Journal of mathematics and computer science 13 (2014), 94-114

Contents list available at JMCS

Journal of Mathematics and Computer Science

Journal Homepage: www.tjmcs.com

Fuzzy semi open soft sets related properties in fuzzy soft topological spaces

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Article history:
Received August 2014
Accepted September 2014
Available online September 2014

Abstract

In the present paper, we continue the study on fuzzy soft topological spaces and investigate the properties of fuzzy semi open (closed) soft sets, fuzzy semi soft interior (closure), fuzzy semi continuous (open) soft functions and fuzzy semi separation axioms which are important for further research on fuzzy soft topology. In particular, we study the relationship between fuzzy semi soft interior fuzzy semi soft closure. Moreover, we study the properties of fuzzy soft semi regular spaces and fuzzy soft semi normal spaces. This paper, not only can form the theoretical basis for further applications of topology on soft sets, but also lead to the development of information systems.

Keywords: Soft set, Fuzzy soft set, Fuzzy soft topological space, Fuzzy semi soft interior, Fuzzy semi soft closure, Fuzzy semi open soft, Fuzzy semi closed soft, Fuzzy semi continuous soft functions, Fuzzy soft semi separation axioms, Fuzzy soft semi T_i-spaces $(i = 1,2,3,4)$, Fuzzy soft semi regular, Fuzzy soft semi normal.

1. Introduction

The concept of soft sets was first introduced by Molodtsov [25] in 1999 as a general mathematical tool for dealing with uncertain objects. In [25, 26], Molodtsov successfully applied the soft theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement, and so on. After presentation of the operations of soft sets [23], the properties and applications of soft set theory have been studied increasingly [4,18,26]. Xiao et al.[37] and Pei and Miao [29] discussed the relationship between soft sets and information systems. They showed that soft sets are a class of special information systems. In recent years, many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets [1,6,9,13,21,22,23,24,26,27,40]. To develop soft set theory, the operations of the soft sets are redefined and a uni-int decision making method was constructed by using these new operations [10].

Recently, in 2011, Shabir and Naz [33] initiated the study of soft topological spaces. They defined soft topology on the collection τ of soft sets over X. Consequently, they defined basic notions of soft topological spaces such as open soft and closed soft sets, soft subspace, soft closure, soft nbd of a point, soft separation axioms, soft regular spaces and soft normal spaces and established their several properties. Min in [36] investigate some properties of these soft separation axioms. Kandil et al. [17] introduce the notion of soft semi separation axioms. In particular they study the properties of the soft semi regular spaces and soft semi normal spaces. Maji et. al. [21] initiated the study involving both fuzzy sets and soft sets. In [8], the notion of fuzzy set soft set was introduced as a fuzzy generalization of soft sets and some basic properties of fuzzy soft sets are discussed in detail. Then many scientists such as X. Yang et. al. [38], improved the concept of fuzziness of soft sets. In [1,2], Karal and Ahmed defined the notion of a mapping on classes of (fuzzy) soft sets, which is fundamental important in (fuzzy) soft set theory, to improve this work and they studied properties of (fuzzy) soft images and (fuzzy) soft inverse image s of fuzzy soft sets. Tanay et.al. [35] introduced the definition of fuzzy soft topology over a subset of the initial universe set while Roy and Samanta [32] gave the definition f fuzzy soft topology over the initial universe set. Chang [11] introduced the concept of fuzzy topology τ on a set X by axiomatizing a collection of fuzzy subsets of X.

In the present paper, we introduce the some new concepts in fuzzy soft topological spaces such as fuzzy semi open soft sets, fuzzy semi closed soft sets, fuzzy semi soft interior, fuzzy semi soft closure and fuzzy semi separation axioms. In particular we study the relationship between fuzzy semi soft interior fuzzy semi soft closure. Also, we study the properties of fuzzy soft semi regular spaces and fuzzy soft semi normal spaces. Moreover, we show that if every fuzzy soft point f_e is fuzzy semi closed soft set in a fuzzy soft topological space (X,τ,E), then (X,τ,E), is fuzzy soft semi T_1-space (resp. fuzzy soft semi T_2-space). This paper, not only can form the theoretical basis for further applications of topology on soft sets, but also lead to the development of information systems.

2. Preliminaries

In this section, we present the basic definitions and results of soft set theory which will be needed in the sequel. For more details see [1,4,5,8,11,12,14,21,22,23,24,26,27,30,40].

Definition 2.1 [39] A fuzzy set A of a non-empty set X is characterized by a membership function $\mu_A : X \to [0,1] = I$ whose value $\mu_A(x)$ represents the "degree of membership" of x in A for $x \in X$. Let I^X denotes the family of all fuzzy sets on X. If $A, B \in I^X$, then some basic set operations for fuzzy sets are given by Zadeh [39], as follows:

1- $A \le B \Leftrightarrow \mu_A(x) \le \mu_B(x)\ \forall\ x \in X$.
2- $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)\ \forall\ x \in X$.
3- $C = A \vee B \Leftrightarrow \mu_C(x) = \mu_A(x) \vee \mu_B(x)\ \forall\ x \in X$.
4- $D = A \wedge B \Leftrightarrow \mu_D(x) = \mu_A(x) \wedge \mu_B(x)\ \forall\ x \in X$.
5- $M = A' \Leftrightarrow \mu_M(x) = 1 - \mu_A(x)\ \forall\ x \in X$.

Definition 2.2 [25] Let X be an initial universe and E be a set of parameters. Let $P(X)$ denote the power set of X and A be a non-empty subset of E. A pair (F, A) denoted by F_A is called a soft set over X, where F is a mapping given by $F : A \to P(X)$. In other words, a soft set over X is a parametrized family of subsets of the universe X. For a particular $e \in A$, $F(e)$ may be considered the set of e-approximate elements of the soft set (F, A) and if $e \notin A$, then $F(e) = \phi$ i.e $F_A = \{F(e) : e \in A \subseteq E, F : A \to P(X)\}$. The family of all these soft sets over X denoted by $SS(X)_A$.

Definition 2.3 [20] The union of two soft sets (F, A) and (G, B) over the common universe X is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e), e \in A - B, \\ G(e), e \in B - A, \\ F(e) \cup G(e), e \in A \cap B \end{cases}.$$

Definition 2.4 [23] The intersection of two soft sets (F, A) and (G, B) over the common universe X is the soft set (H, C), where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cap G(e)$. Note that, in order to efficiently discuss, we consider only soft sets (F, E) over a universe X with the same set of parameter E. We denote the family of these soft sets by $SS(X)_E$.

Definition 2.5 [33] Let τ be a collection of soft sets over a universe X with a fixed set of parameters E, then $\tau \subseteq SS(X)_E$ is called a soft topology on X if;

1- $\tilde{X}, \tilde{\phi} \in \tau$, where $\tilde{\phi}(e) = \phi$ and $\tilde{X}(e) = X, \forall e \in E$,
2-the union of any number of soft sets in τ belongs to τ,
3-the intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a soft topological space over X.

Definition 2.6 [21] Let $A \subseteq E$. A pair (f, A), denoted by f_A, is called fuzzy soft set over X, where f is a mapping given by $f : A \to I^X$ defined by $f_A(e) = \mu_{f_A}^e$, where $\mu_{f_A}^e = \bar{0}$ if $e \notin A$ and $\mu_{f_A}^e \neq \bar{0}$ if $e \in A$, where $\bar{0}(x) = 0 \ \forall \ x \in X$. The family of all these fuzzy soft sets over X denoted by $FSS(X)_A$.

Proposition 2.1 [11] Every fuzzy set may be considered a soft set.

Definition 2.7 [31] The complement of a fuzzy soft set (f, A), denoted by $(f, A)'$, is defined by $(f, A)' = (f', A)$, $f'_A : E \to I^X$ is a mapping given by $\mu_{f'_A}^e = \bar{1} - \mu_{f_A}^e \ \forall e \in A$, where $\bar{1}(x) = 1 \ \forall \ x \in X$. Clearly, $(f'_A)' = f_A$.

Definition 2.8 [23] A fuzzy soft set f_A over X is said to be a NULL fuzzy soft set, denoted by $\tilde{0}_A$, if for all $e \in A$, $f_A(e) = \bar{0}$.

Definition 2.9 [23] A fuzzy soft set f_A over X is said to be an absolute fuzzy soft set, denoted by $\tilde{1}_A$, if for all $e \in A$, $f_A(e) = \bar{1}$. Clearly we have $(\tilde{1}_A)' = \tilde{0}_A$ and $(\tilde{0}_A)' = \tilde{1}_A$.

Definition 2.10 [31] Let $f_A, g_B \in FSS(X)_E$. Then f_A is fuzzy soft subset of g_B, denoted by $f_A \,\tilde{\sqsubseteq}\, g_B$, if $A \subseteq B$ and $\mu_{f_A}^e \subseteq \mu_{g_B}^e \ \forall \ e \in A$, i.e.

$\mu_{f_A}^e(x) \leq \mu_{g_B}^e(x) \ \forall \ x \in X \ and \ \forall \ e \in A$.

Definition 2.11 [31] The union of two fuzzy soft sets f_A and g_B over the common universe X is also a fuzzy soft set h_C, where $C = A \cup B$ and for all $e \in C$, $h_C(e) = \mu_{h_c}^e = \mu_{f_A}^e \vee \mu_{g_B}^e \ \forall e \in E$. Here we write $h_C = f_A \,\tilde{\sqcup}\, g_B$.

Definition 2.12 [31] The intersection of two fuzzy soft sets f_A and g_B over the common universe X is also a fuzzy soft set h_C, where $C = A \cap B$ and for all $e \in C$, $h_C(e) = \mu_{h_c}^e = \mu_{f_A}^e \wedge \mu_{g_B}^e \ \forall e \in E$. Here we write $h_C = f_A \,\tilde{\sqcap}\, g_B$.

Theorem 2.1 [1] Let $\{(f, A)_j : j \in J\} \subseteq FSS(X)_E$. Then the following statements hold,

1- $[\tilde{\sqcup}_{j \in J} (f, A)_j]' = \tilde{\sqcap}_{j \in J} (f, A)'_j$,

2- $[\tilde{\sqcap}_{j \in J} (f, A)_j]' = \tilde{\sqcup}_{j \in J} (f, A)'_j$.

Definition 2.13 [31] Let T be a collection of fuzzy soft sets over a universe X with a fixed set of parameters E, then $T \subseteq FSS(X)_E$ is called fuzzy soft topology on X if

1- $\tilde{1}_E, \tilde{0}_E \in T$, where $\tilde{0}_E(e) = \bar{0}$ and $\tilde{1}_E(e) = \bar{1}, \forall e \in E$,

2-the union of any members of T belongs to T.

3-the intersection of any two members of T belongs to T.

The triplet (X,T,E) is called fuzzy soft topological space over X. Also, each member of T is called fuzzy open soft in (X,T,E). We denote the set of all open soft sets by $FOS(X,\mathrm{T},E)$, or $FOS(X)$ and the set of all fuzzy closed soft sets by $FCS(X,\mathrm{T},E)$, or $FCS(X)$.

Definition 2.14 [31] Let (X,T,E) be a fuzzy soft topological space. A fuzzy soft set f_A over X is said to be fuzzy closed soft set in X, if its relative complement f'_A is fuzzy open soft set.

Definition 2.15 [28] Let (X,T,E) be a fuzzy soft topological space and $f_A \in FSS(X)_E$.
1- The fuzzy soft closure of f_A, denoted by $Fcl(f_A)$ is the intersection of all fuzzy closed soft super sets of f_A. i.e.
$Fcl(f_A) = \sqcap\{h_D : h_D \textit{ is fuzzy closed soft set and } f_A \sqsubseteq h_D\}$.
2-The fuzzy soft interior of g_B, denoted by $Fint(f_A)$ is the fuzzy soft union of all fuzzy open soft subsets of f_A.i.e.
$Fint(g_B) = \sqcup\{h_D : h_D \textit{ is fuzzy open soft set and } h_D \sqsubseteq g_B\}$.

Definition 2.16 [20] The fuzzy soft set $f_A \in FSS(X)_E$ is called fuzzy soft point if there exist $x \in X$ and $e \in E$ such that $\mu^e_{f_A}(x) = \alpha$ $(0 < \alpha \le 1)$ and $\mu^e_{f_A}(y) = \overline{0}$ for each $y \in X - \{x\}$, and this fuzzy soft point is denoted by x^α_e or f_e.

Definition 2.17 [20] The fuzzy soft point x^α_e is said to be belonging to the fuzzy soft set (g,A), denoted by $x^\alpha_e \tilde{\in} (g,A)$, if for the element $e \in A,\ \alpha \le \mu^e_{g_A}(x)$.

Theorem 2.2 [20] Let (X,T,E) be a fuzzy soft topological space and f_e be a fuzzy soft point. Then the following properties hold:
1- If $f_e \tilde{\in} g_A$, then $f_e \tilde{\notin} g'_A$;
2- $f_e \tilde{\in} g_A \Leftrightarrow f'_e \tilde{\in} g'_A$;
3-Every non-null fuzzy soft set f_A can be expressed as the union of all the fuzzy soft points belonging to f_A.

Definition 2.18 [20] A fuzzy soft set g_B in a fuzzy soft topological space (X,T,E) is called fuzzy soft neighborhood of the fuzzy soft point x^α_e if there exists a fuzzy open soft set h_C such that $x^\alpha_e \tilde{\in} h_C \sqsubseteq g_B$. A fuzzy soft set g_B in a fuzzy soft topological space (X,T,E) is called fuzzy soft neighborhood of the soft set f_A if there exists a fuzzy open soft set h_C such that $f_A \sqsubseteq h_C \sqsubseteq g_B$. The fuzzy soft neighborhood system of the fuzzy soft point x^α_e, denoted by $N_{\mathrm{T}}(x^\alpha_e)$, is the family of all its fuzzy soft neighborhoods.

Definition 2.19 [20] Let (X,T,E) be a fuzzy soft topological space and $Y \subseteq X$. Let h_E^Y be a fuzzy soft set over (Y,E) such that $h_E^Y : E \to I^Y$ such that $h_E^Y(e) = \mu^e_{h_E^Y}$,

$$\mu^e_{h_E^Y}(x) = \begin{cases} 1, \ x \in Y, \\ \\ 0, \ x \notin Y. \end{cases}$$

Let $T_Y = \{h_E^Y \sqcap g_B : g_B \in T\}$ then the fuzzy soft topology T_Y on (Y,E) is called fuzzy soft subspace topology for (Y,E) and (Y,T_Y,E) is called fuzzy soft subspace of (X,T,E). If $h_E^Y \in T$ (resp. $h_E^Y \in T'$), then (Y,T_Y,E) is called fuzzy open soft subspace (resp. fuzzy closed soft subspace) of (X,T,E).

Definition 2.20 [28] Let $FSS(X)_E$ and $FSS(Y)_K$ be families of fuzzy soft sets over X and Y, respectively. Let $u: X \to Y$ and $p: E \to K$ be mappings. Then the map f_{pu} is called fuzzy soft mapping from X to Y and denoted by,
$f_{pu}: FSS(X)_E \to FSS(Y)_K$ such that,
1- If $f_A \in FSS(X)_E$. Then the image of f_A under the fuzzy soft mapping f_{pu} is the fuzzy soft set over Y defined by $f_{pu}(f_A)$, where $\forall\, k \in p(E), \forall\, y \in Y$,

$$f_{pu}(f_A)(k)(y) = \begin{cases} \bigvee_{u(x)=y} [\vee_{p(e)=k}(f_A(e))](x) & if \ \ x \in u^{-1}(y), \\ 0 & otherwise. \end{cases}$$

2-If $g_B \in FSS(Y)_K$, then the pre-image of g_B under the fuzzy soft mapping f_{pu} is the fuzzy soft set over X defined by $f_{pu}^{-1}(g_B)$, where $\forall\, e \in p^{-1}(K), \forall\, x \in X$,

$$f^{-1}(g_B)(e)(x) = \begin{cases} g_B(p(e))(u(x)) & for \ p(e) \in B, \\ 0 & otherwise. \end{cases}$$

The fuzzy soft mapping f_{pu} is called surjective (resp. injective) if p and u are surjective (resp. injective), also is said to be constant if p and u are constant.

Definition 2.21 *[28] Let* (X,T_1,E) *and* (Y,T_2,K) *be two fuzzy soft topological spaces and* $f_{pu}: FSS(X)_E \to FSS(Y)_K$ *be a fuzzy soft mapping. Then* f_{pu} *is called*

1-Fuzzy continuous soft if $f_{pu}^{-1}(g_B) \in T_1 \ \forall\, (g_B) \in T_2$.

2- Fuzzy open soft if $f_{pu}(g_A) \in T_2 \forall\, (g_A) \in T_1$.

Theorem 2.3 [1] Let $FSS(X)_E$ and $FSS(Y)_K$ be two families of fuzzy soft sets. For the fuzzy soft function $f_{pu}: FSS(X)_E \to FSS(Y)_K$, the following statements hold,
1- $f_{pu}^{-1}((g,B)') = (f_{pu}^{-1}(g,B))' \forall\, (g,B) \in FSS(Y)_K$.

2- $f_{pu}(f_{pu}^{-1}((g,B))) \sqsubseteq (g,B) \forall (g,B) \in FSS(Y)_K$. If f_{pu} is surjective, then the equality holds.

3- $(f,A) \sqsubseteq f_{pu}^{-1}(f_{pu}((f,A))) \forall (f,A) \in FSS(X)_E$. If f_{pu} is injective, then the equality holds.

4- $f_{pu}(\tilde{0}_E) = \tilde{0}_K$, $f_{pu}(\tilde{1}_E) \sqsubseteq \tilde{1}_K$. If f_{pu} is surjective, then the equality holds.

5- $f_{pu}^{-1}(\tilde{1}_K) = \tilde{1}_E$ and $f_{pu}^{-1}(\tilde{0}_K) = \tilde{0}_E$.

6- If $(f,A) \sqsubseteq (g,A)$, then $f_{pu}(f,A) \sqsubseteq f_{pu}(g,A)$.

7- If $(f,B) \sqsubseteq (g,B)$, then $f_{pu}^{-1}(f,B) \sqsubseteq f_{pu}^{-1}(g,B) \forall (f,B),(g,B) \in FSS(Y)_K$.

8- $f_{pu}^{-1}(\sqcup_{j\in J}(f,B)_j) = \sqcup_{j\in J} f_{pu}^{-1}(f,B)_j$ and
$f_{pu}^{-1}(\sqcap_{j\in J}(f,B)_j) = \sqcap_{j\in J} f_{pu}^{-1}(f,B)_j, \forall (f,B)_j \in FSS(Y)_K$.

9- $f_{pu}(\sqcup_{j\in J}(f,A)_j) = \sqcup_{j\in J} f_{pu}(f,A)_j$ and
$f_{pu}(\sqcap_{j\in J}(f,A)_j) \sqsubseteq \sqcap_{j\in J} f_{pu}(f,A)_j \ \forall (f,A)_j \in FSS(X)_E$. If f_{pu} is injective, then the equality holds.

Definition 2.22 [16] Let (X,τ,E) be a soft topological space and $F_A \in SS(X)_E$. If $F_A \tilde{\subseteq} cl(int(F_A))$, then F_A is called semi open soft set. We denote the set of all semi open soft sets by $SOS(X,\tau,E)$, or $SOS(X)$ and the set of all semi closed soft sets by $SCS(X,\tau,E)$, or $SCS(X)$.

3 . Fuzzy semi open (closed) soft sets

Various generalization of closed and open soft sets in soft topological spaces were studied by Kandil et al. [16], but for fuzzy soft topological spaces such generalization have not been studied so far. In this section, we move one step forward to introduce fuzzy semi open and fuzzy semi closed soft sets and study various properties and notions related to these structures.

Definition 3.1 Let (X,T,E) be a fuzzy soft topological space and $f_A \in FSS(X)_E$. If $f_A \sqsubseteq Fcl(Fint(f_A))$, then f_A is called fuzzy semi open soft set. We denote the set of all fuzzy semi open soft sets by $FSOS(X,\mathrm{T},E)$, or $FSOS(X)$ and the set of all fuzzy semi closed soft sets by $FSCS(X,\mathrm{T},E)$, or $FSCS(X)$.

Theorem 3.1 Let (X,T,E) be a fuzzy soft topological space and $f_A \in FSOS(X)$.Then

1- Arbitrary fuzzy soft union of fuzzy semi open soft sets is fuzzy semi open soft.

2-Arbitrary fuzzy soft intersection of fuzzy semi closed soft sets is fuzzy semi closed soft.

Proof.

1- Let $\{(f,A)_j : j \in J\} \subseteq FSOS(X)$. Then $\forall j \in J, (f,A)_j \sqsubseteq Fint(Fcl((f,A)_j))$.
Henc, $\sqcup_j (f,A)_j \sqsubseteq \sqcup_j (Fint(Fcl((f,A)_j))) \sqsubseteq Fint(\sqcup_j Fcl(f,A)_j) = Fint(Fcl(\sqcup_j (f,A)_j))$.
Therefore, $\sqcup_j (f,A)_j \in FSOS(X) \quad \forall j \in J$.

2-By a similar way.

Theorem 3.2 Let (X,T,E) be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then:

1- $f_A \in FSOS(X)$ if and only if $Fcl(f_A) = Fcl(Fint(f_A))$.

2- If $g_B \in \mathrm{T}$, then g_B ſ$Fcl(f_A)$ | $Fcl(g_B$ ſ$g_B)$.

Proof. Immediate.

Theorem 3.3 Let (X,T,E) be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then:

1- $f_A \in FSOS(X)$ if and only if there exists $g_B \in \mathrm{T}$ such that $g_B \hat{o} f_A \hat{o} Fcl(g_B)$.

2- If $f_A \in FSOS(X)$ and $f_A \hat{o} h_D \hat{o} Fcl(f_A)$, then $h_D \in FSOS(X)$.

Proof.

1-Let $f_A \in FSOS(X)$. Then $f_A \hat{o} Fcl(Fint(f_A))$. Take $g_B = Fint(f_A) \in \mathrm{T}$. So, we have $g_B \hat{o} f_A \hat{o} Fcl(g_B)$. Sufficiency, let $g_B \hat{o} f_A \hat{o} Fcl(g_B)$ for some $g_B \in \mathrm{T}$. Then $g_B \hat{o} Fint(f_A)$. It follows that, $Fcl(g_B) \hat{o} Fcl(Fint(f_A))$. Thus, $f_A \hat{o} Fcl(g_B) \hat{o} Fcl(Fint(f_A))$. Therefore, $f_A \in FSOS(X)$.

2-Let $f_A \in FSOS(X)$. Then $g_B \hat{o} f_A \hat{o} Fcl(g_B)$ for some $g_B \in \mathrm{T}$. It follows that, $g_B \hat{o} f_A \hat{o} h_D$. Thus, $g_B \hat{o} h_D \hat{o} Fcl(f_A) \hat{o} Fcl(g_B)$. Hence $g_B \hat{o} h_D \hat{o} Fcl(g_B)$ for some $g_B \in \mathrm{T}$. Therefore, $h_D \in FSOS(X)$ by (1).

Definition 3.2 Let (X,T,E) be a fuzzy soft topological space, $f_A \in FSS(X)_E$ and $f_e \tilde{\in} FSS(X)_E$. Then

1- f_e is called fuzzy semi interior soft point of f_A if $\exists$ $g_B \in FSOS(X)$ such that $f_e \tilde{\in} g_B \hat{o} f_A$. The set of all fuzzy semi interior soft points of f_A is called the fuzzy semi soft interior of f_A and is denoted by $FSint(f_A)$ consequently,
$FSint(f_A) = \int \{g_B : g_B | f_A, g_B \in FSOS(X)\}$

2- f_e is called fuzzy semi cluster soft point of f_A if f_A ſ$h_D \neq \tilde{0}_E$ $\forall\, h_D \in FSOS(X)$. The set of all fuzzy semi cluster soft points of f_A is called fuzzy semi soft closure of f_A and denoted by $FScl(f_A)$. Consequently, $FScl(f_A) =$ ſ$\{h_D : h_D \in FSCS(X), f_A | h_D\}$.

Theorem 3.4 Let (X,T,E) be a fuzzy soft topological space and $f_A, g_B \in FSS(X)_E$. Then the following properties are satisfied for the fuzzy semi interior operator, denoted by $FSint$.

1- $FSint(\tilde{1}_E) = \tilde{1}_E$ and $FSint(\tilde{0}_E)) = \tilde{0}_E$.

2- $FSint(f_A) \hat{o} f_A$.

3- $FSint(f_A)$ is the largest fuzzy semi open soft set contained in f_A.
4- If $f_A \sqsubseteq g_B$, then $FSint(f_A) \sqsubseteq FSint(g_B)$.
5- $FSint(FSint(f_A)) = FSint(f_A)$.
6- $FSint(f_A) \sqcup FSint(g_B) \sqsubseteq FSint[(f_A) \sqcup (g_B)]$.
7- $FSint[(f_A) \sqcap (g_B)] \sqsubseteq FSint(f_A) \sqcap FSint(g_B)$.

Proof. Obvious.

Theorem 3.5 Let (X, T, E) be a fuzzy soft topological space and $f_A, g_B \in FSS(X)_E$. Then the following properties are satisfied for the fuzzy semi closure operator, denoted by $FScl$.

1- $FScl(\tilde{1}_E) = \tilde{1}_E$ and $FScl(\tilde{0}_E) = \tilde{0}_E$.
2- $(f_A) \sqsubseteq FScl(f_A)$.
3- $FScl(f_A)$ is the smallest fuzzy semi closed soft set contains f_A.
4- If $f_A \sqsubseteq g_B$, then $FScl(f_A) \sqsubseteq FScl(g_B)$.
5- $FScl(FScl(f_A)) = FScl(f_A)$.
6- $FScl(f_A) \sqcup FScl(g_B) \sqsubseteq FScl[(f_A) \sqcup (g_B)]$.
7- $FScl[(f_A) \sqcap (g_B)] \sqsubseteq FScl(f_A) \sqcap FScl(g_B)$.

Proof. Immediate.

Remark 3.2 Note that the family of all fuzzy semi open soft sets on a fuzzy soft topological space (X, T, E) forms a fuzzy supra soft topology, which is a collection of fuzzy soft sets contains $\tilde{1}_E, \tilde{0}_E$ and closed under arbitrary fuzzy soft union.

Theorem 3.6 Every fuzzy open (resp. closed) soft set in a fuzzy soft topological space (X, T, E) is fuzzy semi open (resp. fuzzy semi closed) soft.

Proof. Let $f_A \in FOS(X)$. Then $Fint(f_A) = f_A$. Since $f_A \sqsubseteq Fcl(f_A)$, then $f_A \sqsubseteq Fcl(Fint(f_A))$. Thus, $f_A \in FSOS(X)$.

Remark 3.3 The converse of Theorem 3.6 is not true in general as shown in the following example.

Example 3.1 Let $X = \{a, b, c\}$, $E = \{e_1, e_2, e_3\}$ and $A, B, C, D \subseteq E$ where $A = \{e_1, e_2\}$, $B = \{e_2, e_3\}$, $C = \{e_1, e_3\}$ and $D = \{e_2\}$. Let $\mathrm{T} = \{\tilde{1}_E, \tilde{0}_E, f_{1A}, f_{2B}, f_{3D}, f_{4E}, f_{5B}, f_{6D}\}$ where $f_{1A}, f_{2B}, f_{3D}, f_{4E}, f_{5B}, f_{6D}$ are fuzzy soft sets over X defined as follows:

$\mu_{f_{1A}}^{e_1} = \{a_{0.5}, b_{0.75}, c_{0.4}\}$, $\mu_{f_{1A}}^{e_2} = \{a_{0.3}, b_{0.8}, c_{0.7}\}$,

$\mu_{f_{2B}}^{e_2} = \{a_{0.4}, b_{0.6}, c_{0.3}\}$, $\mu_{f_{2B}}^{e_3} = \{a_{0.2}, b_{0.4}, c_{0.45}\}$,

$\mu_{f_{3D}}^{e_2} = \{a_{0.3}, b_{0.6}, c_{0.3}\}$,

$\mu_{f_{4E}}^{e_1} = \{a_{0.5}, b_{0.75}, c_{0.4}\}$, $\mu_{f_{4E}}^{e_2} = \{a_{0.4}, b_{0.8}, c_{0.7}\}$, $\mu_{f_{4E}}^{e_3} = \{a_{0.2}, b_{0.4}, c_{0.4}\}$,

$\mu_{f_{5B}}^{e_2} = \{a_{0.4}, b_{0.8}, c_{0.7}\}$, $\mu_{f_{5B}}^{e_3} = \{a_{0.2}, b_{0.4}, c_{0.45}\}$,
$\mu_{f_{6D}}^{e_2} = \{a_{0.3}, b_{0.8}, c_{0.7}\}$.

Then T defines a fuzzy soft topology on X. Hence, the fuzzy soft set k_E where:
$\mu_{k_E}^{e_1} = \{a_{0.4}, b_{0.3}, c_{0.2}\}$, $\mu_{k_E}^{e_2} = \{a_{0.6}, b_{0.9}, c_{0.7}\}$, $\mu_{k_E}^{e_3} = \{a_{0.2}, b_{0.3}, c_{0.1}\}$.
is fuzzy semi open soft set of (X, T, E), but it is not fuzzy open soft.

Theorem 3.7 Let (X, T, E) be a fuzzy soft topological space and $f_A \in FSS(X)$. Then:
1- $FSint(f_A') = \tilde{1} - [FScl(f_A)]$.
2- $FScl(f_A') = \tilde{1} - [FSint(f_A)]$.

Proof.
1- Let $f_e \notin FScl(f_A)$. Then $\exists$ $g_B \in FOS(\tilde{1}, f_e)$ such that $g_B \sqcap f_A = \tilde{0}_E$. Hence $f_e \tilde{\in} g_B \sqsubseteq f_A'$. Thus, $f_e \tilde{\in} FSint(f_A')$. This means that, $\tilde{1}_E - FScl(f_A) \sqsubseteq FSint(f_A')$. Conversely, Let $f_e \tilde{\in} FSint(f_A')$. Since $FSint(f_A') \sqcap f_A = \tilde{0}_E$. So, $f_e \notin FScl(f_A)$. It follows that, $f_e \tilde{\in} \tilde{1} - FScl(f_A)$. Therefore, $FSint(f_A') \sqsubseteq \tilde{1} - FScl(f_A)$, and hence
$FSint(f_A') = \tilde{1} - [FScl(f_A)]$.

2- Let $f_e \tilde{\in} FSint(f_A)$. Then $\forall$ $g_B \in FSO(\tilde{1}, f_e)$, $f_e \tilde{\in} g_B \sqsubseteq f_A$. Hence $g_B \sqcap f_A' = \tilde{0}_E$. Thus, $f_e \notin FSint(f_A')$. This means that, $\tilde{1}_E - FSint(f_A) \sqsubseteq FScl(f_A')$. Conversely, let $f_e \notin FScl(f_A')$. Then $\exists$ $g_B \in FOS(\tilde{1}_E, f_e)$ such that $g_B \sqcap f_A' = \tilde{0}_E$ Hence $f_e \tilde{\in} g_B \sqsubseteq f_A$. It follows that, $f_e \tilde{\in} FSint(f_A)$. This means that, $FScl(f_A') \sqsubseteq \tilde{1}_E - FSint(f_A)$ and hence $FScl(f_A') = \tilde{1}_E - [FSint(f_A)]$. This completes the proof.

Theorem 3.8 Let (X, T, E) be a fuzzy soft topological space and $f_A, g_B \in FSS(X)_E$. If either $f_A \in FSOS(X)$ or $g_B \in FSOS(X)$. Then
$Fint(Fcl(f_A \sqcap g_B)) = Fint(Fcl(f_A)) \sqcap Fint(Fcl(g_B))$.

Proof. Let $f_A, g_B \in FSS(X)_E$ Then, we generally have
$Fint(Fcl(f_A \sqcap g_B)) \sqsubseteq Fint(Fcl(f_A)) \sqcap Fint(Fcl(f_A))$. Suppose that $f_A \in FSOS(X)$. Then
$Fcl(f_A) = Fcl(Fint(f_A))$ from Theorem 3.2 (1).

$$\begin{aligned}
Fint(Fcl(f_A)) \sqcap Fint(Fcl(g_B)) &\sqsubseteq Fint\,[Fint(Fcl(f_A)) \sqcap Fint(Fcl(g_B))] \\
&= Fint\,[Fcl(Fint(f_A)) \sqcap Fint(Fcl(g_B))] \\
&\sqsubseteq Fint\,(Fcl\,[Fint(f_A) \sqcap Fint(Fcl(g_B))]) \\
&\sqsubseteq Fint\,(Fcl\,(Fint\,[Fint(f_A) \sqcap Fcl(g_B)])) \\
&\sqsubseteq Fint\,(Fcl\,(Fint\,(Fcl\,[Fint(f_A) \sqcap (g_B)]))) \\
&\sqsubseteq Fint(Fcl(f_A \sqcap g_B))
\end{aligned}$$

from Theorem 3.2 (2). This completes the proof.

Theorem 3.9 Let (X,T,E) be a fuzzy soft topological space, $f_A \in FOS(X)$ and $g_B \in FSOS(X)$. Then $f_A \sqcap g_B \in FSOS(X)$.

Proof. Let $f_A \in FOS(X)$ and $g_B \in FSOS(X)$. Then $f_A \sqcap g_B \sqsubseteq Fint(f_A) \sqcap Fcl(Fint(g_B)) = Fcl(Fint(f_A) \sqcap Fint(g_B)) = Fcl(Fint((f_A) \sqcap (g_B)))$ from Theorem 3.2 (2). Hence $f_A \sqcap g_B \in FSOS(X)$.

Theorem 3.10 Let (X,T,E) be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then $f_A \in FSCS(X)$ if and only if $Fint(Fcl(f_A)) \sqsubseteq f_A$.

Proof. Let $f_A \in FSCS(X)$, then f'_A is fuzzy semi open soft set. This means that, $f'_A \sqsubseteq Fcl(Fint(\tilde{1}_E - f_A)) = \tilde{1}_E - (Fint(Fcl(f_A)))$. Therefore, $Fint(Fcl(f_A)) \sqsubseteq f_A$. Conversely, let $Fint(Fcl(f_A)) \sqsubseteq f_A$. Then $\tilde{1}_E - f_A \sqsubseteq Fcl(Fint(\tilde{1}_E - f_A))$. Hence, $\tilde{1}_E - f_A$ is fuzzy semi open soft set. Therefore, f_A is fuzzy semi closed soft set.

Corollary 3.1 Let (X,T,E) be a fuzzy soft topological space and $f_A \in FSS(X)_E$. Then $f_A \in FSCS(X)$ if and only if $f_A = f_A \sqcup Fint(Fcl(f_A))$.

Proof. It is obvious from Theorem 3.10.

4. Fuzzy semi continuous soft functions

Kharal et al. [1,2] introduced soft function over classes of (fuzzy) soft sets. The authors also defined and studied the properties of soft images and soft inverse images of (fuzzy) soft sets, and used these notions to the problem of medical diagnosis in medical expert systems. Kandil et al. [17] introduced some types of soft function in soft topological spaces. Here we introduce the notions of fuzzy semi soft function in fuzzy soft topological spaces and study its basic properties.

Definition 4.1 Let (X,T_1,E), (Y,T_2,K) be fuzzy soft topological spaces and $f_{pu} : FSS(X)_E \to FSS(Y)_K$ be a soft function. Then f_{pu} is called;

1- Fuzzy semi continuous soft function if $f_{pu}^{-1}(g_B) \in FSOS(X) \forall g_B \in T_2$.

2- Fuzzy fuzzy semi open soft if $f_{pu}(g_A) \in FSOS(Y) \forall g_A \in T_1$.

3- Fuzzy semi closed soft if $f_{pu}(f_A) \in FSCS(Y) \forall f_A \in T'_1$.

4- Fuzzy irresolute soft if $f_{pu}^{-1}(g_B) \in FSOS(X) \forall g_B \in FSOS(Y)$.

5- Fuzzy irresolute open soft if $f_{pu}(g_A) \in FSOS(Y) \forall g_A \in FSOS(X)$.

6- Fuzzy irresolute closed soft if $f_{pu}(f_A) \in FSCS(Y) \forall f_A \in FSCS(Y)$.

Example 4.1 Let $X = Y = \{a,b,c\}$, $E = \{e_1,e_2,e_3\}$ and $A \subseteq E$ where $A = \{e_1,e_2\}$. Let $f_{pu} : (X,T_1,E) \to (Y,T_2,K)$ be the constant soft mapping where T_1 is the indiscrete fuzzy soft topology and T_2 is the discrete fuzzy soft topology such that $u(x) = a \ \forall x \in X$ and $p(e) = e_1 \ \forall e \in E$. Let f_A be fuzzy soft sets over Y defined as follows:

$\mu_{f_A}^{e_1} = \{a_{0.1}, b_{0.5}, c_6\}$, $\mu_{f_A}^{e_2} = \{a_{0.6}, b_{0.2}, c_{0.5}\}$.

Then $f_A \in \mathrm{T}_2$. Now, we find $f_{pu}^{-1}(f_A)$ as follows:

$f_{pu}^{-1}(f_A)(e_1)(a) = f_A(p(e_1))(u(a)) = f_A(e_1)(a) = 0.1$,

$f_{pu}^{-1}(f_A)(e_1)(b) = f_A(p(e_1))(u(b)) = f_A(e_1)(a) = 0.1$,

$f_{pu}^{-1}(f_A)(e_1)(c) = f_A(p(e_1))(u(c)) = f_A(e_1)(a) = 0.1$,

$f_{pu}^{-1}(f_A)(e_2)(a) = f_A(p(e_2))(u(a)) = f_A(e_1)(a) = 0.1$,

$f_{pu}^{-1}(f_A)(e_2)(b) = f_A(p(e_2))(u(b)) = f_A(e_1)(a) = 0.1$,

$f_{pu}^{-1}(f_A)(e_2)(c) = f_A(p(e_2))(u(c)) = f_A(e_1)(a) = 0.1$,

$f_{pu}^{-1}(f_A)(e_3)(a) = f_A(p(e_3))(u(a)) = f_A(e_1)(a) = 0.1$,

$f_{pu}^{-1}(f_A)(e_3)(b) = f_A(p(e_3))(u(b)) = f_A(e_1)(a) = 0.1$,

$f_{pu}^{-1}(f_A)(e_3)(c) = f_A(p(e_3))(u(c)) = f_A(e_1)(a) = 0.1$.

Hence $f_{pu}^{-1}(f_A) \notin FSOS(X)$. Therefore, f_{pu} is not fuzzy semi continuous soft function.

Theorem 4.1 Every fuzzy continuous soft function is fuzzy semi continuous soft.

Proof. It is obvious from Theorem 3.6.

Theorem 4.2 Let (X, T_1, E), (Y, T_2, K) be fuzzy soft topological spaces and f_{pu} be a soft function such that $f_{pu} : FSS(X)_E \to FSS(Y)_K$. Then the following are equivalent:

1- f_{pu} is fuzzy semi continuous soft function.

2- $f_{pu}^{-1}(h_B) \in FSCS(X)\ \forall\ h_B \in FCS(Y)$.

3- $f_{pu}(FScl(g_A) \,|\, Fcl_{\mathrm{T}_2}(f_{pu}(g_A))\ \forall\ g_A \in FSS(X)_E$.

4- $FScl(f_{pu}^{-1}(h_B)) \,|\, f_{pu}^{-1}(FScl_{\mathrm{T}_2}(h_B))\ \forall\ h_B \in FSS(Y)_K$.

5- $f_{pu}^{-1}(Fint_{\mathrm{T}_2}(h_B)) \,|\, FSint(f_{pu}^{-1}(h_B))\ \forall\ h_B \in FSS(Y)_K$.

Proof.

$\mathbf{1 \to 2}$ Let h_B be a fuzzy closed soft set over Y. Then $h'_B \in FOS(Y)$ and $f_{pu}^{-1}(h'_B) \in FSOS(X)$ from Definition 4.1. Since $f_{pu}^{-1}(h'_B) = (f_{pu}^{-1}(h_B))'$ from Theorem 2.3. Thus, $f_{pu}^{-1}(h_B) \in FSCS(X)$.

$\mathbf{2 \to 3}$ Let $g_A \in FSS(X)_E$. Since $g_A \,|\, f_{pu}^{-1}(f_{pu}(g_A)) \,|\, f_{pu}^{-1}(Fcl_{\mathrm{T}_2}(f_{pu}(g_A))) \in FSCS(X)$ from (2) and Theorem 2.3. Then $g_A \,|\, FScl(g_A) \,|\, f_{pu}^{-1}(Fcl_{\mathrm{T}_2}(f_{pu}(g_A)))$. Hence, $f_{pu}(FScl(g_A)) \,|\, f_{pu}(f_{pu}^{-1}(Fcl_{\mathrm{T}_2}(f_{pu}(g_A)))) \,|\, Fcl_{\mathrm{T}_2}(f_{pu}(g_A)))$ from Theorem 2.3. Thus, $f_{pu}(FScl(g_A)) \,|\, Fcl_{\mathrm{T}_2}(f_{pu}(g_A))$.

$\mathbf{3 \to 4}$ Let $h_B \in FSS(Y)_K$ and $g_A = f_{pu}^{-1}(h_B)$. Then

$f_{pu}(FSclf_{pu}^{-1}(h_B)) | Fcl_{\mathrm{T}_2}(f_{pu}(f_{pu}^{-1}(h_B)))$ From (3). Hence,

$FScl(f_{pu}^{-1}(h_B)) | f_{pu}^{-1}(f_{pu}(FScl(f_{pu}^{-1}(h_B)))) | f_{pu}^{-1}(Fcl_{\mathrm{T}_2}(f_{pu}(f_{pu}^{-1}(h_B)))) | f_{pu}^{-1}(Fcl_{\mathrm{T}_2}(h_B))$

from Theorem 2.3. Thus, $FScl(f_{pu}^{-1}(h_B)) | f_{pu}^{-1}(Fcl_{\mathrm{T}_2}(h_B))$.

4 → 2 Let h_B be a fuzzy closed soft set over Y. Then $FScl(f_{pu}^{-1}(h_B)) | f_{pu}^{-1}(Fcl_{\mathrm{T}_2}(h_B)) = f_{pu}^{-1}(h_B) \ \forall h_B \in FSS(Y)_K$ from (4). But clearly, $f_{pu}^{-1}(h_B) | FScl(f_{pu}^{-1}(h_B))$. This means that, $f_{pu}^{-1}(h_B) = FScl(f_{pu}^{-1}(h_B)) \in FSCS(X)$.

1 → 5 Let $h_B \in FSS(Y)_K$. Then, $f_{pu}^{-1}(Fint_{\mathrm{T}_2}(h_B)) \in FSOS(X)$ from (1). Hence, $f_{pu}^{-1}(Fint_{\mathrm{T}_2}(h_B)) = FSint(f_{pu}^{-1}Fint_{\mathrm{T}_2}(h_B)) | FSint(f_{pu}^{-1}(h_B))$. Thus, $f_{pu}^{-1}(Fint_{\mathrm{T}_2}(h_B)) | FSint(f_{pu}^{-1}(h_B))$.

5 → 1 Let h_B be a fuzzy open soft set over Y. Then $Fint_{\mathrm{T}_2}(h_B) = h_B$ and $f_{pu}^{-1}(Fint_{\mathrm{T}_2}(h_B)) = f_{pu}^{-1}((h_B)) | FSint(f_{pu}^{-1}(h_B))$ from (5). But we have $FSint(f_{pu}^{-1}(h_B)) | f_{pu}^{-1}(h_B)$. This means that, $FSint(f_{pu}^{-1}(h_B)) = f_{pu}^{-1}(h_B) \in FSOS(X)$. Thus, f_{pu} is fuzzy semi continuous soft function.

Theorem 4.3 Let (X, T_1, E) and (Y, T_2, K) be fuzzy soft topological spaces and f_{pu} be a soft function such that $f_{pu} : FSS(X)_E \to FSS(Y)_K$. Then the following are equivalent:

1- f_{pu} is fuzzy semi open soft function.

2- $f_{pu}(Fint_{\mathrm{T}_1}(g_A)) | FSint(f_{pu}(g_A)) \ \forall g_A \in FSS(X)_E$.

Proof.

1 → 2 Let $g_A \in \mathrm{T}_1$. Then $f_{pu}(g_A) \in FSOS(Y) \forall g_A \in \mathrm{T}_1$ by (1). It follow that, $f_{pu}(Fint_{\mathrm{T}_1}(g_A)) = FSint(f_{pu}Fint_{\mathrm{T}_1}(g_A)) | FSint(f_{pu}(g_A))$. Therefore, $f_{pu}(Fint_{\mathrm{T}_1}(g_A)) | FSint(f_{pu}(g_A)) \ \forall g_A \in FSS(X)_E$.

2 → 1 Let $g_A \in \mathrm{T}_1$. By hypothesis, $f_{pu}(Fint_{\mathrm{T}_1}(g_A)) = f_{pu}(g_A) | FSint(f_{pu}(g_A)) \in FSOS(Y)$. Then, $f_{pu}(g_A) \in FSOS(Y) \ \forall g_A \in \mathrm{T}_1$. Hence, f_{pu} is fuzzy semi open soft function.

Theorem 4.4 Let $f_{pu} : FSS(X)_E \to FSS(Y)_K$ be a fuzzy semi open soft function. If $k_D \in FSS(Y)_K$ and $l_C \in \mathrm{T}'_1$ such that $f_{pu}^{-1}(k_D) | l_C$, then there exists $h_B \in FSCS(Y)$ such that $k_D | h_B$ and $f_{pu}^{-1}(h_B) | l_C$.

Proof. Let $k_D \in FSS(Y)_K$ and $l_C \in T'_1$ such that $f_{pu}^{-1}(k_D) \,|\, l_C$. Then, $f_{pu}(l'_C) \,|\, k'_D$ from Theorem 2.3 where $l'_C \in T_1$. Since f_{pu} is fuzzy semi open soft function. Then $f_{pu}(l'_C) \in FSOS(Y)$. Take $h_B = [f_{pu}(l'_C)]'$. Hence $h_B \in FSCS(Y)$ such that $k_D \,|\, h_B$ and $f_{pu}^{-1}(h_B) = f_{pu}^{-1}([f_{pu}(l'_C)]') \,|\, f_{pu}^{-1}(k'_D)' = f_{pu}^{-1}(k_D) \,|\, l_C$. This completes the proof.

Theorem 4.5 Let (X, T_1, E) and (Y, T_2, K) be fuzzy soft topological spaces and f_{pu} be a soft function such that $f_{pu} : FSS(X)_E \to FSS(Y)_K$. Then the following are equivalent:

1- f_{pu} is fuzzy semi closed soft function.

2- $FScl(f_{pu}(h_A)) \,|\, f_{pu}(Fcl_{T_1}(h_A)) \; \forall \; h_A \in FSS(X)_E$.

Proof.

1 → 2 Let $h_A \in T'_1$. Then $f_{pu}(h_A) \in FSCS(Y) \forall \; f_A \in T'_1$ by (1). Hence, $FScl(f_{pu}(h_A)) = f_{pu}(h_A) \,|\, f_{pu}(Fcl_{T_1}(h_A)) \; \forall \; h_A \in FSS(X)_E$.

2 → 1 Let $g_A \in T'_1$. By hypothesis, $FScl(f_{pu}(h_A)) \,|\, f_{pu}(Fcl_{T_1}(h_A)) = f_{pu}(h_A)$. Hence, $f_{pu}(h_A) \in FSCS(Y) \forall \; h_A \in T'_1$. Therefore, f_{pu} is fuzzy semi closed soft function.

5. Fuzzy soft semi separation axioms

Soft separation axioms for soft topological spaces were studied by Shabir and Naz [33]. Kandil et al. [17] introduced the notions of soft semi separation axioms in soft topological spaces. Here we introduce the notions of fuzzy semi connectedness in fuzzy soft topological spaces and study its basic properties.

Definition 5.1 Two fuzzy soft points $f_e = x_e^{\alpha}$ and $g_e = y_e^{\alpha}$ are said to be distinct if and only if $x \neq y$.

Definition 5.2 A fuzzy soft topological space (X, T, E) is said to be a fuzzy soft semi T_o-space if for every pair of distinct fuzzy soft points f_e, g_e there exists a fuzzy semi open soft set containing one but not the other.

Examples 5.1

1- Let $X = \{a, b\}$, $E = \{e_1, e_2\}$ and T be the discrete fuzzy soft topology on X. Then (X, T, E) is fuzzy soft semi T_o-space.

2- Let $X = \{a, b\}$, $E = \{e_1, e_2\}$ and T be the indiscrete fuzzy soft topology on X. Then T is not fuzzy soft semi T_o-space.

Theorem 5.1 A soft subspace (Y, T_Y, E) of a fuzzy soft semi T_o-space (X, T, E) is fuzzy soft semi T_o.

Proof. Let h_e, g_e be two distinct fuzzy soft points of T_Y. Then these fuzzy soft points are also in T. Hence there exists a fuzzy semi open soft set f_A in T containing one fuzzy soft point but not the other. Thus, $h_E^Y \tilde{\sqcap} f_A$ is fuzzy semi open soft set in (Y, T_Y, E) containing one fuzzy soft point but not the other from Definition 2.19. Therefore, (Y, T_Y, E) is fuzzy soft semi T_o.

Definition 5.3 A fuzzy soft topological space (X, T, E) is said to be a fuzzy soft semi T_1-space if for every pair of distinct fuzzy soft points f_e, g_e there exist fuzzy semi open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$, $g_e \tilde{\notin} f_A$; $f_e \tilde{\notin} g_B$, $g_e \tilde{\in} g_B$.

Example 5.1 Let $X = \{a,b\}$, $E = \{e_1, e_2\}$ and T be the discrete fuzzy soft topology on X. Then (X, T, E) is fuzzy soft semi T_1-space.

Theorem 5.2 A fuzzy soft subspace (Y, T_Y, E) of a fuzzy soft semi T_1-space (X, T, E) is fuzzy soft semi T_1.

Proof. It is similar to the proof of Theorem 5.1.

Theorem 5.3 If every fuzzy soft point of a fuzzy soft topological space (X, T, E) is fuzzy semi closed soft, then (X, T, E) is fuzzy soft semi T_1.

Proof. Suppose that $f_e = x_{e_i}^{\alpha}$ and $g_e = y_{e_j}^{\beta}$ $(i \neq j)$ be two distinct fuzzy soft points of T. Then we have two cases:

1. $\alpha, \beta \leq 0.5$.

Then we can always find some γ, δ such that $\alpha \leq \gamma$ and $\beta \leq \delta$. Hence $\alpha \leq 1-\gamma$ and $\beta \leq 1-\delta$. Thus, $x_{e_i}^{1-\gamma}$ and $y_{e_j}^{1-\delta}$ $(i \neq j)$ are two fuzzy semi open soft sets containing $x_{e_i}^{\alpha}$ and $y_{e_j}^{\beta}$ $(i \neq j)$ respectively. Therefore, (X, T, E) is fuzzy soft semi T_1.

2. $\alpha, \beta > 0.5$.

Then, we can always find some γ, δ such that $\alpha \leq \gamma$ and $\beta \leq \delta$. Hence $\alpha \leq 1-\gamma$ and $\beta \leq 1-\delta$. Thus, $x_{e_i}^{1-\gamma}$ and $y_{e_j}^{1-\delta}$ $(i \neq j)$ are two fuzzy semi open soft sets containing $x_{e_i}^{\alpha}$ and $y_{e_j}^{\beta}$ $(i \neq j)$ respectively. Therefore, (X, T, E) is fuzzy soft semi T_1.

Definition 5.4 A fuzzy soft topological space (X, T, E) is said to be a fuzzy soft semi T_2-space if for every pair of distinct fuzzy soft points f_e, g_e there exist disjoint fuzzy semi open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$ and $g_e \tilde{\in} g_B$.

Example 5.2 Let $X = \{a,b\}$, $E = \{e_1, e_2\}$ and T be the discrete fuzzy soft topology on X. Then (X, T, E) is fuzzy soft semi T_2-space.

Proposition 5.1 For a fuzzy soft topological space (X,T,E) we have:
fuzzy soft semi T_2-space $\Rightarrow$ fuzzy soft semi T_1-space $\Rightarrow$ fuzzy soft semi-T_o-space.

Proof.
1-Let (X,T,E) be a fuzzy soft semi T_2-space and f_e, g_e be two distinct fuzzy soft points. Then, there exist disjoint fuzzy semi open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$ and $g_e \tilde{\in} g_B$. Since $f_A \tilde{\cap} g_B = \tilde{0}_E$. Then $f_e \tilde{\notin} g_B$ and $g_e \tilde{\notin} f_A$. Therefore, there exist fuzzy semi open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$, $g_e \tilde{\notin} f_A$; and $f_e \tilde{\notin} g_B$, $g_e \tilde{\in} g_B$. Thus, (X,T,E) is fuzzy soft semi T_1-space.

2- Let (X,T,E) be a fuzzy soft semi T_1-space and f_e, g_e be two distinct fuzzy soft points. Then there exist fuzzy semi open soft sets f_A and g_B such that $f_e \tilde{\in} f_A$, $g_e \tilde{\notin} f_A$; and $f_e f_e \tilde{\notin} g_B$, $g_e \tilde{\in} g_B$. Then we have a fuzzy semi open soft set containing one of the fuzzy soft point but not the other. Thus, (X,T,E) is fuzzy soft semi T_o -space.

Theorem 5.4 Let (X,T,E) be a fuzzy soft topological space. If (X,T,E) is fuzzy soft semi T_2-space, then for every pair of distinct fuzzy soft points f_e, g_e there exists a fuzzy semi closed soft set b_A such that containing one of the fuzzy soft points $g_e \tilde{\in} b_A$, but not the other $f_e \tilde{\notin} b_A$ and $g_e \tilde{\notin} FScl(b_A)$.
,

Proof. Let f_e, g_e be two distinct fuzzy soft points. By assumption, there exists a fuzzy semi open soft sets b_A and h_B such that $f_e \tilde{\in} b_A$, $g_e \tilde{\in} h_B$. Hence, $g_e \tilde{\in} b_A'$ and $f_e \tilde{\notin} b_A'$ from Theorem 2.2. Thus, b_A' is a fuzzy semi closed soft set containing g_e but not f_e and $f_e \tilde{\notin} FScl\,(b_A') = b_A'$.

Theorem 5.5 A fuzzy soft subspace (Y,T_Y,E) of fuzzy soft semi T_2-space (X,T,E) is fuzzy soft semi T_2.

Proof. Let j_e, k_e be two distinct fuzzy soft points of T_Y. Then these fuzzy soft points are also in T. Hence there exist distinct fuzzy semi open soft sets f_A and g_B in T such that $j_e \in f_A$ and $k_e \in g_B$. Thus, $h_E^Y \tilde{\cap} f_A$ and $h_E^Y \tilde{\cap} g_B$ are disjoint fuzzy semi open soft sets in (Y,T_Y,E) such that $j_e \in h_E^Y \tilde{\cap} f_A$ and $k_e \in h_E^Y \tilde{\cap} g_B$. Therefore, (Y,T_Y,E) is fuzzy sssoft semi T_2.

Theorem 5.6 If every fuzzy soft point of a fuzzy soft topological space (X,T,E) is fuzzy semi closed soft, then (X,T,E) is fuzzy soft semi T_2.

Proof. It similar to the proof of Theorem 5.3.

Definition 5.5 Let (X,T,E) be a fuzzy soft topological space, h_C be a fuzzy semi closed soft set and g_e be a fuzzy soft point such that $g_e \tilde{\notin} h_C$. If there exist fuzzy semi open soft sets f_S and f_W such that $g_e \tilde{\in} f_S$, $g_B \sqsubseteq f_W$ and $f_S \sqcap f_W = \tilde{0}_E$, then (X, T, E) is called fuzzy soft semi regular space. A fuzzy soft semi regular T_1-space is called fuzzy soft semi T_3-space.

Proposition 5.2 Let (X, T, E) be a fuzzy soft topological space, h_C be a fuzzy semi closed soft set and g_e be a fuzzy soft point such that $g_e \tilde{\notin} h_C$. If (X, T, E) is fuzzy soft semi regular space, then there exists a fuzzy semi open soft set f_A such that $g_e \tilde{\in} f_A$ and $f_A \sqcap h_C = \tilde{0}_E$.

Proof. Obvious from Definition 5.5.

Theorem 5.7 Let (X, T, E) be a fuzzy soft semi regular space in which every fuzzy soft point is fuzzy semi closed soft. Then for a fuzzy semi open soft set g_B containing fuzzy soft point f_e, there exists a fuzzy semi open soft set f_S containing f_e such that $FScl(f_S) \sqsubseteq g_B$.

Proof. Let g_B be a fuzzy semi open soft set containing fuzzy soft point f_e in a fuzzy soft semi regular space (X, T, E). Then g'_B is fuzzy semi closed soft such that $f_e \tilde{\notin} g'_B$ from Theorem 2.2. By hypothesis, there exists disjoint fuzzy semi open soft sets f_S and f_W such that $f_e \tilde{\in} f_S$ and $g'_B \sqsubseteq f_W$. It follows that $f'_W \sqsubseteq g_B$ and $f_S \sqsubseteq f'_W$. Thus, $FScl(f_S) \sqsubseteq f'_W \sqsubseteq g_B$. Therefore, there exists a fuzzy semi open soft set f_S containing f_e such that $FScl(f_S) \sqsubseteq g_B$.

Theorem 5.8 A fuzzy soft subspace (Y, T_Y, E) of a fuzzy soft semi T_3-space (X, T, E) is fuzzy soft semi T_3.

Proof. By Theorem 5.2, (Y, T_Y, E) is fuzzy soft semi T_1-space. Now we want to prove that (Y, T_Y, E) is fuzzy soft semi regular space. Let k_E be a fuzzy semi closed soft set in T_Y and g_e be a fuzzy soft point in T_Y such that $g_e \tilde{\notin} k_E$. Then k_E is also fuzzy semi closed soft set in T and g_e is also fuzzy soft point in T such that $g_e \tilde{\notin} k_E$. Since (X, T, E) is fuzzy soft semi T_3. Then there exist disjoint fuzzy semi open soft sets f_S and f_W in T such that $g_e \tilde{\in} f_S$ and $k_E \sqsubseteq f_W$. It follows that $h_E^Y \sqcap f_S$ and $h_E^Y \sqcap f_W$ are disjoint fuzzy semi open soft sets in T_Y such that $g_e \tilde{\in} h_E^Y \sqcap f_S$ and $k_E \sqsubseteq h_E^Y \sqcap f_W$. Therefore, (Y, T_Y, E) is fuzzy soft semi T_3.

Definition 5.6 Let (X, T, E) be a fuzzy soft topological space, h_C, g_B be disjoint fuzzy semi closed soft sets. If there exist disjoint fuzzy semi open soft sets f_S and f_W such that $h_C \sqsubseteq f_S$, $g_B \sqsubseteq f_W$. Then (X, T, E) is called fuzzy soft semi normal space. A fuzzy soft semi normal T_1-space is called fuzzy soft semi T_4-space.

Theorem 5.9 Let (X,T,E) be a fuzzy soft topological space. Then the following are equivalent:

1- (X,T,E) is fuzzy soft semi normal space.

2- For every fuzzy semi closed soft set h_C and fuzzy semi open soft set g_B such that $h_C \sqsubseteq g_B$, there exists a fuzzy semi open soft set f_S such that $h_C \sqsubseteq f_S$, $FScl(f_S) \sqsubseteq g_B$.

Proof.

$\mathbf{1 \to 2}$ Let h_C be a semi closed soft set and g_B be a fuzzy semi open soft set such that $h_C \sqsubseteq g_B$. Then h_C, g'_B are disjoint fuzzy semi closed soft sets. It follows by (1), there exist disjoint fuzzy semi open soft sets f_S and f_W such that $h_C \sqsubseteq f_S$, $g'_B \sqsubseteq f_W$. Now, $f_S \sqsubseteq f'_W$, so $FScl(f_S) \sqsubseteq FSclf'_W = f'_W$, where g_B is fuzzy semi open soft set. Also, $f'_W \sqsubseteq g_B$. Hence $FScl(f'_S) \sqsubseteq f'_W \sqsubseteq g_B$. Thus, $h_C \sqsubseteq f_S$, $FScl(f_S) \sqsubseteq g_B$.

$\mathbf{2 \to 1}$ Let g_A and g_B be disjoint fuzzy semi closed soft sets. Then, $g_A \sqsubseteq g'_B$. By hypothesis, there exists a fuzzy semi open soft set f_S such that $g_A \sqsubseteq f_S$, $FScl(f_S) \sqsubseteq g'_B$. So $g_B \sqsubseteq [FScl(f_S)]'$, $g_A \sqsubseteq f_S$ and $[FScl(f_S)]' \sqcap f_S = \tilde{0}_E$, where f_S and $[FScl(f_S)]'$ are fuzzy semi open soft sets. Thus, (X,T,E) is fuzzy soft semi normal space.

Theorem 5.10 A fuzzy semi closed soft subspace (Y,T_Y,E) of a fuzzy soft semi normal space (X,T,E) is fuzzy soft semi normal.

Proof. Let g_A and g_B be disjoint fuzzy semi closed soft sets in T_Y. Then g_A and g_B are disjoint fuzzy semi closed soft sets in T. Since (X,T,E) is fuzzy soft semi normal. Then there exist disjoint fuzzy semi open soft sets f_S and f_W in T such that $g_A \sqsubseteq f_S$, $g_B \sqsubseteq f_W$. It follows that $h_E^Y \sqcap f_S$ and $h_E^Y \sqcap f_W$ are disjoint fuzzy semi open soft sets in T_Y such that $g_e \tilde{\in} h_E^Y \sqcap f_S$ and $k_E \sqsubseteq h_E^Y \sqcap f_W$. Therefore, (Y,T_Y,E) is fuzzy soft semi normal.

Theorem 5.11 Let (X,T_1,E) and (Y,T_2,K) be fuzzy soft topological spaces and $f_{pu}: SS(X)_E \to SS(Y)_K$ be a fuzzy soft function which is surjective, injective, fuzzy irresolute soft and fuzzy irresolute open soft. If (X,T_1,E) is soft semi normal space, then (Y,T_2,K) is also fuzzy soft semi normal space.

Proof. Let f_A, g_B be disjoint fuzzy semi closed soft sets in Y. Since f_{pu} is fuzzy irresolute soft, then $f_{pu}^{-1}(f_A)$ and $f_{pu}^{-1}(g_B)$ are fuzzy semi closed soft set in X such that $f_{pu}^{-1}(f_A) \sqcap f_{pu}^{-1}(g_B) = f_{pu}^{-1}[f_A \sqcap g_B] = f_{pu}^{-1}[\tilde{0}_K] = \tilde{0}_E$ from Theorem 2.3. By hypothesis, there exist disjoint fuzzy semi open soft sets k_C and h_D in X such that $f_{pu}^{-1}(f_A) \sqsubseteq k_C$ and $f_{pu}^{-1}(g_B) \sqsubseteq h_D$. It follows that $f_A = f_{pu}[f_{pu}^{-1}(f_A)] \sqsubseteq f_{pu}(k_C)$, $g_B = f_{pu}[f_{pu}^{-1}(g_B)] \sqsubseteq f_{pu}(h_D)$ from Theorem 2.3 and

$f_{pu}(k_C) \tilde{\sqcap} f_{pu}(h_D) = f_{pu}[k_C \tilde{\sqcap} h_D] = f_{pu}[\tilde{0}_E] = \tilde{0}_K$ from Theorem 2.3. Since f_{pu} is fuzzy irresolute open soft function. Then $f_{pu}(k_C), f_{pu}(h_D)$ are fuzzy semi open soft sets in Y . Thus, (Y, T_2, K) is fuzzy soft semi normal space.

6 . Conclusion

Topology is an important and major area of mathematics and it can give many relationships between other scientific areas and mathematical models. Recently, many scientists have studied the soft set theory, which is initiated by Molodtsov [25] and easily applied to many problems having uncertainties from social life. In the present work, we have continued to study the properties of fuzzy soft topological spaces. We introduce the some new concepts in fuzzy soft topological spaces such as fuzzy semi open soft sets, fuzzy semi closed soft sets, fuzzy semi soft interior, fuzzy semi soft closure and fuzzy semi separation axioms and have established several interesting properties. Since the authors introduced topological structures on fuzzy soft sets [8,15, 35], so the semi topological properties, which introduced by Kandil et al.[17], is generalized here to the fuzzy soft sets which will be useful in the fuzzy systems. Because there exists compact connections between soft sets and information systems [29,37], we can use the results deducted from the studies on fuzzy soft topological space to improve these kinds of connections. We hope that the findings in this paper will help researcher enhance and promote the further study on fuzzy soft topology to carry out a general framework for their applications in practical life.

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