The Journal of
Mathematics and Computer Science

Available online at
http://www.TJMCS.com

The Journal of Mathematics and Computer Science Vol .3 No.3 (2011) 301 - 305

ϕ -PSEUDO $\tilde{W}_4$ FLAT LP-SASAKIAN MANIFOLDS

AMIT PRAKASH

Received: August 2011, Revised: November 2011
Online Publication: December 2011

ABSTRACT. The object of the present paper is to study pseudo $\tilde{W}_4$ curvature tensor in a Lorentzian para-Sasakian manifolds.

1. PRELIMINARIES

An n-dimensional differentiable manifold M^n is Lorentzian para-Sasakian (LP-Sasakian) manifold, if it admits a (1, 1)-tensor field ϕ,a vector field ξ, a 1-form η and a Lorentzian metric g, which satisfies

$$\phi^2 X = X + \eta(X)\xi, \tag{1.1}$$

$$\eta(\xi) = -1, \tag{1.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{1.3}$$

$$g(X, \xi) = \eta(X), \tag{1.4}$$

$$(D_X\phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{1.5}$$

$$D_X\xi = \phi X, \tag{1.6}$$

for arbitrary vector fields X and Y; where D denotes the operator of covariant differentiation with respect to the Lorentzian metric g ([5], [6]).

In an LP-Sasakian manifold M^n with structure (ϕ, ξ, η, g) it is easily seen that

$$(a) \quad \phi\xi = 0, (b) \quad \eta(\phi X) = 0, (c) \quad rank\phi = (n-1). \tag{1.7}$$

Let us put

$$F(X, Y) = g(\phi X, Y), \tag{1.8}$$

then the tensor field F is symmetric (0, 2)-tensor field. Thus we have

$$F(X, Y) = F(Y, X), \tag{1.9}$$

$$F(X, Y) = (D_X\eta)(Y), \tag{1.10}$$

$$(D_X\eta)(Y) - (D_Y\eta)(X) = 0. \tag{1.11}$$

An LP-Sasakian manifold M^n is said to be Einstein manifold if its Ricci tensor S is of the form

$$S(X, Y) = kg(X, Y), \tag{1.12}$$

where k is some scaler functions on M^n.

An LP-Sasakian manifold M^n is said to be $\eta-$Einstein manifold if its Ricci tensor S is of the form

$$S(X, Y) = \alpha g(X, , Y) + \beta\eta(X)\eta(Y), \tag{1.13}$$

for any vector fields X and Y, where α, β are functions on M^n.

2000 *Mathematics Subject Classification.* 53B05, 53B15, 53C25.

Key words and phrases. LP-Sasakian manifold, pseudo $\tilde{W}_4$ curvature tensor, pseudo $\tilde{W}_4$ conservative , $\phi-$pseudo $\tilde{W}_4$ flat, $\eta-$ Einstein manifold.

Let M^n be an n-dimensional LP-Sasakian manifold with structure (ϕ, ξ, η, g). Then we have ([6], [7]).

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{1.14}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{1.15}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{1.16}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{1.17}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{1.18}$$

and

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\,\eta(Y), \tag{1.19}$$

for any vector fields X, Y, Z where $R(X, Y)Z$ is the Riemannian curvature tensor of type (1, 3), S is the Ricci tensor of type (0, 2), Q is (1, 1) Ricci tensor and r is the scalar curvature.

$$g\,(QX, Y) = S(X, Y) \qquad \text{for all } X, Y.$$

Recently Prasad, Narain and Mourya defined Pseudo $\tilde{W}_4$ curvature tensor on a Riemannian manifold $(M^n, g)\,(n > 2)$ of type (1, 3) as follows

$$\begin{aligned}\tilde{W}_4(X,Y)Z &= aR(X,Y)Z + b[g(X,Y)QZ - g(X,Z)QY] \\ &\quad -\frac{r}{n}[\frac{a}{n-1} + b][g(X,Y)Z - g(X,Z)Y],\end{aligned} \tag{1.20}$$

where a and b are constants such that $a, b \neq 0$. If $a = 1$ and $b = -\frac{1}{(n-1)}$ then (1.20) takes the form

$$\tilde{W}_4(X,Y)Z = R(X,Y)Z + \frac{1}{(n-1)}[g(X,Z)QY - g(X,Y)QZ] = W_4(X,Y)Z,$$

where $\tilde{W}_4$ is the pseudo $\tilde{W}_4$ curvature tensor ([11]). Hence W_4([13]) curvature tensor is a particular case of the tensor $\tilde{W}_4$. For the reason $\tilde{W}_4$ is called pseudo $\tilde{W}_4$ curvature tensor.

2. Einstein LP-Sasakian manifold satisfying (div)$\tilde{W}_4(X,Y)Z = 0$

Definition 2.1: A manifold $(M^n, g)(n > 2)$ is called pseudo $\tilde{W}_4$ conservative if (div)$\tilde{W}_4(X,Y)Z = 0$, ([3])

In this section we assume that

$$div\tilde{W}_4 = 0, \tag{2.1}$$

where div denotes divergence.

Now differentiating (1.20) covariantly with respect to U, we get

$$\begin{aligned}(D_U\tilde{W}_4)(X,Y)Z &= a(D_U R)(X,Y)Z + b[g(X,Y)(D_U Q)(Z) - g(X,Z)(D_U Q)(Y)] \\ &\quad -\frac{1}{n}\Big[\frac{a}{n-1} + b\Big][D_U r][g(X,Y)Z - g(X,Z)Y].\end{aligned} \tag{2.2}$$

Contraction of (2.2) gives

$$\begin{aligned}(div\tilde{W}_4)(X,Y)Z &= a(divR)(X,Y)Z + b[g(X,Y)(divQ)(Z) - g(X,Z)(divQ)(Y)] \\ &\quad -\frac{1}{n}\Big[\frac{a}{n-1} + b\Big][g(X,Y)dr(Z) - g(X,Z)dr(Y)].\end{aligned} \tag{2.3}$$

But from [2], we have

$$(\text{div}\, R)(X,Y)Z = (D_X S)(Y,Z) - (D_Y S)(X,Z), \tag{2.4}$$

If LP-Sasakian manifold is an Einstein manifold, then from (1.12) and (2.4), we get

$$(\text{div} R)(X,Y)Z = (D_X S)(Y,Z) - (D_Y S)(X,Z) = 0. \tag{2.5}$$

From (2.3) and (2.5), we get

$$(\text{div}\tilde{W}_4)(X,Y)Z = \Big[\frac{b(n-1)(n-2) - 2a}{2n(n-1)}\Big][g(X,Y)dr(Z) - g(X,Z)dr(Y)]. \tag{2.6}$$

From (2.1) and (2.6), we get

$$g(X,Y)dr(Z) - g(X,Z)dr(Y) = 0, \quad \text{provided } 2a - b(n-1)(n-2) \neq 0. \tag{2.7}$$

which shows that r is constant. Again if r is constant then from (2.6), we get

$$(\text{div}\tilde{W}_4)(X,Y)Z = 0. \tag{2.8}$$

Hence, we can state the following theorem :

Theorem 2.1. *An Einstein LP-Sasakian manifold* $(M^n, g)(n > 2)$ *is pseudo* $\tilde{W}_4$ *conservative if and only if the scalar curvature is constant, provided* $2a - b(n-1)(n-2) \neq 0$.

3. $\phi-$ Pseudo $\tilde{W}_4$ flat LP-Sasakian manifold.

Definition 3.1: A differentiable manifold $(M^n, g)(n > 2)$ satisfying the condition ([1])

$$\phi^2 \tilde{W}_4(\phi X, \phi Y)\phi Z = 0, \tag{3.1}$$

is called $\phi-$ pseudo $\tilde{W}_4$ flat LP-Saskian manifold.

Suppose that $(M^n,\, g)(n > 2)$, is a $\phi-$pseudo $\tilde{W}_4$ flat LP-Sasakian manifold. It is easy to see that

$\phi^2 \tilde{W}_4(\phi X, \phi Y)\phi Z = 0$, holds if and only if

$g(\tilde{W}_4(\phi X, \phi Y)\phi Z, \phi W) = 0$,

for any vector fields X, Y, Z, W.

By the use of (1.20), $\phi-$ pseudo $\tilde{W}_4$ flat means,

$$a'R(\phi X, \phi Y, \phi Z, \phi W) = -b[S(\phi Z, \phi W)g(\phi X, \phi Y) - S(\phi Y, \phi W)g(\phi X, \phi Z)]$$
$$+\frac{r}{n}\Big[\frac{a}{n-1} + b\Big][g(\phi X, \phi Y)g(\phi Z, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)], \tag{3.2}$$

where $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M^n. By using the fact that $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (3.2) and sum up with respect to i, then we have

$$\sum_{i=1}^{n-1} a'R(\phi e_i, \phi Y, \phi Z, \phi e_i) = -b\sum_{i=1}^{n-1}[S(\phi Z, \phi e_i)g(\phi e_i, \phi Y) - S(\phi Y, \phi e_i)g(\phi e_i, \phi Z)]$$

$$+\frac{r}{n}\Big[\frac{a}{n-1} + b\Big]\sum_{i=1}^{n-1}[g(\phi e_i, \phi Y)g(\phi Z, \phi e_i) - g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)] \tag{3.3}$$

On an LP-Sasakian manifold, we have ([9])

$$\sum_{i=1}^{n-1}{}'R(\phi e_i,\phi Y,\phi Z,\phi e_i)=S(\phi Y,\phi Z)+g(\phi Y,\phi Z), \tag{3.4}$$

$$\sum_{i=1}^{n-1}S(\phi e_i,\phi e_i)=r+n-1, \tag{3.5}$$

$$\sum_{i=1}^{n-1}g(\phi e_i,\phi Z)S(\phi Y,\phi e_i)=S(\phi Y,\phi Z), \tag{3.6}$$

$$\sum_{i=1}^{n-1}g(\phi e_i,\phi e_i)=n+1, \tag{3.7}$$

$$\sum_{i=1}^{n-1}g(\phi e_i,\phi Z)g(\phi Y,\phi e_i)=g(\phi Y,\phi Z). \tag{3.8}$$

So by virtue of (3.4)-(3.8) and then using (1.3) and (1.19), the equation (3.3) takes the form,

$$a[S(Y,Z)+g(Y,Z)+n\eta(Y)\eta(Z)]=0. \tag{3.9}$$

Since $a\neq 0$,then from (3.9) ,we have

$$S(Y,Z)=-g(Y,Z)-n\eta(Y)\eta(Z). \tag{3.10}$$

which shows that, M^n is an $\eta-$Einstein manifold.

Again contracting (3.10), we get $r=0$.

Hence, we can state the following theorem :

Theorem 3.1 *Let M^n be an n-dimensional $(n>2)$ $\phi-$pseudo $\tilde{W}_4$ flat LP-Sasakian manifold, then M^n is an $\eta-$Einstein manifold with zero scalar curvature.*

References

[1] Cabrerizo, J. L., Fernandez, L. M., Fernandez, M. and Zhen, G., The structure of a class of k-contact manifolds, Acta Math. Hungar, 82 (4),(1999), 331-340.

[2] Eisenhart, L. P., Riemannian Geometry, Princeton Univ. Press,(1926), 82-91.

[3] Hicks, N. J., Notes on Differential Geometry, Affiliated East West Press Pvt. Ltd.(1969), p. 95.

[4] Khan, Q., On an Einstein projective Sasakian manifolds, NOVI SAD J. Math., 36(1),(2006), 97-102.

[5] Matsumoto, K. , On Lorentzian para-contact manifolds, Bull. of Yamagata Univ., Nat. Sci. 12(2) (1989), 151-156.

[6] Matsumoto, K. and Mihai, I. , On a certain transformation in a Lorentzian para-Sasakian manifold, Tensor N. S., 47 (1988), 189-197.

[7] Mihai, I., Shaikh, A. A. and De, U. C., On Lorentzian para-Sasakian manifolds, Korean J. Math. Sciences. 6, 1-13.

[8] Mishra, R. S., Structure on a differentiable manifold and their applications, Chandrama Prakashan, 50 A, Balrampur House, Allahabad, India (1984).

[9] Özgür, C., ϕ-conformally flat LP-Sasakian manifolds, Radovi matematicki, 12 (2003), 99-106.

[10] Prasad, B., A pseudo projective curvature tensor on a Riemannian manifold, Bull. Cal. Math. Soc. 94 (3)(2002), 163-166.

[11] Prasad, B.,Narain,S. and Mourya A., A pseudo $\tilde{W}_4$ curvature tensor on a Riemannian manifold, Acta Math cincia (Communicated).

[12] Narain, D.,Prakash, A., and Prasad, B., A pseudo projective curvature tensor on an Lorentzian para-Sasakian manifold,,ANALELE STIINTIFICE ALE UNIVERSITATII "AL.I.CUZA"DIN IASI(S.N.)MATHEMATICA,Tomul LV,(2009),fase.2, 275-284.

[13] Pokhariyal, G. P., Curvature tensor and their relativistic significanceIII, Yokohama mathematical Journal,20 ,(1972),115-119.

Department of Mathematics
Allenhouse Institute of Technology,
Rooma Kanpur U.P. India .
E-mail address: apoct0185@rediffmail.com