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Homomorphism of intuitionistic $(\alpha,\beta)-$fuzzy H_v-submodule

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Abstract: The notion of intuitionistic fuzzy sets was introduced by Atanassov as a generalization of the notion of fuzzy sets. Using the notion of ”belongingness $(\in)$” and ”quasi-coincidence (q)” of fuzzy points with fuzzy sets, we introduce the concept of an intuitionistic (α,β)-fuzzy H_v-submodules of an H_v-modules, where $\alpha\in\{\in,q\}$, $\beta\in\{\in,q,\in\vee q,\in\wedge q\}$. The concept of a homomorphism of intuitionistic (α,β)-fuzzy H_v-submodule is considered, and some interesting properties are investigated.

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1. Introduction

Hyperstructures represent a natural extension of classical algebraic structures and they were introduced by the French mathematician Marty in 1934 [26].
Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Since then, hundreds of papers and several books have been written on this topic, see [15, 17, 22, 29]. A short review of this theory appears in [15]. A recent book on hyperstructures [17] points out their applications in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Vougiouklis [29] introduced a new class of hyperstructures,

the so-called H_v-structures. The H_v-structures are hyperstructures where equality is replaced by non-empty intersection.
Another book [22] is devoted especially to the study of hyperring theory. Several kinds of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. The theory of suitable modified hyperstructures can serve as a mathematical background in the field of quantum communication systems.

Given a set H, a fuzzy subset of H (or a fuzzy set in H) is, by definition, an arbitrary mapping $\mu : H \rightarrow [0,1]$ where $[0,1]$ is the closed interval in reals whose endpoints are 0 and 1. This important concept of a fuzzy set has been introduced by Zadeh in [31]. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications (see, for example, [1, 14, 16, 24]).
After the introduction of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [2, 6] is one among them. An intuitionistic fuzzy set gives both a membership degree and a non-membership degree. The membership and non-membership values induce an indeterminacy index, which models the hesitancy of deciding the degree to which an object satisfies a particular property. As the basis for the study of intuitionistic fuzzy set theory, many operations and relations over intuitionistic fuzzy sets were introduced [3-6]. Many concepts in fuzzy set theory were also extended to intuitionistic fuzzy set theory, such as intuitionistic fuzzy relations, intuitionistic L-fuzzy sets, intuitionistic fuzzy implications, intuitionistic fuzzy grade of hypergroups, intuitionistic fuzzy logics, and the degree of similarity between intuitionistic fuzzy sets, etc., [12, 13, 18]. Cristea and Davvaz in [18] introduced connections between hypergroupoids and Atanassov's intuitionistic fuzzy. In [7] Biswas applied the concept of intuitionistic fuzzy sets to the theory of groups and studied intuitionistic fuzzy subgroups of a group. Davvaz et al. [23] considered the intuitionistic fuzzy sets for H_v-modules. Recently, Dudek et al. [24] considered the intuitionistic fuzzification of sub-hyperquasigroups in a hyperquasigroup and investigated some properties of such hyperquasigroups.
The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [27], played a vital role to generate some different types of fuzzy subgroups. Bhakat and Das [9, 10] gave the concepts of (α, β)-fuzzy subgroups by using the notion of "belongingness $(\in)$" and "quasi-coincidence (q)" between a fuzzy point and a fuzzy subgroup, where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \vee q,$ and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. In [11] $(\in, \in \vee q)$-fuzzy subrings and ideals defined. In [20] Davvaz defined $(\in, \in \vee q)$-fuzzy subnearring and ideals of a near ring. In [25] Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup.
In [8] Bhakat defined $(\in \vee q)$-level subsets of a fuzzy set. In [28] Shabir, Jun et al. studied characterizations of regular semigroups by (α, β)-fuzzy ideals.

In [30] Yuan, Li et al. redefined (α,β)-intuitionistic fuzzy subgroups. Davvaz and Corsini initiated the study of (α,β)-Fuzzy H_v-Ideals of H_v-Rings in [21]. This paper continues this line of research.

The paper is organized as follows: in Section 2 some fundamental definitions on fuzzy sets and IF sets are explored, and in Section 3 we present some fundamental definitions on H_v-structures, in Section 4 we define intuitionistic (α,β)-fuzzy with H_v-submodules. Section 5 we define homomorphisms of intuitionistic (α,β)-fuzzy H_v-submodules, and give several examples and then establish some useful theorems.

2. Fuzzy sets and intuitionistic fuzzy sets

The concept of a fuzzy set in a non-empty set was introduced by Zadeh [31] in 1965. Let H be a non-empty set. A mapping $\mu : H \to [0,1]$ is called a fuzzy set in H. The complement of μ, denoted by μ^c, is the fuzzy set in H given by $\mu^c(x) = 1-\mu(x)$ for all $x \in H$.

For any $t \in [0,1]$ and fuzzy set μ of H, the set

$$U(\mu,t) = \{x \in H \mid \mu(x) \geq t\} \text{ (respectively, } L(\mu,t) = \{x \in H \mid \mu(x) \leq t\}\text{)}$$

is called an upper (respectively, lower) t-level cut of μ.

Definition 2.1. Let f be a mapping from a set X into a set Y. Let μ be a fuzzy set in X and λ be a fuzzy set in Y. Then the inverse image $f^{-1}(\lambda)$ of λ is a fuzzy set in X defined by

$$f^{-1}(\lambda)(x) = \lambda(f(x)) \text{ for all } x \in X.$$

The image $f(\mu)$ of μ is the fuzzy set in Y defined by

$$f(\mu)(y) = \begin{cases} \sup_{x\in f^{-1}(y)} \mu(x), & f^{-1}(y) \neq \phi \\ 0, & f^{-1}(y) \neq \phi \end{cases}$$

for all $y \in Y$.

We have always $f(f^{-1}(\lambda)) \leq \lambda$ and $\mu \leq f^{-1}(f(\mu))$.

Definition 2.2. An intuitionistic fuzzy set A in a non-empty set X is an object having the form

$$A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\},$$

where the functions $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\lambda_A(x)$) of each element $x \in X$ with respect to the set A, respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$. For the sake of simplicity, we shall use the symbol $A=(\mu_A,\lambda_A)$ for the intuitionistic fuzzy set $A=\{(x,\mu_A(x),\lambda_A(x))\,|\,x\in X\}$.

3. H_v-Structures

Let H be a nonempty set and let $P^{\bullet}(H)$ be the set of all nonempty subsets of H. A hyperoperation on H is a map $.: H\times H\to P^{\bullet}(H)$ and the couple $(H,\cdot)$ is called a hypergroupoid (or hyperstructure). If *A* and *B* are nonempty subsets of *H*, then we denote

$$x\cdot B=\{x\}\cdot B \quad ; \quad A\cdot x=A\cdot\{x\} \quad ; \quad A\cdot B=\bigcup_{\substack{a\in A\\ b\in B}} a\cdot b$$

A hypergroupoid $(H,\cdot)$ is called a semihypergroup if for all $x,y,z\in H,$ we have $x\cdot(y\cdot z)=(x\cdot y)\cdot z,$ which means that

$$\bigcup_{u\in x.y} u.z=\bigcup_{v\in y.z} x.v$$

We say that a semihypergroup $(H,\cdot)$ is a hypergroup if for all $x\in H,$ we Have $x\cdot H=H\cdot x=H.$

A hyperstructure $(H,\cdot)$ is called an H_v-semigroup if

$$(x\cdot y)\cdot z\cap x\cdot(y\cdot z)\neq\phi,$$

for all $x,y,z\in H.$

Definition 3.1. [29] An H_v-ring is a system $(R,+,\cdot)$ with two hyperoperations satisfying the following axioms:

(i) $(R,+)$ is an H_v-group, i.e.,

$$(x+y)+z\cap x+(y+z)\neq\phi,\ \text{for all } x,y,z\in R,$$
$$x+R=R+x=R\ \text{ for all } x\in R,$$

(ii) $(R,\cdot)$ is an H_v-semigroup,

(iii) "." is weak distributive with respect to " + ", i.e., for all $x,y,z\in R,$

$$x.(y+z)\cap(x.y)+(x.z)\neq\phi\ ,\ (x+y).z\cap(x.z)+(y.z)\neq\phi.$$

An H_v-group $(R,+)$ is called a weak commutative H_v-group if

$$(y+x)\cap(x+y)\neq\phi,\ \text{for all } x,y\in R.$$

Definition 3.2. [29] A nonempty set *M* is called an H_v-module over an H_v-ring *R* if $(M,+)$ is a weak commutative H_v-group and there exists a map

$$.: R\times M\to P^{\bullet}(M),\ (r,x)\mapsto r.x$$

such that for all $a,b\in R$ and $x,y\in M,$ we have

$$(a.(x+y))\cap(a.x+a.y)\neq\phi,$$
$$((x+y).a)\cap(x.a+y.a)\neq\phi,$$
$$(a.(b.x))\cap((a.b).x)\neq\phi.$$

We note that an H_v-module is a generalization of a module. For more definitions, results and applications on H_v-ring, we refer the reader to [29]. Note that by using fuzzy sets, we can consider the structure of H_v-module on any ordinary module.

Example 3.3. [19] Let M be an ordinary module over an ordinary ring R, and let μ_A be a fuzzy set in M and μ_B be a fuzzy set in R. We define Hyperoperations "$\oplus,\otimes,*$" and "$.$" as follows:

$$x\oplus y=\{z\in R\mid \mu_A(z)=\mu_A(x+y)\}\text{ for all } x,y\in R,$$
$$x\otimes y=\{z\in R\mid \mu_A(z)=\mu_A(x.y)\}\text{ for all } x,y\in R,$$
$$x*y=\{z\in M\mid \mu_B(z)=\mu_B(x+y)\}\text{ for all } x,y\in M,$$
$$x.y=\{z\in M\mid \mu_B(z)=\mu_B(r.x)\}\text{ for all } r\in R\text{ and } x\in M,$$

respectively. Then

(i) $(R,\oplus,\otimes)$ is an H_v-ring.

(ii) $(R,*,.)$ is an H_v-module over the H_v-ring $(R,\oplus,\otimes)$.

Let M be an H_v-module over an H_v-ring R. A non-empty subset S of M is called an left (right) H_v-submodule if the following conditions hold:

(i) $(S,+)$ is an H_v-subgroup of $(M,+)$,

(ii) $R.S\subseteq S$ $(S.R\subseteq S)$.

Definition 3.4. [23] An intuitionistic fuzzy set $A=(\mu_A,\lambda_A)$ in M is called an intuitionistic fuzzy H_v-submodule of M if

(1) $\mu_A(x)\wedge\mu_A(y)\leq\bigwedge_{z\in x+y}\mu_A(z)$ for all $x,y\in M$,

(2) for all $a,x\in M$, there exist $y,z\in M$ such that $x\in(a+y)\cap(z+a)$ and $\mu_A(x)\wedge\mu_A(a)\leq\mu_A(y)\wedge\mu_A(z)$,

(3) $\mu_A(y)\leq\bigwedge_{z\in x.y}\mu_A(z)$ for all $y\in M$ and $x\in R$,

(4) $\lambda_A(x)\vee\lambda_A(y)\geq\bigvee_{z\in x+y}\lambda_A(z)$ for all $x,y\in M$,

(5) for all $a,x\in M$, there exist $y,z\in M$ such that $x\in(a+y)\cap(z+a)$ and $\lambda_A(x)\vee\lambda_A(a)\geq\lambda_A(y)\vee\lambda_A(z)$,

(6) $\lambda_A(y)\geq\bigvee_{z\in x.y}\lambda_A(z)$ for all $y\in M$ and $x\in R$.

Notation 3.5. Let μ_A be a fuzzy set in M. We defind

$$\mathrm{Im}(\mu_A) = \{y \in M \mid \exists x \in M, \mu_A(x) = y\}.$$

4. Intuitionistic $(\alpha,\beta)-$Fuzzy H_v-Submodules

Definition 4.1. [9] Let μ be a fuzzy subset of R. If there exist a $t \in (0,1]$ and an $x \in R$ such that

$$\mu(y) = \begin{cases} t, & y = x \\ 0, & y \neq x \end{cases}$$

Then μ is called a fuzzy point with support x and value t and is denoted by x_t.

Definition 4.2. [9] Let μ be a fuzzy subset of R and x_t be a fuzzy point.
(1) If $\mu(x) \geq t$, then we say x_t belongs to μ, and write $x_t \in \mu$.
(2) If $\mu(x) + t > 1$, then we say x_t is quasi-coincident with μ, and write $x_t q\mu$.
(3) $x_t \in \vee q\mu \Leftrightarrow x_t \in \mu$ or $x_t q\mu$.
(4) $x_t \in \wedge q\mu \Leftrightarrow x_t \in \mu$ and $x_t q\mu$.

In what follows, unless otherwise specified, α and β will denote any one of $\in, q, \in\vee q$ or $\in\wedge q$ with $\alpha \neq \in\wedge q$, which was introduced by Bhakat and Das [10]. Indeed, the most viable generalization of Rosenfelds fuzzy subgroup is the (α,β)-fuzzy subgroup. Based on [9], we can extend the concept of (α,β)-fuzzy subgroups to the concept of intuitionistic (α,β)-fuzzy H_v-submodule.

Definition 4.3. [21] Let R be an H_v-ring. A fuzzy subset A of R is said to be an (α,β)-fuzzy left (right) H_v-ideals of R if for all $t, r \in (0,1]$,
(1) $x_r, y_t \alpha A$ implies $z_{r\wedge t}\beta A$ for all $z \in x + y$,
(2) $x_r, a_t \alpha A$ implies $y_{r\wedge t}\beta A$ for some $y \in R$ with $x \in a + y$,
(3) $x_r, a_t \alpha A$ implies $z_{r\wedge t}\beta A$ for some $z \in R$ with $x \in z + a$,
(4) $y_t \alpha A$ and $x \in R$ imply $z_t \beta A$ for all $z \in x.y$
($x_t \alpha A$ and $y \in R$ imply $z_t \beta A$ for all $z \in x.y$).

In what follows, let M denote an H_v-module over an H_v-ring R unless other wise specified. We start by defining the notion of intuitionistic (α,β)-fuzzy H_v-submodules.

Definition 4.4. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in M is said to be an intuitionistic (α,β)-fuzzy left (right) H_v-submodule of M if for all $t,r\in(0,1]$,

(1) For all $x,y\in M$, $x_t,y_r\alpha\mu_A$ implies $z_{t\wedge r}\beta\mu_A$ for all $z\in x+y$,

(2) For all $x,a\in M$, $x_t,a_r\alpha\mu_A$ implies $(y\wedge z)_{t\wedge r}\beta\mu_A$ for some $y,z\in M$ with $x\in(a+y)\bigcap(z+a)$,

(3) For all $x\in R, y\in M$, $y_t\alpha\mu_A$ implies $z_t\beta\mu_A$ for all $z\in x.y$,

(For all $x\in R, y\in M$, $y_t\alpha\mu_A$ implies $z_t\beta\mu_A$ for all $z\in y.x$),

(4) For all $x,y\in M$, $x_t,y_r\overline{\alpha}\lambda_A$ implies $z_{t\wedge r}\overline{\beta}\lambda_A$ for all $z\in x+y$,

(5) For all $x,a\in M$, $x_t,a_r\overline{\alpha}\lambda_A$ implies $(y\wedge z)_{t\wedge r}\overline{\beta}\lambda_A$ for some $y,z\in M$ with $x\in(a+y)\bigcap(z+a)$,

(6) For $x\in R, y\in M$, $y_t\overline{\alpha}\lambda_A$ implies $z_t\overline{\beta}\lambda_A$ for all $z\in x.y$,

(For all $x\in R, y\in M$, $y_t\overline{\alpha}\lambda_A$ implies $z_t\overline{\beta}\lambda_A$ for all $z\in y.x$),

where $(y\wedge z)_{t\wedge r}$, i.e., $y_{t\wedge r}$ and $z_{t\wedge r}$. Also, the symbol $\overline{\beta}$ means β does not hold for all $\beta\in\{\in,q,\in\vee q,\in\wedge q\}$.

Let R be an H_v-ring. Then a fuzzy subset λ_A of M is said to be an anti (α,β)-fuzzy left (right) H_v-submodule of M if it satisfies the conditions (4)-(6) of Definition 4.4.

In this paper we present all the proofs for left H_v-submodules. Similar results hold for right H_v-submodules.

Example 4.5. Let $M=\{a,b,c,d\}$ and $R=\{a,b,c\}$. Let operation "." and hyperoperation "+" and defied by the following tables

0	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	c	c	c
d	a	d	d	d

$+$	a	b	c	d
a	a	b	c	d
b	b	$\{a,b\}$	d	c
c	c	d	$\{a,c\}$	b
d	d	c	b	$\{a,d\}$

Let μ and λ be two fuzzy subset of M such that $\mu(a)=0.6$ $\mu(d)=\mu(c)=\mu(b)=0.8$ and $\lambda(a)=\lambda(b)=\lambda(c)=\lambda(d)=0.3$. Then (μ,λ) is an intuitionistic $(\in,\in\vee q)$-fuzzy H_v-submodule of M.

Proof. μ is an $(\in,\in\vee q)$-fuzzy H_v-ideal of M (see [21]). So, it is easy to see that λ satisfies the conditions (4)-(6) of Definition 4.4.

Lemma 4.6. Let $A=(\mu_A,\lambda_A)$ be an intuitionistic fuzzy set in M. Then for all $x\in M$ and $t\in(0,1]$, we have

(1) $x_t q\mu_A \Leftrightarrow x_t \,\overline{\in}\, \mu_A^C$.

(2) $x_t \in\vee q\mu_A \Leftrightarrow x_t \overline{\in\wedge q}\mu_A^C$.

Proof.

(1) Let $x\in M$ and $t\in(0,1]$. Then, we have

$$x_t q\mu_A \Leftrightarrow \mu_A(x)+t>1$$
$$\Leftrightarrow 1-\mu_A(x)<t$$
$$\Leftrightarrow \mu_A^C(x)<t$$
$$\Leftrightarrow x_t \,\overline{\in}\, \mu_A^C$$

(2) Let $x\in M$ and $t\in(0,1]$. Then, we have

$$x_t \in\vee q\mu_A \Leftrightarrow x_t \in\mu_A \text{ or } x_t q\mu_A$$
$$\Leftrightarrow \mu_A(x)\geq t \text{ or } \mu_A(x)+t>1$$
$$\Leftrightarrow 1-\mu_A^C(x)\geq t \text{ or } \mu_A^C(x)<t$$
$$\Leftrightarrow x_t\overline{q}\mu_A^C \text{ or } x_t \,\overline{\in}\, \mu_A^C$$
$$\Leftrightarrow x_t \overline{\in\wedge q}\,\mu_A^C.$$

Let $\beta=\in,q,\in\vee q,\in\wedge q.$ We write $\beta'=q,\in,\in\wedge q,\in\vee q,$ respectively. It is obvious that $\beta''=\beta.$

Definition 4.7. Let $t\in[0,1]$ and μ is a fuzzy set in M. Then, the set

$U(\alpha\mu,t)=\{x\in M \mid x_t\alpha\mu\}$ (respectively, $L(\alpha\mu,t)=\{x\in M \mid x_t\overline{\alpha}\mu\}$),

is called an upper (respectively, lower) t-level cut of $\alpha\mu$.

Theorem 4.8. Let $A=(\mu_A,\lambda_A)$ is an intuitionistic (α,β)-fuzzy H_v-submodule of M, then the set $U(\alpha\mu_A,t)$ $(U(\alpha'\mu_A,t))$ is an H_v-submodule of M for all $t\in \mathrm{Im}(\mu_A)$, where

$(\alpha,\beta)\in\{(\in,\in),(q,q),(\in,\in\wedge q),(q,\in\wedge q)\}$ $((\alpha,\beta)\in\{(\in,\in\wedge q),(q,\in\wedge q)\})$.

Proof. We only prove the case of $(\alpha,\beta)=(\in,\in\wedge q).$ The others are analogous. We must show that

(i) $a+U(\in\mu_A,t)=U(\in\mu_A,t)+a=U(\in\mu_A,t)$ for all $U(\in\mu_A,t)$,

(ii) $R.U(\in \mu_A, t) \subseteq U(\in \mu_A, t)$.

Case (i). Suppose that $t \in \mathrm{Im}(\mu_A) \subseteq [0,1]$ and let $a, x \in U(\in \mu_A, t)$. By definition, we have $a_t \in \mu_A$ and $x_t \in \mu_A$. Hence $\mu_A(a) \geq t$ and $\mu_A(x) \geq t$.
Since μ_A is an $(\in, \in \wedge q)$-fuzzy H_v-submodule of M. It follows from condition (1) of Definition 4.4 that $z_t \in \wedge q \mu_A$ for all $z \in a + x$ and $z \in x + a$. Which implies

$$z_t \in \mu_A \text{ for all } z \in a + x \text{ and } z \in x + a.$$

Therefore

$$a + x \subseteq U(\in \mu_A, t) \text{ and } x + a \subseteq U(\in \mu_A, t).$$

On the other hand, since $a, x \in U(\in \mu_A, t)$. Thus $a_t, x_t \in \mu_A$. By condition (2) of Definition 4.4, we have

$$(y \wedge z)_t \in \wedge q \mu_A \text{ for some } y, z \in M \text{ with } x \in (a + y) \cap (z + a),$$

which implies

$$y_t \in \mu_A \text{ and } z_t \in \mu_A.$$

Thus $y \in U(\in \mu_A, t)$ and $z \in U(\in \mu_A, t)$. This proves that

$$U(\in \mu_A, t) \subseteq a + U(\in \mu_A, t) \text{ and } U(\in \mu_A, t) \subseteq U(\in \mu_A, t) + a,$$

for all $a \in U(\in \mu_A, t)$.
Case (ii). Let $x \in R$, $y \in U(\in \mu_A, t)$. Hence $y_t \in \mu_A$. Since $A = (\mu_A, \lambda_A)$ is an intuitionistic $(\in, \in \wedge q)$-fuzzy H_v-submodule of M. It follows from conditions (3) of Definition 4.4 that

$$z_t \in \wedge q \mu_A \text{ for all } z \in x.y.$$

Thus $z_t \in \mu_A$. Then $z \in U(\in \mu_A, t)$ and so

$$x.y \subseteq U(\in \mu_A, t).$$

Therefore

$$R.U(\in \mu_A, t) \subseteq U(\in \mu_A, t).$$

This completes the proof.

Corollary 4.9. Let μ_A is an (α, β)-fuzzy H_v-submodule of M, then the set $U(\alpha \mu_A, t)$ $(U(\alpha' \mu_A, t))$ is an H_v-submodule of M for all $t \in \mathrm{Im}(\mu_A)$, where $(\alpha, \beta) \in \{(\in, \in), (q, q), (\in, \in \wedge q), (q, \in \wedge q)\}$ $((\alpha, \beta) \in \{(\in, \in \wedge q), (q, \in \wedge q)\})$.

Theorem 4.10. Let $A = (\mu_A, \lambda_A)$ is an intuitionistic (α, β)-fuzzy H_v-submodule of M, then the set $L(\alpha \lambda_A, t)$ $(L(\alpha' \lambda_A, t))$ is an H_v-submodule of M for all $t \in \mathrm{Im}(\lambda_A)$, where
$(\alpha, \beta) \in \{(\in, \in), (q, q), (\in, \in \vee q), (q, \in \vee q)\}$ $((\alpha, \beta) \in \{(\in, \in \vee q), (q, \in \vee q)\})$.

Proof. We only show $(\alpha, \beta) = (\in, \in \vee q)$. We must show that
(i) $a + L(\in \lambda_A, t) = L(\in \lambda_A, t) + a = L(\in \lambda_A, t)$ for all $L(\in \lambda_A, t)$,
(ii) $R.L(\in \lambda_A, t) \subseteq L(\in \lambda_A, t)$.

Case (i). Suppose that $t \in \mathrm{Im}(\lambda_A) \subseteq [0,1]$ and let $a, x \in L(\in \mu_A, t)$. By definition, we have $a_t \overline{\in} \lambda_A$ and $x_t \overline{\in} \lambda_A$. Hence $\lambda_A(a) < t$ and $\lambda_A(x) < t$. Since $A = (\mu_A, \lambda_A)$ is an intuitionistic $(\in, \in \vee q)$ -fuzzy H_v -submodule of *M*. It follows from condition (4) of Definition 4.4 that

$$z_t \overline{\in \vee q} \lambda_A \text{ for all } z \in a + x \text{ and } z \in x + a.$$

Which implies

$$z_t \overline{\in} \lambda_A \text{ for all } z \in a + x \text{ and } z \in x + a.$$

Therefore

$$a + x \subseteq L(\in \mu_A, t) \text{ and } x + a \subseteq L(\in \mu_A, t).$$

On the other hand, since $a, x \in L(\in \mu_A, t)$. Thus $a_t, x_t \overline{\in} \lambda_A$. By condition (5) of Definition 4.4, implies

$$(y \wedge z)_t \overline{\in \vee q} \lambda_A \text{ for some } y, z \in M \text{ with } x \in (a + y) \cap (z + a),$$

which implies

$$y_t \overline{\in} \lambda_A \text{ and } z_t \overline{\in} \lambda_A.$$

Thus $y \in L(\in \lambda_A, t)$ and $z \in L(\in \lambda_A, t)$. This proves that

$$L(\in \lambda_A, t) \subseteq a + L(\in \lambda_A, t) \text{ and } L(\in \lambda_A, t) \subseteq L(\in \lambda_A, t) + a,$$

for all $a \in L(\in \lambda_A, t)$.

Case (ii). Let $x \in R$, $y \in L(\in \lambda_A, t)$. Hence $y_t \overline{\in} \lambda_A$. Since λ_A is an anti $(\in, \in \vee q)$ -fuzzy H_v -submodule of *M*. It follows from conditions (6) of Definition 4.4 that

$$z_t \overline{\in \vee q} \lambda_A \text{ for all } z \in x.y.$$

Then $z \in L(\in \lambda_A, t)$ and so

$$x.y \subseteq L(\in \lambda_A, t).$$

Therefore

$$R.L(\in \lambda_A, t) \subseteq L(\in \lambda_A, t).$$

This completes the proof.

Corollary 4.11. Let λ_A is an anti (α, β) -fuzzy H_v-submodule of *M*, then the set $L(\alpha \lambda_A, t)$ $(L(\alpha' \lambda_A, t))$ is an H_v -submodule of *M* for all $t \in \mathrm{Im}(\lambda_A)$, where $(\alpha, \beta) \in \{(\in, \in), (q, q), (\in, \in \vee q), (q, \in \vee q)\}$ $((\alpha, \beta) \in \{(\in, \in \vee q), (q, \in \vee q)\})$.

5. Homomorphisms of Intuitionistic (α, β) -Fuzzy H_v -Submodules

Definition 5.1. [29] Let M_1 and M_2 be two H_v-modules over an H_v-ring *R*. A mapping *f* from M_1 into M_2 is called a *homomorphism* if

$$f(x + y) = f(x) + f(y) \text{ and } f(r.x) = r.f(x),$$

for all $x, y \in M_1$ and $r \in R$.

Definition 5.2. A fuzzy set μ in a set X is said to have sup property if for every non-empty subset S of X, there exists $x_0 \in S$ such that

$$\mu(x_0) = \sup_{x \in S}\{\mu(x)\}$$

Theorem 5.3. Let M_1 and M_2 be two H_v-modules over an H_v-ring of R and mapping f from M_1 into M_2 be a surjection. Let $A = (\mu_A, \lambda_A)$ is an intuitionistic (α, β)-fuzzy H_v-submodule of M_1 such that μ_A and λ_A have sup property, then for all $t \in (0,1]$ we have

(1) $U(\alpha\, f(\mu_A), t) = f(U(\alpha\mu_A, t))$,

(2) $L(\alpha\, f(\lambda_A), t) \subseteq f(L(\alpha\lambda_A, t))$,

where $\alpha \in \{\in, q\}$.

Proof. (1) We only prove the case of $\alpha = \in$. The others are analogous.

$$\begin{aligned} y \in U(\in f(\mu_A), t) &\Leftrightarrow y_t \in f(\mu_A) \\ &\Leftrightarrow f(\mu_A)(y) \geq t \\ &\Leftrightarrow \sup_{x \in f^{-1}(y)} \{\mu_A(x)\} \geq t \\ &\Leftrightarrow \exists x' \in f^{-1}(y),\ \mu_A(x') \geq t \\ &\Leftrightarrow f(x') = y,\ x'_t \in \mu_A \\ &\Leftrightarrow f(x') = y,\ x' \in U(\in \mu_A, t) \\ &\Leftrightarrow y \in f(U(\in \mu_A, t)). \end{aligned}$$

(2) We only prove the case of $\beta = q$. The others are analogous.

$$\begin{aligned} y \in L(q\ f(\lambda_A), t) &\Rightarrow y_t \overline{q} f(\lambda_A) \\ &\Rightarrow f(\lambda_A)(y) + t \leq 1 \\ &\Rightarrow \sup_{x \in f^{-1}(y)} \{\lambda_A(x)\} + t \leq 1 \\ &\Rightarrow \lambda_A(x) + t \leq 1 \ \ for\ all\ \ x \in f^{-1}(y) \\ &\Rightarrow x_t \overline{q} \lambda_A \ \ for\ all\ \ x \in f^{-1}(y) \\ &\Rightarrow x \in L(q\lambda_A, t) \ \ for\ all\ \ x \in f^{-1}(y) \\ &\Rightarrow y \in f(L(q\lambda_A, t)). \end{aligned}$$

Corollary 5.4. Let M_1 and M_2 be two H_v-modules over an H_v-ring of

R and mapping f from M_1 into M_2 be a surjection. Let $A=(\lambda_A^c,\lambda_A)$ is an intuitionistic $(\in,q)$-fuzzy H_v-submodule of M_1 such that λ_A have sup property, then for all $t\in(0,1]$ we have

$$L(\alpha\, f(\lambda_A),t)=U(\alpha' f(\lambda_A^c),t),$$

where $\alpha\in\{\in,q\}$.

Corollary 5.5. Let M_1 and M_2 be two H_v-modules over an H_v-ring of R and mapping f from M_1 into M_2 be a surjection. Let $A=(\mu_A,\mu_A^c)$ is an intuitionistic $(\in,q)$-fuzzy H_v-submodule of M_1 such that μ_A have sup property, then for all $t\in(0,1]$ we have

$$U(\alpha\, f(\mu_A),t)=L(\alpha' f(\mu_A^c),t),$$

where $\alpha\in\{\in,q\}$.

Theorem 5.6. Let M_1 and M_2 be two H_v-modules over an H_v-ring of R and f from M_1 into M_2 be a map. Let $A=(\mu_A,\lambda_A)$ is an intuitionistic (α,β)-fuzzy H_v-submodule of M_2 such that μ_A and λ_A have sup property, then for all $t\in(0,1]$ we have

(1) $U(\alpha\, f^{-1}(\mu_A),t)=f^{-1}(U(\alpha\mu_A,t))$,

(2) $L(\alpha\, f^{-1}(\lambda_A),t)\subseteq f^{-1}(L(\alpha\lambda_A,t))$,

where $\alpha\in\{\in,q\}$.

Proof. (1) Let $\alpha=\in$. We have

$$\begin{aligned} x\in U(\in f^{-1}(\mu_A),t) &\Leftrightarrow x_t \in f^{-1}(\mu_A)\\ &\Leftrightarrow f^{-1}(\mu_A)(x)\ge t\\ &\Leftrightarrow \mu_A(f(x))\ge t\\ &\Leftrightarrow f(x)_t\in\mu_A\\ &\Leftrightarrow f(x)\in U(\in\mu_A,t)\\ &\Leftrightarrow x\in f^{-1}(U(\in\mu_A,t)). \end{aligned}$$

The other the cases of (1) can be proven analogously.

(2) Let $\beta=q$. We have

$$
\begin{aligned}
x \in L(q\ f^{-1}(\lambda_A), t) &\Leftrightarrow x_t \overline{q} f^{-1}(\lambda_A) \\
&\Leftrightarrow f^{-1}(\lambda_A)(y) + t \le 1 \\
&\Leftrightarrow \lambda_A(f(x)) + t \le 1 \\
&\Leftrightarrow f(x)_t \overline{q} \lambda_A \\
&\Leftrightarrow f(x) \in L(q\lambda_A, t) \\
&\Leftrightarrow x \in f^{-1}(L(q\lambda_A, t)).
\end{aligned}
$$

The other the cases of (2) can be proven analogously.

Corollary 5.7. Let M_1 and M_2 be two H_v-modules over an H_v-ring of R and f from M_1 into M_2 be a map. Let $A = (\lambda_A^c, \lambda_A)$ is an intuitionistic $(\in, q)$ -fuzzy H_v -submodule of M_2 such that λ_A have sup property, then for all $t \in (0,1]$ we have

$$L(\alpha\ f^{-1}(\lambda_A), t) = U(\alpha' f^{-1}(\lambda_A^c), t),$$

where $\alpha \in \{\in, q\}$.

Corollary 5.8. Let M_1 and M_2 be two H_v-modules over an H_v-ring of R and f from M_1 into M_2 be a map. Let $A = (\mu_A, \mu_A^c)$ is an intuitionistic $(\in, q)$ -fuzzy H_v -submodule of M_2 such that μ_A have sup property, then for all $t \in (0,1]$ we have

$$U(\alpha\ f^{-1}(\mu_A), t) = L(\alpha' f^{-1}(\mu_A^c), t),$$

where $\alpha \in \{\in, q\}$.

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