = I$

$(ii)\alpha_{t+s} = \alpha_t\alpha_s(t, s \in \mathbb{R})$

α is said to be a strongly continuous one-parameter group of unitaries if , in addition ,$\alpha_t(x) \to x$ where $t \to 0$ in the norm of E for all $x \in E$. In this case we call α a dynamical system on E. We can define the infinitesimal generator of a dynamical system as follows:

Definition 4.2. Let $\alpha : \mathbb{R} \to U(E)$ be a dynamical system on E , we define the infinitesimal generator δ of α as mapping $\delta : D(\delta) \subseteq E \to E$,where

$$D(\delta) = \{x \in E, \lim_{t\to 0} \frac{\alpha_t x - x}{t} exists\}$$

and

$$\delta(x) = \lim_{t\to 0} \frac{\alpha_t x - x}{t}, x \in D(\delta)$$

Now we are ready to prove the main theorem of this paper

Theorem 4.3. *Let E be Finsler A-module , α be dynamical system on E and δ be the infinitesimal generator of α . Then $D(\delta)$ is a dense subspace of E there exists a derivation $d : D(d) \subseteq A \to A$ such that $D(\delta)$ is a left $D(d)$-module and $\delta(ax) = a\delta(x) + d(a)x$ for all $x \in D(\delta)$ and all $a \in D(d)$.*

Proof. By Hille-Yosida theorem [2] $D(\delta)$ is a dense subspace of E, since α is a dynamical system on E ,for each $t \in \mathbb{R}$, the mapping $\alpha_t : E \to E$ is a unitary. So there exists

$*$-isomorphism $\acute{\alpha}_t : A \to A$ such that $\rho_A(\alpha_t(x)) = \acute{\alpha}_t(\rho_A(x))$ and $\alpha_t(ax) = \acute{\alpha}_t(a)\alpha_t(x)$ $(a \in A, x \in E)$.Now we show that $\acute{\alpha} : \mathbb{R} \to Aut(A)$ is a C^* -dynamical system. For each $a \in A, x \in E$ we have $ax = \alpha_0(ax) = \acute{\alpha}_0(a)\alpha_0(x) = \acute{\alpha}_0(a)x$, thus by lemma 2.6 $\acute{\alpha}_0(a) = a$ for all $a \in A$ .Therefor $\acute{\alpha}_0 = I$. Also for all $t, s \in \mathbb{R}$ we have

$$\acute{\alpha}_{t+s}(a)\alpha_{t+s}(x) = \alpha_{t+s}(ax)$$

$$= \alpha_t(\alpha_s(ax))$$

$$= \alpha_t(\acute{\alpha}_s(a)\alpha_s(x))$$

$$= \acute{\alpha}_t(\acute{\alpha}_s(a))\alpha_{t+s)}(x)$$

and so $\acute{\alpha}_{t+s}(a) = \acute{\alpha}_t\acute{\alpha}_s(a)$. Thus $\acute{\alpha}_{t+s} = \acute{\alpha}_t\acute{\alpha}_s$. Since for each $x \in E$

$$\lim_{t \to 0} \|\alpha_t(x) - x\|_E = \lim_{t \to 0} \|\rho_A(\alpha_t(x) - x)\| = 0$$

we have

$$\|\acute{\alpha}_t(a)x - ax\|_E$$

$$= \|\rho_A(\acute{\alpha}_t(a)x - ax)\|$$

$$= \|\rho_A(\acute{\alpha}_t(a)x - \acute{\alpha}_t(a)\alpha_t(x) + \acute{\alpha}_t(a)\alpha_t(x) - ax)\|$$

$$\leq \|\rho_A(\acute{\alpha}_t(a)x - \acute{\alpha}_t(a)\alpha_t(x))\| + \|\rho_A(\acute{\alpha}_t(a)\alpha_t(x) - ax)\|$$

Thus $\lim_{t\to 0} \acute{\alpha}_t(a)x = ax$ for all $x \in E$, whence $\lim_{t\to 0} \acute{\alpha}_t(a) = a$ for all $a \in A$. Therefor $\acute{\alpha} : \mathbb{R} \to Aut(A)$ is a C^*-dynamical system on A. If d is the infinitesimal generator of $\acute{\alpha}$ then for each $a \in D(d), x \in D(\delta)$ we have

$$\lim_{t\to 0}\ \frac{\alpha_t(ax)-ax}{t}$$

$$=\lim_{t\to 0}\frac{a\alpha_t(x)-ax}{t}+\lim_{t\to 0}\frac{\acute{\alpha}_t(a)\alpha_t(x)-a\alpha_t(x)}{t}$$

$$=a\lim_{t\to 0}\frac{\alpha_t(x)-x}{t}+\lim_{t\to 0}\frac{\acute{\alpha}_t(a)-a}{t}\alpha_t(x)$$

$$=a\delta(x)+d(a)x.$$

Hence $ax \in D(\delta)$ and $\delta(ax) = a\delta(x) + d(a)x$. Furthermore, $D(\delta)$ is a left $D(d)$- module.

□

References

[1] M.Amyari and A.Niknam, *On homomorphisms of Finsler modules*,Intern.Math. Journal, Vol.3, (2003)

[2] M.Amyari and A.Niknam, *A not on Finsler modules*,.Bull.Iran Math Soc, Vol.29,(2003),77-81

[3] E. Hille and R. Phillips , *Functional Anal. and Semi-group*,,Proc. of Symp. in Pure Math, vol 31 Amer. Math .Soc. Rhode Island 1957

[4] E.C. Lance, *Hilbert C^*-module,LMS Lecture Note Series 210*,Cambridge University Press,1995.

[5] W.L, *Inner product module over B^*-algebras*, Trans Amer Math. 1973.443-468.

[6] N.Phillips and N.Weaver, *modules with norms which take values in a C^*-algebra* , J.of Maths vol. 185 ,(1998),163-181.

[7] M. Mathieu, *Elementary operators and Applications*,Proceeding of the International work shop, singapore ,1992

[8] Taghvi and M.jafarzadeh, *A note on Modules maps over Finsler Modules*,Advances in Applied Math. AnaL.Vol2,2007,89-95.

[9] S.sakai, *Operator Algebra in Dynamical Systems* , Cambridge Univ. Preess Cambridge, 1991

Department of Mathematics, Faculty of sciences, Mashhad Branch,Islamic Azad University, Mashhad, Iran

E-mail address: hassani@mshdiau.ac.ir