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A remark on the coupled fixed point theorems for mixed monotone operators in partially ordered metric spaces

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Abstract

We present a coupled fixed point theorems for mixed monotone operators in partially ordered metric spaces.

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1 Introduction

Mixed monotone operators were introduced by D.Guo and V.Lakshmikantham in 1987[3]. Existence of fixed point in partially ordered set has been considered recently in [1-19]. The

purpose of this paper is to present some coupled fixed point theorems for mixed monotone operator in the context of ordered metric spaces involving following functions with the method of [2].
Let δ denote the class of those functions $\beta : [0, \infty) \to [0, \infty)$which satisfies the condition $\beta(t_n) \longrightarrow 1$ implies $t_n \to 0$.

Now we briefly recall various basic definition and facts.

Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. we say that F has the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is , for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping F if

$$F(x, y) = x \quad and \quad F(y, x) = y.$$

Theorem 1. ([1]) Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space.Let $f : X \to X$ be an increasing mapping such that there exists an element $x_0 \in X$ with $x_0 \leq f(x_0)$. suppose that there exists $\beta \in \delta$ such that

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y), \text{ for each } x, y \in X, \text{ with } x \geq y.$$

Assume that either f is continuous or X is such that

$$\text{if an increasing sequence } \{x_n\} \to x \text{ in } X, \text{ then } \ x_n \leq x, \forall n.$$

Besides, if

$$\text{for each } x, y \in X \text{ there exists } z \in M \ \text{ which is comparable to } x \text{ and } y.$$

Then f has a unique fixed point.
Bhaskar and Lakshmikantham [3] introduced the notions of mixed monotone mapping and a coupled fixed point.

Theorem 2.([3]) Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space.Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X and assume that there exists $k \in [0, 1)$ with

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u), d(y, v)], \text{ for any } x \geq u \text{ and } y \leq v.$$

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \quad \text{and} \quad y_0 \geq F(y_0, x_0)$$

and suppose either F us continuous or X satisfies the following property:

$$\text{if } (x_n) \text{ is a nondecreasing sequence with } x_n \to x \text{ then } x_n \leq x \text{ for each } n \in N$$

and

$$\text{if } (y_n) \text{ is a nonincreasing sequence with } y_n \to y \text{ then } y \leq y_n \text{ for each } n \in N$$

then F has a coupled fixed point.

2 Main result

let $(X, \leq)$ be a partially ordered set and d a metric on X such that (X, d) is a complete metric space.Further, we consider in the product space $X \times X$ the following partial order:

$$\text{if } (x, y), (u, v) \in X \times X, (x, y) \leq (u, v) \Longleftrightarrow x \leq u \quad \text{and} \quad y \geq v.$$

Now we present the following theorem which is a version of Theorem2.1 of [1] in the context of mapping with the mixed monotone property.

Theorem 3. Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space.Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X and continuous such that

$$d(F(x, y), F(u, v)) \leq \beta(max(d(x, u), d(y, v)))max(d(x, u), d(y, v)),$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$, where $\beta \in \delta$.
If there exist $x_0, y_0 \in X$ with

$$x_0 \leq F(x_0, y_0) \quad \text{and} \quad y_0 \geq F(y_0, x_0)$$

then F has a coupled fixed point.

Proof. We construct sequences (x_n) and (y_n) putting

$$x_{n+1} = F(x_n, y_n) \quad \text{and} \quad y_{n+1} = F(y_n, x_n), \text{ for } n \geq 0.$$

In order to the proof is more comprehensive we will divide it in several steps.

Step 1 $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$, for $n \geq 0$.
In fact, we use the mathematical induction.

As $x_0 \leq F(x_0, y_0) = x_1$ and $y_0 \geq F(y_0, x_0) = y_1$ our claim is satisfied for $n = 0$.
Suppose that our claim holds for some fixed $n > 0$.Then, since $x_{n-1} \leq x_n$ and $y_n \leq y_{n-1}$ and, as F has the mixed monotone property, we get

$$x_{n+1} = F(x_n, y_n) \geq F(x_{n-1}, y_n) \geq F(x_{n-1}, y_{n-1}) = x_n$$

and

$$y_{n+1} = F(y_n, x_n) \leq F(y_{n-1}, x_n) \leq F(y_{n-1}, x_{n-1}) = y_n$$

and this proves our claim.

Step 2 $\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.
In facet, using the contractive condition and, since $x_n \geq x_{n-1}$ and $y_n \leq y_{n-1}$ (step1), we obtain

$$\begin{aligned} d(x_{n+1}, x_n) &= d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\leq \beta(max(d(x_n, x_{n-1}), d(y_n, y_{n-1})))max(d(x_n, x_{n-1}), d(y_n, y_{n-1})) \qquad (1) \\ &\leq max(d(x_n, x_{n-1}), d(y_n, y_{n-1})) \end{aligned}$$

and consequently

$$d(x_{n+1}, x_n) \leq max(d(x_n, x_{n-1}), d(y_n, y_{n-1})) \qquad (2)$$

Similarly, since $x_{n-1} \leq x_n$ and $y_{n-1} \geq y_n$, we get

$$\begin{aligned} d(y_{n+1}, y_n) &= d(F(y_n, x_n), F(y_{n-1}, x_{n-1})) \\ &\leq \beta(max(d(y_{n-1}, y_n), d(x_{n-1}, x_n)))max(d(y_{n-1}, y_n), d(x_{n-1}, x_n)) \qquad (3) \\ &\leq max(d(x_n, x_{n-1}), d(y_n, y_{n-1})) \end{aligned}$$

and, consequently,

$$d(y_{n+1}, y_n) \leq max(d(x_n, x_{n-1}), d(y_n, y_{n-1})) \tag{4}$$

By (2) and (4)

$$max(d(x_{n+1}, x_n), d(y_{n+1}, y_n)) \leq max(d(x_n, x_{n-1}), d(y_n, y_{n-1}))$$

and, thus, the sequence $(max(d(x_{n+1}, x_n), d(y_{n+1}, y_n)))$ is nonnegative decreasing.This implies that there exists $r \geq 0$ such that

$$\lim_{n\to\infty} max(d(x_{n+1}, x_n), d(y_{n+1}, y_n)) = r \tag{5}$$

assume that $r > 0$ taking account this and (1) and (3) we get

$$max(d(x_{n+1}, x_n), d(y_{n+1}, y_n)) \leq \beta(max(d(x_n, x_{n-1}), d(y_n, y_{n-1})))max(d(x_n, x_{n-1}), d(y_n, y_{n-1}))$$

then we have

$$\frac{max(d(x_{n+1}, x_n), d(y_{n+1}, y_n))}{max(d(x_n, x_{n-1}), d(y_n, y_{n-1}))} \leq \beta(max(d(x_n, x_{n-1}), d(y_n, y_{n-1}))).$$

Letting $n \to \infty$ in the last inequality and taking into account (5), we get

$$\lim_{n\to\infty} \beta(max(d(x_n, x_{n-1}), d(y_n, y_{n-1}))) = 1,$$

and since $\beta \in \delta$ this implies $r = 0$. and, consequently,

$$\lim_{n\to\infty} (max(d(x_{n+1}, x_n), d(y_{n+1}, y_n))) = 0.$$

This proves our claim.

Step 3 (x_n) , (y_n) are Cauchy sequence.
In fact, assume that at least one of the sequence (x_n) or (y_n) is not a Cauchy sequence. This implies that $\lim_{n,m\to\infty} d(x_n, x_m) \not\to 0$ or $\lim_{n,m\to\infty} d(y_n, y_m) \not\to 0$, and, consequently,

$$\lim_{n,m\to\infty}(max(d(x_n, x_m), d(y_n, y_m))) \not\to 0.$$

This means that there exist $\epsilon > 0$ for which we can find subsequences $(x_{m(k)})$ and $(x_{n(k)})$ with $n(k) > m(k) > k$ such that

$$\lim_{n,m\to\infty}(max(d(x_{m(k)}, x_{n(k)}), d(y_{m(k)}, y_{n(k)}))) \geq \epsilon \tag{6}$$

Further, corresponding to $m(k)$ we can choose $n(k)$ in such way that it is the smallest integer with $n(k) > m(k)$ and satisfying (6).
Then

$$(max(d(x_{m(k)}, x_{n(k)-1}), d(y_{m(k)}, y_{n(k)-1}))) < \epsilon \tag{7}$$

Since $x_{n(k)-1} \geq x_{m(k)-1}$ and $y_{n(k)-1} \leq y_{m(k)-1}$, using the contractive condition we can obtain

$$\begin{aligned} d(x_{n(k)}, x_{m(k)}) &= d(F((x_{n(k)-1}, y_{n(k)-1})), F((x_{m(k)-1}, y_{m(k)-1}))) \\ &\leq \beta(max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))) \\ &\times max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})) \end{aligned} \tag{8}$$

and

$$\begin{aligned} d(y_{n(k)}, y_{m(k)}) &= d(F((y_{n(k)-1}, x_{n(k)-1})), F((y_{m(k)-1}, x_{m(k)-1}))) \\ &= d(F((y_{m(k)-1}, x_{m(k)-1}))), F((y_{n(k)-1}, x_{n(k)} - 1)) \\ &\leq \beta(max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))) \\ &\times max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})). \end{aligned} \tag{9}$$

By (8) and (9) we get

$$\begin{aligned} & max(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) \qquad (10) \\ \leq \ & \beta(max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})))max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})). \end{aligned}$$

On the other hand, the triangle inequality and (7) gives us

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) < d(x_{n(k)}, x_{n(k)-1}) + \epsilon \qquad (11)$$

and

$$d(y_{n(k)}, y_{m(k)}) \leq d(y_{n(k)}, y_{n(k)-1}) + d(y_{m(k)-1}, y_{m(k)}) < d(y_{n(k)}, y_{n(k)-1}) + \epsilon \qquad (12)$$

From (6),(11) and (12) we get

$$\epsilon \leq max(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) \leq max(d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)}, y_{n(k)-1})) + \epsilon$$

Letting $k \to \infty$ in the last inequality and taking into account the step 2 we have

$$\lim_{k\to\infty} max(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) = \epsilon \qquad (13)$$

Again, the triangle inequality and (7) gives us

$$d(x_{n(k)-1}, x_{m(k)-1}) \leq d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}) < \epsilon + d(x_{m(k)}, x_{m(k)-1}) \qquad (14)$$

and

$$d(y_{n(k)-1}, y_{m(k)-1}) \leq d(y_{n(k)-1}, y_{m(k)}) + d(y_{m(k)}, y_{m(k)-1}) < \epsilon + d(y_{m(k)}, y_{m(k)-1}) \qquad (15)$$

By (14) and (15) we obtain

$$\begin{aligned} &max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})) \\ &\leq max(d(x_{m(k)}, x_{m(k)-1}), d(y_{m(k)}, y_{m(k)-1})) + \epsilon \end{aligned} \tag{16}$$

Using the triangle inequality we have

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)})$$

and

$$d(y_{n(k)}, y_{m(k)}) \leq d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)})$$

and by the last inequality and (6) we get

$$\begin{aligned} \epsilon &\leq max(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) \\ &\leq max(d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)}, y_{n(k)-1})) \\ &+ max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})) \\ &+ max(d(x_{m(k)-1}, x_{m(k)}), d(y_{m(k)-1}, y_{m(k)})) \end{aligned} \tag{17}$$

By (16) and (17) we have

$$\begin{aligned} &\epsilon - max(d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)}, y_{n(k)-1})) - max(d(x_{m(k)-1}, x_{m(k)}), d(y_{m(k)-1}, y_{m(k)})) \\ &\leq max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})) < max(d(x_{m(k)}, x_{m(k)-1}), d(y_{m(k)}, y_{m(k)-1})) + \epsilon \end{aligned}$$

Letting $k \to \infty$ in the last inequality and using step 2 we obtain

$$\lim_{k\to\infty} max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})) = \epsilon \tag{18}$$

Finally, letting $k \to \infty$ in (10) and using (13),(18) we get

$$\epsilon \leq \beta(max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})))\epsilon \quad or \quad \varepsilon \leq \beta(\epsilon)\epsilon.$$

we obtain $\lim_{n,m\to\infty} \beta(max(d(x_n, x_m), d(y_n, y_m))) = 1$.
But since $\beta \in \delta$, we get $\lim_{n,m\to\infty} max(d(x_n, x_m), d(y_n, y_m)) = 0$.This is a contradiction.
This proves our claim.

Since (X, d) is a complete metric there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x \quad and \quad \lim_{n\to\infty} y_n = y \tag{19}$$

Now we prove that (x, y) is a couple fixed point of F.
In fact, as $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$ and using continuity of F and (19) we have

$$x = \lim_{n\to\infty} F(x_n, y_n) = F(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = F(x, y)$$

$$y = \lim_{n\to\infty} F(y_n, x_n) = F(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = F(y, x)$$

and this proves that (x,y) is a coupled fixed point of F. □

In what follows, we prove that Theorem 3 is still valid for F not necessarily continuous, assuming the following hypothesis hold,

if (x_n) is a nondecreasing sequence with $x_n \to x$ then $x_n \leq x$, for all $n \in N$.

if (y_n) is a nonincreasing sequence with $y_n \to y$ then $y_n \geq y$, for all $n \in N$.

Theorem 4. If in Theorem 3 we substitute the continuity of F by the condition above mentioned we also obtain the existence of a coupled fixed point of F.

Proof. Following the proof of Theorem 3 we only have to check that (x, y) is a coupled fixed point of F.
In fact, since (x_n) is nondecreasing sequence and $x_n \to x$ and (y_n) is nonincreasing and $y_n \to y$, by our assumption,$x_n \leq x$ and $y_n \geq y$ for every $n \in N$.
Applying the contractive condition we have

$$\begin{aligned} d(F(x,y),F(x_n,y_n)) &\leq \beta(max(d(x,x_n),d(y,y_n)))max(d(x,x_n),d(y,y_n)) \\ &\leq max(d(x,x_n),d(y,y_n)). \end{aligned} \tag{20}$$

On the other hand by triangular inequality and (20) we get

$$\begin{aligned} d(x,F(x,y)) &\leq d(x,x_{n+1}) + d(x_{n+1},F(x,y) \\ &= d(x,x_{n+1}) + d(F(x,y),F(x_n,y_n)) \\ &\leq d(x,x_{n+1}) + max(d(x,x_n),d(y,y_n)). \end{aligned}$$

As $x_n \to x$ and $y_n \to y$ and taking $n \to \infty$ in the last inequality, we have

$$d(x, F(x, y)) = 0,$$

and, consequently, $F(x, y) = x$.
Using a similar argument it can be proved that $y = F(y, x)$ and this finishes the proof. □

In what follows, we give a sufficient condition for the uniqueness of coupled fixed point in Theorem 3 and Theorem 4. This condition is

$$\text{for } (x,y),(u,v) \in X \times X \text{ there exists } (z,t) \in X \times X$$

$$\text{which is comparable to } (x,y) \text{ and } (u,v). \qquad (*)$$

Notice that in $X \times X$ we consider the partial order relation given by

$$(x, y) \leq (u, v) \Leftrightarrow x \leq u \quad \text{and} \quad y \geq v.$$

Theorem 5. Adding condition (*) to the hypotheses of Theorem 3 (resp.Theorem 4) we obtain uniqueness of the coupled fixed point of F.

Proof. Suppose that (x, y) and (z, t) are coupled fixed point of F, that is $x = F(x, y)$, $y = F(y, x)$, $z = F(z, t)$ and $t = F(t, z)$.
Let (u, v) be an element of $X \times X$ and comparable to (x, y) and (z, t). Suppose that $(x, y) \geq (u, v)$ (the proof is similar in other cases).
We construct the sequences (u_n) and (v_n) defined by

$$u_0 = u,\ v_0 = v,\ u_{n+1} = F(u_n, v_n),\ v_{n+1} = F(v_n, u_n).$$

We claim that $(x, y) \geq (u_n, v_n)$ for each $n \in N$.
In fact, we will use the mathematical induction.
For $n = 0$, as $(x, y) \geq (u_n, v_n)$, this means $u_0 = u \leq x$ and $y \geq v = v_0$ and, consequently, $(u_0, v_0) \leq (x, y)$.
Suppose that $(x, y) \geq (u_n, v_n)$, then using mixed monotone property of F, we get

$$u_{n+1} = F(u_n, v_n) \leq F(x, v_n) \leq F(x, y) = x$$

$$v_{n+1} = F(v_n, u_n) \geq F(y, u_n) \geq F(y, x) = y$$

and this proves our claim.
Now, since $u_n \leq x$ and $v_n \geq y$, using the contractive condition we have

$$\begin{aligned} d(x, u_n) &= d(f(x, y), F(u_{n-1}, v_{n-1})) \\ &\leq \beta(max(d(x, u_{n-1}), d(y, v_{n-1})))max(d(x, u_{n-1}), d(y, v_{n-1})) \\ &\leq max(d(x, u_{n-1}), d(y, v_{n-1})), \end{aligned} \tag{21}$$

and, analogously

$$\begin{aligned} d(y, v_n) &= d(F(y, x), F(v_{n-1}, u_{n-1})) \\ &= d(F(v_{n-1}, u_{n-1}), F(y, x)) \\ &\leq \beta(max(d(x, u_{n-1}), d(y, v_{n-1}))).max(d(x, u_{n-1}), d(y, v_{n-1})) \\ &\leq max(d(x, u_{n-1}), d(y, v_{n-1})). \end{aligned} \tag{22}$$

From (22) and (23), we obtain

$$\begin{aligned} max(d(x, u_n), d(y, v_n)) &\leq \beta(max(d(x, u_{n-1}), d(y, v_{n-1})))max(d(x, u_{n-1}), d(y, v_{n-1})) \\ &\leq max(d(x, u_{n-1}), d(y, v_{n-1})). \end{aligned} \tag{23}$$

This last inequality implies that

$$max(d(x,u_n),d(y,v_n)) \leq max(d(x,u_{n-1}),d(y,v_{n-1})),$$

and, consequently, the sequence $(max(d(x,u_n),d(y,v_n)))$ is decreasing and nonnegative , and so,

$$\lim_{n\to\infty}(max(d(x,u_n),d(y,v_n))) = r, \tag{24}$$

for certain $r \geq 0$.
Now we show that $r = 0$, on the contrary assume $r > 0$.
Using (24) and letting $n \to \infty$ in (23) we have

$$r \leq \beta(r)r,$$

and consequently $\lim_{n\to\infty}\beta(r) = 1$, since $\beta \in \delta$, then $\lim_{n\to\infty} max(d(x,u_n),d(y,v_n)) = 0$.
This contradiction, proves $r = 0$, this gives us $u_n \to x$ and $v_n \to y$.
using a similar argument for (z,t) we can obtain

$$u_n \to z \quad and \quad v_n \to t$$

and the uniqueness of limit gives $x = z$ and $y = t$.
This finishes the proof. □

Theorem 6. Under assumption of Theorems 3 (or Theorem 4), suppose that x_0 and y_0 are comparable then the coupled fixed point $(x,y) \in X \times X$ satisfies $x = y$.

Proof. Assume $x_0 \leq y_0$ (similar argument for $y_0 \leq x_0$).
We claim that $x_n \leq y_n$ for all $n \in N$, where $x_{n+1} = F(x_n,y_n)$ and $y_{n+1} = F(y_n,x_n)$.
Obviously,the inequality is satisfied for $n = 0$.
Suppose $x_n \leq y_n$.
Using the mixed monotone property of F, we have

$$x_{n+1} = F(x_n,y_n) \leq F(y_n,y_n) \leq F(y_n,x_n) = y_{n+1}$$

and this proves our claim.
Now, using the contractive condition, as $x_n \leq y_n$, we get

$$\begin{aligned} d(x_{n+1},y_{n+1}) &= d(y_{n+1},x_{n+1}) = d(F(y_n,x_n),F(x_n,y_n)) \\ &\leq \beta(d(x_n,y_n))max(x_n,y_n) \leq max(x_n,y_n) \end{aligned} \tag{25}$$

then, we have

$$d(x_{n+1}, y_{n+1}) \leq d(x_n, y_n).$$

Thus, $\lim_{n\to\infty} d(x_n, y_n) = r$, for certain $r \geq 0$.
Taking $n \to \infty$ in (25), we have

$$r \leq \beta(r).r \quad or \quad \beta(r) = 1,$$

since $\beta \in \delta$ then $\lim_{n\to\infty} d(x_n, y_n) = 0$, this gives us $r = 0$.
As $x_n \to x$ and $y_n \to y$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$, we have $0 = \lim_{n\to\infty} d(x_n, y_n) = d(lim_{n\to\infty} x_n, lim_{n\to\infty} y_n) = d(x, y)$ and thus $x = y$.
This finishes the proof. □

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