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A note on the existence of positive solution for a class of Laplacian nonlinear system with sign-changing weight

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Abstract

This study concerns the existence of positive solution for the system

$$\begin{cases} -\Delta u = \lambda\, a(x)\, f(v), & x \in \Omega, \\ -\Delta v = \lambda\, b(x)\, g(u), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where $\lambda > 0$ is a parameter, Ω is a bounded domain in $R^N (N > 1)$ with smooth boundary $\partial\Omega$ and Δ is the Laplacian operator. Here $a(x)$ and $b(x)$ are C^1 sign-changing functions that maybe negative near the boundary and f, g are C^1 nondecresing functions such that $f, g : [0, \infty) \to [0, \infty)$; $f(s),\, g(s) > 0$; $s > 0$ and

$$\lim_{x\to\infty} \frac{f(Mg(x))}{x} = 0,$$

for every $M > 0$.

We discuss the existence of positive solution when f, g, $a(x)$ and $b(x)$ satisfy certain additional conditions. We use the method of sub-super solutions to establish our results.

Keywords: Laplacian system; Sign-changing weight.
AMS Subject Classification: 35J55, 35J65.

1 Introduction

In this paper we consider the existence of positive solution for the system

$$\begin{cases} -\Delta u = \lambda\, a(x)\, f(v), & x \in \Omega, \\ -\Delta v = \lambda\, b(x)\, g(u), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\lambda > 0$ is a parameter, Δ is the Laplacian operator, Ω is a bounded domain in $R^N (N > 1)$ with smooth boundary $\partial\Omega$, $a(x)$ and $b(x)$ are C^1 sign-changing functions that maybe negative near the boundary and $f, g : [0, \infty) \to [0, \infty)$ are C^1 nondecreasing functions such that $f(s)$, $g(s) > 0$ for $s > 0$.

Systems of the form (1) arise in several context in biology and engineering (see [12]). It provides a simple model to describe, for instance, the interaction of two diffusing biological species. u and v represent the densities of two species. See [13] for details on the physical models involving more general elliptic system. We refer to [1, 2, 3, 9, 10] for additional results on elliptic systems.

For the single-equation, namely equation of the form

$$\begin{cases} -\Delta u = \lambda\, a(x)\, f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

with sign-changing weight function has been studied by several authors (see [11, 7]). In a recent paper [4], the authors established the existence results to the problem (2) for the case when the Laplacian operator is replaced by a p-Laplacian operator.

This paper extends the recent works in [5, 10], where the authors studied the existence of positive solution of the system (1) without the weight functions. Here we focus on sign-changing weight functions $a(x)$ and $b(x)$. Due to this weights functions, the extensions are challenging and nontrivial. Our approach is based on the method of sub-super solutions, see [6, 8].

To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} -\Delta\phi = \lambda\,\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{3}$$

Let $\phi_1 \in C^1(\overline{\Omega})$ be the eigenfunction corresponding to the first eigenvalue λ_1 of (3) such that $\phi_1(x) > 0$ in Ω, and $||\phi_1||_\infty = 1$. It can be shown that $\frac{\partial \phi_1}{\partial n} < 0$ on $\partial\Omega$. Here n is the outward normal. This result is well known and hence, depending on Ω, there exist positive constants m, δ, σ such that

$$\lambda_1\,\phi_1^2 - |\nabla\phi_1|^2 \,\le\, -m, \qquad x \in \bar{\Omega}_\delta, \tag{4}$$

$$\phi_1 \ge \sigma, \qquad x \in \Omega_0 = \Omega \setminus \bar{\Omega}_\delta, \tag{5}$$

with $\bar{\Omega}_\delta = \{x \in \Omega \mid d(x, \partial\Omega) \le \delta\}$. We will also consider the unique solution, $e(x) \in C^1(\overline{\Omega})$, of the boundary value problem

$$\begin{cases} -\Delta e(x) = 1, & x \in \Omega, \\ e(x) = 0, & x \in \partial\Omega, \end{cases}$$

to discuss our existence result. It is known that $e(x) > 0$ in Ω and $\frac{\partial e(x)}{\partial n} < 0$ on $\partial\Omega$.

Here we assume that the weight functions $a(x)$ and $b(x)$ take negative values in $\bar{\Omega}_\delta$, but require $a(x)$ and $b(x)$ be strictly positive in $\Omega - \bar{\Omega}_\delta$. To be precise we assume that there exist positive constants a_0, a_1, b_0 and b_1 Such that $a(x) \ge -a_0$, $b(x) \ge -b_0$ on $\bar{\Omega}_\delta$ and $a(x) \ge a_1$, $b(x) \ge b_1$ on $\Omega - \bar{\Omega}_\delta$.

2 Existence results

In this section, we shall establish our existence result via the method of sub and supersolutions. A pair of nonnegative functions (ψ_1, ψ_2), (z_1, z_2) are called a subsolution and supersolution of (1) if they satisfy $(\psi_1, \psi_2) = (0, 0) = (z_1, z_2)$ on $\partial\Omega$ and

$$\begin{cases} -\Delta\psi_1 \le \lambda\, a(x)\, f(\psi_2), \\ -\Delta\psi_2 \le \lambda\, b(x)\, g(\psi_1), \end{cases}$$

and

$$\begin{cases} -\Delta z_1 \ge \lambda\, a(x)\, f(z_2), \\ -\Delta z_2 \ge \lambda\, b(x)\, g(z_1). \end{cases}$$

Then the following result holds:

Lemma 2.1. (See [6]) Suppose there exist sub and super- solutions (ψ_1, ψ_2) and (z_1, z_2) respectively of (1) such that $(\psi_1, \psi_2) \le (z_1, z_2)$. Then (1) has a solution (u, v) such that $(u, v) \in [(\psi_1, \psi_2), (z_1, z_2)]$.

We make the following assumptions:

(H1) $f, g : [0,\infty) \to [0,\infty)$ are C^1 nondecreasing functions such that $f(s),\ g(s) > 0$ for $s > 0$.

(H2) For all $M > 0$,

$$\lim_{x\to\infty} \frac{f(Mg(x))}{x} = 0.$$

(H3) Suppose that there exists $\epsilon > 0$ such that:

$$\frac{\lambda_1\, a_0}{m}\, f(\epsilon) < \min\Big\{b_1\, g(\frac{1}{2}\epsilon\, \sigma^2), a_1\, f(\frac{1}{2}\epsilon\, \sigma^2)\Big\},$$

$$\frac{\lambda_1\, b_0}{m}\, g(\epsilon) < \min\Big\{b_1\, g(\frac{1}{2}\epsilon\, \sigma^2), a_1\, f(\frac{1}{2}\epsilon\, \sigma^2)\Big\},$$

and

$$\max\Big\{\frac{\epsilon\,\lambda_1}{b_1\, g(\frac{1}{2}\epsilon\,\sigma^2)}, \frac{\epsilon\,\lambda_1}{a_1\, f(\frac{1}{2}\epsilon\,\sigma^2)}\Big\} \le \frac{1}{||b||_\infty}.$$

Now we are ready to state our existence result.

Theorem 2.2. Let $(H1) - (H3)$ hold. Then there exists a positive solution of (1) for every $\lambda \in [\lambda_*(\epsilon), \lambda^*(\epsilon)]$, where

$$\lambda^* = \min\Big\{\frac{\epsilon\, m}{a_0\, f(\epsilon)}, \frac{m\,\epsilon}{b_0\, g(\epsilon)}, \frac{1}{||b||_\infty}\Big\}, \tag{6}$$

$$\lambda_* = \max\Big\{\frac{\epsilon\,\lambda_1}{b_1\, g(\frac{1}{2}\epsilon\,\sigma^2)}, \frac{\epsilon\,\lambda_1}{a_1\, f(\frac{1}{2}\epsilon\,\sigma^2)}\Big\}. \tag{7}$$

Remark 2.3. Note that $(H3)$ implies $\lambda_* < \lambda^*$.

Example 2.4. Let $\alpha > 0$, $f(x) = e^{\frac{\alpha\, x}{\alpha + x}}$ and $g(x) = e^x$. Clearly f, g satisfy $(H1)$ and $(H2)$ as

$$\lim_{x\to\infty} \frac{f(Mg(x))}{x} = \lim_{x\to\infty} \frac{e^{\frac{\alpha\, M\, e^x}{\alpha + M\, e^x}}}{x} = 0.$$

We can choose $\epsilon > 0$ so small that f, g satisfy $(H3)$.

Proof of Theorem 2.2. We shall verify that $(\psi_1, \psi_2) = (\psi, \psi)$ where $\psi = \frac{1}{2}\epsilon\,\phi_1^2$ is a sub-solution of (1). Since $\nabla\psi = \epsilon\,\phi_1\,\nabla\phi_1$, a calculation shows that

$$\begin{aligned} -\Delta\psi &= -(\frac{1}{2})\epsilon\,\Delta\phi_1^2 \\ &= -\epsilon\,(|\nabla\phi_1|^2 + \phi_1\,\Delta\phi_1) \\ &= \epsilon\,(\lambda_1\phi_1^2 - |\nabla\phi_1|^2). \end{aligned}$$

Thus ψ is a sub-solution if

$$\epsilon\,(\lambda_1\phi_1^2 - |\nabla\phi_1|^2) \le \lambda\,a(x)\,f(\frac{1}{2}\epsilon\,\phi_1^2).$$

First we consider the case when $x \in \bar{\Omega}_\delta$. We have $\lambda_1\,\phi_1^2 - |\nabla\phi_1|^2 \le -m$ on $\bar{\Omega}_\delta$ and since $\lambda \le \lambda^*$, then $\lambda \le \frac{m\,\epsilon}{a_0\,f(\epsilon)}$. Hence

$$\begin{aligned} -\Delta\,\psi = \epsilon\,(\lambda_1\,\phi_1^2 - |\nabla\phi_1|^2) &\le -m\,\epsilon \\ &\le -\lambda\,a_0\,f(\epsilon) \\ &\le -\lambda\,a_0\,f(\epsilon\,\frac{1}{2}\,\phi_1^2) \\ &\le \lambda\,a(x)\,f(\psi). \end{aligned}$$

A similar argument shows that

$$-\Delta\,\psi \le \lambda\,b(x)\,g(\psi)$$

when $x \in \bar{\Omega}_\delta$.

On the other hand, on $\Omega \setminus \bar{\Omega}_\delta$, we note that $\phi_1 \ge \sigma > 0$, $a(x) \ge a_1$, $b(x) \ge b_1$ and since $\lambda \ge \lambda_*$, we have $\lambda \ge \frac{\epsilon\,\lambda_1}{a_1\,f(\frac{1}{2}\epsilon\sigma^2)}$. Hence

$$\begin{aligned} -\Delta\,\psi = \epsilon\,(\lambda_1\,\phi_1^2 - |\nabla\phi_1|^2) &\le \epsilon\,\lambda_1 \\ &\le \lambda\,a_1\,f(\frac{1}{2}\epsilon\,\sigma^2) \\ &\le \lambda\,a(x)\,f(\psi). \end{aligned}$$

A similar argument shows that:

$$-\Delta\,\psi \le \lambda\,b(x)\,g(\psi).$$

Those we have shown that (ψ_1, ψ_2) is sub-solution.

Now, we will prove there exist a c large enough so that $(z_1, z_2) = (\frac{c}{l}\, e, g(c)\, e)$ is a super-solution of (1) where $l = ||e(x)||_\infty$. A calculation shows that:

$$-\Delta\, z_1 = \frac{c}{l}.$$

By **(H2)** we can choose c large enough so that

$$c\,(\lambda\,||a(x)||_\infty)\,l)^{-1} \geq f(l\,g(c)).$$

Hence

$$\begin{aligned} -\Delta\, z_1 = \frac{c}{l} &\geq \lambda\,||a(x)||_\infty\, f(l\,g(c)) \\ &\geq \lambda\, a(x)\, f(e(x)\, g(c)) \\ &= \lambda\, a(x)\, f(z_2). \end{aligned}$$

Next, since $\lambda \leq \lambda^*$ we have $\lambda \leq \frac{1}{||b||_\infty}$. Hence

$$\begin{aligned} -\Delta\, z_2 &= g(c) \\ &\geq g(c\,\frac{e}{l}) \\ &\geq \lambda\,||b||_\infty\, g(c\,\frac{e}{l}) \\ &\geq \lambda\, b(x)\, g(z_1), \end{aligned}$$

i.e. (z_1, z_2) is a super-solution of (1) with $z_i \geq \psi_i$ for c large, $i = 1, 2$. (Note $|\nabla e(x)| \neq 0$ on $\partial\Omega$). Thus, there exists a positive solution (u, v) of (1) such that $(\psi, \psi) \leq (u, v) \leq (z_1, z_2)$. This completes the proof of Theorem 2.2. □

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