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Fourth order Volterra integro-differential equations using modified homotopy-perturbation method

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Abstract. This paper compare modified homotopy perturbation method with the exact solution for solving Fourth order Volterra integro-differential equations. From the computational viewpoint, the modified homotopy perturbation method is more efficient and easy to use.

Keywords. Fourth order integro-differential equations; modification of homotopy-perturbation method (MHPM); Nonlinear; exact solution; boundary value problems(BVP).
AMS (MOS) subject classification: 35B30, 35B40

1 Introduction

In recent years, the application of the homotopy perturbation method (HPM) [1-8] in nonlinear problems has been developed by scientists and engineers, because this method deforms the difficult problem under study into a simple problem which is easy to solve. Most perturbation methods assume a small parameter exists, but most nonlinear problems have no small parameter at all. Many new methods, such as variational method [9,10], variational iterations method [11-13], and others [14,15], are proposed to eliminate the shortcomings arising in the small parameter assumption. A review of recently developed nonlinear analysis methods can be found in [16]. Recently, the applications of homotopy perturbation theory have appeared in the works of many scientists [17-20], which has become a powerful mathematical tool [21,22]. Recently, S. Abbasbandy [18] applied this method to functional integral equations. In this paper, we propose MHPM to solve Fourth order Volterra integro-differential equations and comparisons are made between the exact solutions and the modified homotopy perturbation method.

2 Homotopy-perturbation method

In this letter, we apply the Homotopy-perturbation method to the discussed problems. To illustrate the basic ideas of the new method, we consider the following nonlinear differential equation,

$$A(u) - f(r) = 0, \tag{1}$$

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with the boundary condition of:

$$B(u, \frac{\partial u}{\partial n}) = 0, \tag{2}$$

where $A(u)$ is defined as follows:

$$A(u) = L(u) + N(u). \tag{3}$$

Homotopy-perturbation structure is shown as:

$$H(v,p) = L(v) - L(u_0) + p\ L(u_0) + p\ [N(v) - \ f(r)] = 0, \tag{4}$$

or

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{5}$$

where,

$$v(r,p) : \Omega \times [0,1] \longrightarrow \mathbb{R}. \tag{6}$$

Obviously, considering Eqs. (4) and (5) we have:

$$H(v,0) = L(v) - L(u_0) = 0, \qquad H(v,1) = A(v) - f(r) = 0, \tag{7}$$

where $p \in [0,1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. The process of the changes in p from zero to unity is that of $v(r,p)$ changing from u_0 to u_r. We consider v as:

$$v = v_0 + p\ v_1 + p^2 v_2 + \ \ldots, \tag{8}$$

and the best approximation is:

$$u = \lim_{p \longrightarrow 1} v = v_0 + \ v_1 + \ v_2 + \ \ldots. \tag{9}$$

the above convergence is discussed in [30,31].

3 Description of modified homotopy perturbation method

This section is devoted to reviewing MHPM for solving fourth order Volterra integro-differential equation:

$$f^{(4)}(x) = g(x) + \int_0^x k(t, f(t), f'(t), ..., f^{(4)}(x))\ dt. \tag{10}$$

To explain HPM, we consider the above integro-differential equation as

$$L(u) = u^{(4)}(x) - g(x) - \int_0^x k(t, f(t), f'(t), ..., f^{(4)}(x))\ dt = 0, \tag{11}$$

with solution $f(x)$. As a possible remedy, we can define homotopy $H(u,p)$ by

$$H(u,0) = F(u), \qquad H(u,1) = L(u),$$

where $F(u)$ is a functional operator with known solution v_0, which can be obtained easily. In MHPM, we difine

$$v_0(x) = a + b\ x + c\ x^2 + d\ x^3,$$

which is dependent on the order of differentiation. Typically, we may choose a convex homotopy by

$$H(u,p) = (1-p)\ F(u) + p\ L(u) = 0, \tag{12}$$

and continuously trace an implicitly defined curve from a starting point $H(v_0,0)$ to a solution function $H(f,1)$.

The embedding parameter p monotonously increases from zero to unit as trivial problem $F(u) = 0$ is continuously deformed to the original problem $L(u) = 0$. The embedding parameter $p \in (0,1]$ can be considered as an expanding parameter [5].

The HPM uses the homotopy parameter p as an expanding parameter to obtain [23]

$$u = v_0 + p\ v_1 + p^2 v_2 + + \ \ldots. \tag{13}$$

When $p \to 1$, (13) corresponds to (12) and becomes the approximate solution of (11), i.e.,

$$f = \lim_{p \longrightarrow 1} u = v_0 + \ v_1 + \ v_2 + \ \ \ldots. \tag{14}$$

Series (14) is convergent for most cases, and the rate of convergence depends on $L(u)$ [5].

4 Numerical example

In this section, three examples are presented. The examples are linear and nonlinear fourth order volterra integro-differential equations that using MHPM and the results are compared with the exact solutions.

Example 4.1. Consider the following linear fourth order integro-differential equation with the exact solution as $u(x) = 1 + x\ e^x$.

$$u^{(4)}(x) = x\ (1 + e^x) + 3\ e^x + u(x) - \int_0^x \ u(t)\ dt, \tag{15}$$

with the boundary conditions:

$$u(0) = 1, \quad u(1) = 1 + e, \quad u''(0) = 2, \quad u''(1) = 3e. \tag{16}$$

let $F(u) = u^{(4)}(x) - g(x) = 0$,In order to solve Eq.(15) using MHPM, we construct the following homotopy we need a initially equation

$$v_0(x) = a + b\ x + c\ x^2 + d\ x^3,$$

Hence, we may choose a convex homotopy such that

$$H(u,p) = (1-p)\ (u^{(4)}(x) - x\ (1+e^x) - 3\ e^x - u(x))$$
$$+p\ (u^{(4)}(x) - x\ (1+e^x) - 3\ e^x - u(x) + \int_0^x u(t)\ dt) = 0. \tag{17}$$

Substituting (13) into (17), and equating the terms with identical powers of p, we have

$$p^0 :\ v_0^{(4)}(x) = x\ e^x + x + v_0(x) + 3\ e^x, \tag{18}$$
$$p^1 :\ v_1^{(4)}(x) = v_1(x) - \int_0^x v_0(t)\ dt), \tag{19}$$
$$p^2 :\ v_2^{(4)}(x) = v_2(x) - \int_0^x v_1(t)\ dt), \tag{20}$$
$$\vdots$$

The solutions of Eqs. (18)-(20) may be written as follows:

$$v_0(x) = a + b\ x + c\ x^2 + d\ x^3, \tag{21}$$
$$v_1(x) = 6d + a\ x + \frac{1}{2}\ b\ x^2 + \frac{1}{3}\ c\ x^3 + \frac{1}{4}\ d\ x^4, \tag{22}$$
$$v_2(x) = 2\ c + 12\ d\ x + \frac{1}{2}\ a\ x^2 + \frac{1}{6}\ b\ x^3 + \frac{1}{12}\ c\ x^4 + \frac{1}{20}\ d\ x^5, \tag{23}$$

The solution of linear fourth order volterra intogro-differential equation, when$p \longrightarrow 1$, will be follows:

$$f(x) = v_0(x) + v_1(x) + v_2(x). \tag{24}$$

Incorporating the boundary conditions, Eq. (15), into $f(x)$, we have:

$$f(0) = a + b + 2\ c = 2, \tag{25}$$
$$f(0) = a + 6\ d + 2\ c = 1, \tag{26}$$
$$f(1) = 5\ c + 10\ d + 2\ b + a = 3e, \tag{27}$$
$$f(1) = \frac{5}{2}\ a + \frac{5}{3}\ b + \frac{41}{12}\ c + \frac{193}{10}\ d = 1 + e, \tag{28}$$

Solving Eqs. (24)-(27) simultaneously, we obtain:

$b = 1.599415315,\ a = -1.970351743,\ d = 0.999025525e^{-1},\ c = 1.185468215.$

Therefore, the approximate solution of Example 4.1 can be readily obtained by

$$u(x) = f(x) = 1.000000002 + 0.827894202x + 1.000000001x^2$$

$$+ 0.7616278434x^3 + 0.1237646560x^4 + 0.004995127625x^5. \quad (29)$$

In practice, all terms of series $f(x) = \sum_{n=0}^{\infty} v_n(x)$ cannot be determined and so we use an approximation of the solution by the following truncated series:

$$\varphi_m(x) = \sum_{n=0}^{m-1} v_m(x) \quad with \quad f(x) = \lim_{m\to\infty} \varphi_m(x). \quad (30)$$

The results of which are shown in Table 1 (with three terms). fig1. shows the numerical result of exact solution and MHPM solution, it is clear that the results are in excellent agreement.

Table 1

Numerical results of Example 4.1

x	$Exactsolution$	$MHPM$	$error$
0	1	1.000000002	0.2E-8
0.1	1.110517092	1.093563476	0.16953616E-1
0.2	1.244280552	1.211871486	0.32409066E-1
0.3	1.404957642	1.359946847	0.45010795E-1
0.4	1.596729879	1.543121390	0.53608489E-1
0.5	1.824360636	1.767041972	0.57318664E-1
0.6	2.093271280	2.037676457	0.55594823E-1
0.7	2.409626895	2.361319719	0.48307176E-1
0.8	2.780432742	2.744599627	0.35833115E-1
0.9	3.213642800	3.194483047	0.19159753E-1

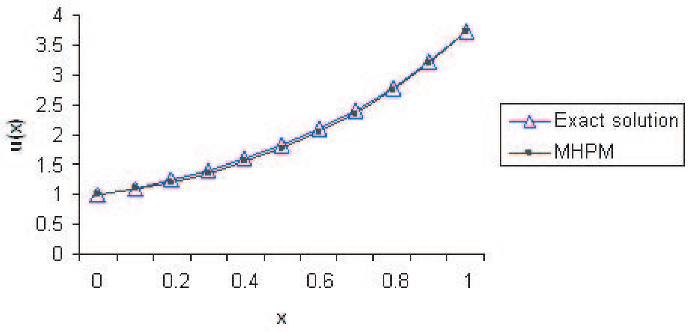

Figure 1: The numerical result of $u(x)$ of Eq. (15) with MHPM which is equal to the exact solution.

Example 4.2. Consider the following nonlinear integro-differential equation with the exact solution as $u(x) = e^x$.:

$$u^{(4)}(x) = e^x - \frac{1}{2}\, e^{2x} + \frac{1}{2} - \int_0^x u(t)\, u''(t)\, dt, \tag{31}$$

with the boundary conditions:

$$u(0) = 1, \quad u(1) = e, \quad u'(0) = 1, \quad u''(0) = 1. \tag{32}$$

let $F(u) = u^{(4)}(x) - g(x) = 0$, In order to solve Eq.(31) using MHPM, we construct the following homotopy we need a initially equation

$$v_0(x) = a + b\, x + c\, x^2 + d\, x^3,$$

Hence, we may choose a convex homotopy such that

$$H(u,p) = (1-p)\, (u^{(4)}(x) - e^x + \tfrac{1}{2}\, e^{2x} - \tfrac{1}{2})$$

$$+p\, (u^{(4)}(x) - e^x + \frac{1}{2}\, e^{2x} - \frac{1}{2} + \int_0^x u(t)\, u''(t)\, dt) = 0. \tag{33}$$

Substituting (13) into (33), and equating the terms with identical powers of p, we have

$$p^0 : \; v_0^{(4)}(x) = \frac{1}{2} + e^x - \frac{1}{2}\, e^{2x}, \tag{34}$$

$$p^1 : \; v_1^{(4)}(x) = -\int_0^x v_0(t)\, v_0''(t)\, dt, \tag{35}$$

$$p^2: \ v_2^{(4)}(x) = -\int_0^x \ v_1(t) \ v_1''(t) \ dt, \tag{36}$$

$$\vdots$$

The solutions of Eqs. (34)-(36) may be written as follows:

$$v_0(x) = a + b \ x + c \ x^2 + d \ x^3, \tag{37}$$

$$\begin{aligned} v_1(x) = & \frac{-1}{60} \ a \ c \ x^5 - \frac{1}{360} \ x^6 \ b \ c - \frac{1}{120} \ x^6 \ a \ d - \frac{1}{420} \ x^7 \ b \ d \\ & -\frac{1}{1260} \ x^7 \ c^2 - \frac{1}{840} \ c \ d \ x^8 - \frac{1}{2520} \ d^2 \ x^9, \end{aligned} \tag{38}$$

$$\begin{aligned} v_2(x) = & \frac{-1}{27799200} \ a^2c^2x^{13} - \frac{1}{27799200} \ a^2c^2x^{13} - \frac{1}{1854391226688000} \\ & (53603550 \ a^2c \ d + 17867850 \ b \ a \ c^2) \ x^{14} - \frac{1}{2528715309120000} \ (1624350 \\ & b^2 \ c^2 + 27010620 \ b \ a \ c \ d + 5754840 \ a \ c^3) \ x^{15} - \frac{1}{3371620412160000} \ (1021020 \\ & b \ c^3 + 14619150 \ a^2d^2 + 3063060 \ b^2 \ c \ d + 12762750 \ a \ c^2d + 9189180 \ b \ a \ d^2) \ x^{16} \\ & -\frac{1}{4409042077440000} \ (1413720 \ b^2d^2 + 157080 \ c^4 + 2631090 \ b \ c^2 \ d \\ & -\frac{1}{5668768385280000} \ (2151435 \ b \ c \ d^2 + 510510 \ c^3d + 1859715 \ a \ d^3) \ x^{18} \\ & +8678670 \ a \ c \ d^2) \ x^{17} - \frac{1}{7180439954688000} \ (554268 \ b \ d^3 + 593164 \ c^2d^2) \ x^{19} \\ & -\frac{1}{30767688000} \ cd^3 \ x^{20} - \frac{1}{215373816000} \ d^4 \ x^{21}. \end{aligned} \tag{39}$$

The solution of nonlinear fourth order volterra intogro-differential equation, when$p \longrightarrow 1$, will be follows:

$$f(x) = v_0(x) + v_1(x) + v_2(x). \tag{40}$$

Incorporating the boundary conditions, Eq. (31), into $f(x)$, and solving equations simultaneously, we obtain:

$$b = 1, \ c = 0.5, \ a = 1, \ d = 0.2308342493.$$

Therefore, the approximate solution of Example 4.2 can be readily obtained by

$$u(x) = f(x) = 1 + x + 0.5\ x^2 + 0.2308342493\ x^3 - 0.008333333334\ x^5$$

$$-0.003312507633\ x^6 - 0.0007480180539\ x^7 - 0.0001374013388\ x^8$$

$$-0.00002114462327\ x^9 - 0.8993064548 \times 10^{-8}\ x^{13}$$

$$-0.574513617 \times 10^{-8}\ x^{14} - 0.1985950713 \times 10^{-8}\ x^{15}$$

$$-0.5063789010 \times 10^{-9}\ x^{16} - 0.1061913876 \times 10^{-9}\ x^{17}$$

$$-0.1674502694 \times 10^{-10}\ x^{18} - 0.2049878884 \times 10^{-11}\ x^{19}$$

$$-0.1998830099 \times 10^{-12}\ x^{20} - 0.1318281272 \times 10^{-13}\ x^{21}. \tag{41}$$

In practice, all terms of series $f(x) = \sum_{n=0}^{\infty} v_n(x)$ cannot be determined and so we use an approximation of the solution by the following truncated series:

$$\varphi_m(x) = \sum_{n=0}^{m-1} v_m(x) \quad with \quad f(x) = \lim_{m\to\infty} \varphi_m(x). \tag{42}$$

The results of which are shown in Table 2 (with three terms). fig2. shows the numerical result of exact solution and MHPM solution, it is clear that the results are in excellent agreement.

Table 2

Numerical results of Example 4.2

x	$Exactsolution$	$MHPM$	$error$
0	1	1	0
0.1	1.105170918	1.105230748	0.59830E-4
0.2	1.221402758	1.221843785	0.441027E-3
0.3	1.349858808	1.351209687	0.1350879E-2
0.4	1.491824698	1.494673169	0.2848471E-2
0.5	1.648721271	1.653535684	0.4814413E-2
0.6	1.822118800	1.829034189	0.6915389E-2
0.7	2.013752707	2.022315475	0.8562768E-2
0.8	2.225540928	2.234405354	0.8864426E-2
0.9	2.459603111	2.466171899	0.6568788E-2
1	2.718281828	2.718281826	0.2E-8

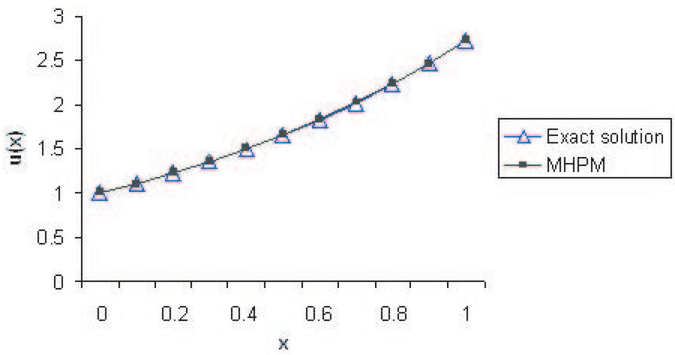

Figure 2: The numerical result of $u(x)$ of Eq. (31) with MHPM which is equal to the exact solution.

Example 4.3. Consider the following nonlinear integro-differential equation with the exact solution as $u(x) = e^{-x}$.

$$u^{(4)}(x) = e^{-x} + e^{-3x} - 1 + 3 \int_0^x u^3(t)\ dt, \tag{43}$$

with the boundary conditions:

$$u(0) = 1, \quad u'(0) = -1, \quad u''(0) = 1, \quad u'''(0) = 1. \tag{44}$$

let $F(u) = u^{(4)}(x) - g(x) = 0$, In order to solve Eq.(43) using MHPM, we construct the following homotopy we need a initially equation

$$v_0(x) = a + b\ x + c\ x^2 + d\ x^3,$$

Hence, we may choose a convex homotopy such that

$$H(u,p) = (1-p)\ (u^{(4)}(x) - e^{-x} - e^{-3x} + 1)$$
$$+p\ (u^{(4)}(x) - e^{-x} - e^{-3x} + 1 - 3 \int_0^x u^3(t)\ dt) = 0. \tag{45}$$

Substituting (13) into (45), and equating the terms with identical powers of p, we have

$$p^0 :\ v_0^{(4)}(x) = e^{-x} + e^{-3x} - 1, \tag{46}$$
$$p^1 :\ v_1^{(4)}(x) = 3 \int_0^x v_0^3(t)\ dt, \tag{47}$$
$$p^2 :\ v_2^{(4)}(x) = 3 \int_0^x v_1^3(t)\ dt, \tag{48}$$

$$\vdots$$

The solutions of Eqs. (46)-(48) may be written as follows:

$$v_0(x) = a + b\ x + c\ x^2 + d\ x^3, \tag{49}$$

$$v_1(x) = \frac{1}{40}\ a^3x^5 + \frac{1}{80}a^2b\ x^6 + \frac{1}{235200}(840b^2a + 840ca^2)x^7$$
$$+\frac{1}{470400}(1260cab+210b^3+630da^2)x^8+\frac{1}{846720}(504ac^2+504b^2c+1008dab)x^9$$
$$+\frac{1}{1411200}(420bc^2+840acd+420b^2d)x^{10}+\frac{1}{2217600}(720bcd+360ad^2+120c^3)x^{11}$$
$$+\frac{1}{3326400}(315c^2d + 315\ bd^2)x^{12} + \frac{1}{17160}cd^2x^{13} + \frac{1}{80080}d^3x^{14}, \tag{50}$$

$$v_2(x) = \frac{1}{134400}a^4x^{10} + \frac{3}{492800}a^3bx^{11} + \frac{1}{2131024896000}$$
$$(5765760ca^3 + 3243240b^2a^2)x^{12} + \frac{1}{3078147072000}(3363360ca^2b + 720720ab^3$$
$$+4724720da^3)x^{13} + \frac{1}{4309405900800}(2426424da^2b + 72072b^4 + 1105104cb^2a$$
$$+672672c^2a^2)x^{14} + \frac{1}{5876462592000}(524160bac^2 + 152880b^3c + 742560db^2a$$
$$+808080dca^2)x^{15} + \frac{1}{7835283456000}(202020d^2a^2 + 87360ac^3 + 644280dcab$$
$$+120120b^2c^2 + 100100db^3)x^{16} + \frac{1}{10246139904000}(151200db^2c + 43680bc^3$$
$$+159600dac^2 + 179760d^2ab)x^{17} + \frac{1}{13173608448000}(6240c^4 + 45240b^2d^2$$
$$+82680dbc^2 + 94080acd^2)x^{18} + \frac{1}{16686570700800}(51408bcd^2 + 16016c^3d$$
$$+18816ad^3)x^{19} + +\frac{1}{20858213376000}(10815bd^3 + 15435c^2d^2)x^{20}$$
$$+\frac{1}{3834230400}cd^3x^{21} + \frac{1}{28117689600}d^4x^{22}. \tag{51}$$

The solution of nonlinear fourth order volterra intogro-differential equation, when$p \longrightarrow 1$, will be follows:

$$f(x) = v_0(x) + v_1(x) + v_2(x). \tag{52}$$

Incorporating the boundary conditions, Eq. (43), into $f(x)$, and solving equations simultaneously, we obtain:

$$b = -1,\ c = 0.5,\ d = 0.1666666667,\ a = 1.$$

Therefore, the approximate solution of Example 4.3 can be readily obtained by

$$\begin{gathered} u(x) = f(x) = 1 - x + 0.5x^2 + 0.1666666667x^3 + 0.025x^5 - 0.0125x^6 \\ +0.005357142857x^7 - 0.0015625x^8 + 0.2480158730 \times 10^{-3}x^9 \\ +0.3224206351 \times 10^{-4}x^{10} - 0.2187049063 \times 10^{-4}x^{11} \\ +0.4189965126 \times 10^{-5}x^{12} + 0.2847268481 \times 10^{-6}x^{13} \\ +0.1479382060 \times 10^{-6}x^{14} - 0.2787391178 \times 10^{-8}x^{15} - 0.1757234772x^{19} \\ -0.3039030663 \times 10^{-8}x^{16} + 0.8585347015 \times 10^{-9}x^{17} - 0.3732209500 \times 10^{-10}x^{18} \\ +0.2738396362 \times 10^{-11}x^{20} + 0.6037234528 \times 10^{-12}x^{21} + 0.2744197514 \times 10^{-13}x^{22}. \end{gathered} \tag{53}$$

In practice, all terms of series $f(x) = \sum_{n=0}^{\infty} v_n(x)$ cannot be determined and so we use an approximation of the solution by the following truncated series:

$$\varphi_m(x) = \sum_{n=0}^{m-1} v_m(x) \quad with \quad f(x) = \lim_{m \to \infty} \varphi_m(x). \tag{54}$$

The results of which are shown in Table 3 (with three terms). fig3. shows the numerical result of exact solution and MHPM solution, it is clear that the results are in excellent agreement.

Table 3
Numerical results of Example 4.3

x	$Exactsolution$	$MHPM$	$error$
0	1	1	0
0.04	0.9607894392	0.9608106692	0.212300E-4
0.08	0.9231163464	0.9232854120	0.1690656E-3
0.12	0.8869204367	0.8874885866	0.5681499E-3
0.16	0.8521437890	0.8534850921	0.13413031E-2
0.2	0.8187307531	0.8213405980	0.26098449E-2
0.24	0.7866278611	0.7911217470	0.44938859E-2
0.28	0.7557837415	0.7628963355	0.71125940E-2
0.32	0.7261490371	0.7367334755	0.105844384E-1
0.36	0.6976763261	0.7127037390	0.150274129E-1

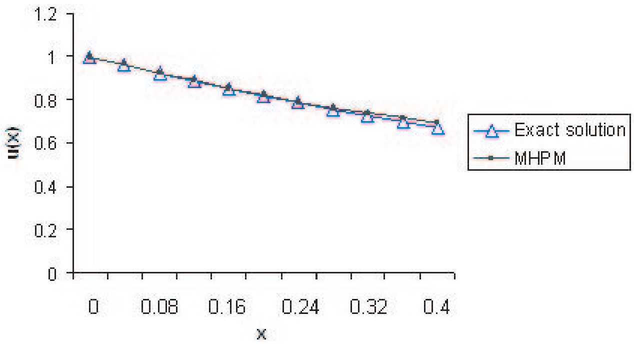

Figure 3: The numerical result of $u(x)$ of Eq. (43) with MHPM which is equal to the exact solution.

5 Conclusion

In this work, we proposed the MHPM for solving the linear and nonlinear integro-differential equations and compared our results with the exact solution. The figures clearly show that result by MHPM are in excellent agreement with the exact solutions. MHPM provides highly accurate numerical solutions in comparison with other methods and this is powerful mathematical tool can solve a large class of nonlinear differential systems,especially nonlinear integral systems and equations used in engineering and physics.

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