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Weak and strong convergence theorems of a new iterative process
with errors for common fixed points of a finite families of
asymptotically nonexpansive mappings in the intermediate sense
in Banach spaces

Shrabani Banerjee*, B.S.Choudhury†

Department of Mathematics

Bengal Engineering and Science University, Shibpur, Howrah-711103, India.

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Abstract

In this paper we study the weak and strong convergence results for a new multi-step iterative scheme with errors to a common fixed point for a finite family of asymptotically nonexpansive mappings in the intermediate sense in a uniformly convex Banach space. Our results generalize a number of results.

Keywords: Multi-step iterative process with errors; Asymptotically nonexpansive mappings in the intermediate sense; Opial's condition; Kadec-lee property; uniformly convex Banach space; common fixed point; Condition (\overline{B}) ; weak and strong convergence.

AMS Subject Classification 2000 : 47H10

1 Introduction

Let C be a nonempty subset of real normed space X . A self mapping $T : C \rightarrow C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C$$

A self mapping $T : C \rightarrow C$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for all } x, y \in C \text{ and } n \geq 1.$$

T is said to be uniformly L -Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\| \text{ for all } x, y \in C \text{ and } n \geq 1.$$

*corresponding author, E-mail: banerjee.shrabani@yahoo.com

†E-mail: binayak12@yahoo.co.in

T is called asymptotically nonexpansive mapping in the intermediate sense [1] provided T is uniformly continuous and

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk[4] in 1972 as an important generalization of the class of nonexpansive self mappings. After that Bruck et. al.[1] introduced the class of asymptotically nonexpansive mappings in the intermediate sense. It is known that if T is a self map on C where C is a nonempty closed convex subset of a real uniformly convex Banach space, then T has a fixed point. The above definitions make it clear that asymptotically nonexpansive mapping must be asymptotically nonexpansive mapping in the intermediate sense and uniformly L -Lipschitzian mapping, but the converse need not be true:

Example [8]: Let $X = R, C = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $|k| < 1$. for each $x \in C$, define

$$T(x) = \begin{cases} kx \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

then T is asymptotically nonexpansive mapping in the intermediate sense, but is not a Lipschitzian mapping but $T^n x \rightarrow 0$ uniformly so that it is not asymptotically nonexpansive mapping.

To proceed we shall need the following well known Lemmas and definitions:

A Banach space X is said to satisfy Opial's condition [10] if $x_n \rightharpoonup x$ (i.e. $x_n \rightarrow x$ weakly) and $x \neq y$ imply

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$$

A Banach space X is said to satisfy τ -Opial condition [1] if for every bounded $\{x_n\} \in X$ that τ -converges to $x \in X$ then

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$$

for every $x \neq y$, where τ is a Housdorff linear topology on X .

A Banach space X has the uniform τ -Opial property [1] if for each $c > 0$ there exists $r > 0$ with the property that for each $x \in X$ and each sequence $\{x_n\}$ such that $\{x_n\}$ is τ -convergent to 0 and

$$1 \leq \limsup_{n \rightarrow \infty} \|x_n\| < \infty, \|x\| \geq c$$

imply that $\limsup_{n \rightarrow \infty} \|x_n - x\| \geq 1 + r$. Clearly uniform τ -Opial condition implies τ -Opial condition. Note that a uniformly convex space which has the τ -Opial property necessarily has the uniform τ -Opial property, where τ is a Housdorff linear topology on X .

Let T be a self-mapping of a nonempty subset C of a Banach space X . A sequence $\{x_n\}$ in C is called an almost orbit[3] of T if $\lim_{n \rightarrow \infty} [\sup_{m \geq 0} \|x_{n+m} - T^m x_n\|] = 0$

A Banach space X is said to satisfy Kadec-Klee property, if for every sequence $\{x_n\} \in X, x_n \rightharpoonup x$ and $\|x_n\| \rightarrow \|x\|$ together imply that $x_n \rightarrow x$ as $n \rightarrow \infty$. There are uniformly convex Banach spaces which have neither a *Fréchet* differentiable norm nor satisfy Opial's property but their duals do have the Kadec-Klee property (see [3],[5]).

Khan et al.[7] introduced an iterative process for a finite family of mappings as follows: let C be a convex subset of a Banach space X and let $\{T_i : i = 1, 2, \dots, k\}$ be a family self-mappings of C . Let $\{x_n\}$ be the sequence generated by the following algorithm:

$$\left\{ \begin{array}{l} x_1 \in C \\ x_{n+1} = (1 - \alpha_{kn})x_n + \alpha_{kn}T_k^n y_{(k-1)n} \\ y_{(k-1)n} = (1 - \alpha_{(k-1)n})x_n + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n} \\ \cdot \\ \cdot \\ \cdot \\ y_{2n} = (1 - \alpha_{2n})x_n + \alpha_{2n}T_2^n y_{1n} \\ y_{1n} = (1 - \alpha_{1n})x_n + \alpha_{1n}T_1^n y_{0n} \end{array} \right. \tag{1.1}$$

where $y_{0n} = x_n$ for all n and $\{\alpha_{in}\}$ are appropriate real sequences in $[0, 1]$ for $i = \{1, 2, \dots, k\}$. Very recently Kettapun et al.[6] introduced a new multistep iteration process for approximating common fixed points for finite families of asymptotically quasi-nonexpansive mappings which is defined as follows:

$$\left\{ \begin{array}{l} x_1 \in C \\ x_{n+1} = (1 - \alpha_{kn})y_{(k-1)n} + \alpha_{kn}T_k^n y_{(k-1)n} \\ y_{(k-1)n} = (1 - \alpha_{(k-1)n})y_{(k-2)n} + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n} \\ \cdot \\ \cdot \\ \cdot \\ y_{2n} = (1 - \alpha_{2n})y_{1n} + \alpha_{2n}T_2^n y_{1n} \\ y_{1n} = (1 - \alpha_{1n})y_{0n} + \alpha_{1n}T_1^n y_{0n} \end{array} \right. \tag{1.2}$$

where $y_{0n} = x_n$ for all n and $\{\alpha_{in}\}$ are appropriate real sequences in $[0, 1]$ for $i = \{1, 2, \dots, k\}$. We study the convergence of the following iteration process with errors for approximating common fixed points for finite families of asymptotically nonexpansive mappings in the intermediate sense which is defined as follows:

$$\left\{ \begin{array}{l} x_1 \in C \\ x_{n+1} = (1 - \alpha_{kn} - \beta_{kn})y_{(k-1)n} + \alpha_{kn}T_k^n y_{(k-1)n} + \beta_{kn}u_{kn} \\ y_{(k-1)n} = (1 - \alpha_{(k-1)n} - \beta_{(k-1)n})y_{(k-2)n} + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n} + \beta_{(k-1)n}u_{(k-1)n} \\ \cdot \\ \cdot \\ \cdot \\ y_{2n} = (1 - \alpha_{2n} - \beta_{2n})y_{1n} + \alpha_{2n}T_2^n y_{1n} + \beta_{2n}u_{2n} \\ y_{1n} = (1 - \alpha_{1n} - \beta_{1n})y_{0n} + \alpha_{1n}T_1^n y_{0n} + \beta_{1n}u_{1n} \end{array} \right. \tag{1.3}$$

where $y_{0n} = x_n$ for all n , $\{\alpha_{jn}\}, \{\beta_{jn}\}$ are appropriate real sequences in $[0, 1]$ with $\alpha_{jn} + \beta_{jn} \leq 1$ for $j \in I$ and $\{u_{jn}\}$ are bounded sequences in C for $j \in I$ and $I = \{1, 2, \dots, k\}$.

The purpose of this paper is to establish several strong and weak convergence theorems of the multi-step iterative scheme with errors (1.3) for a finite family of asymptotically nonexpansive mappings in the intermediate sense in a uniformly convex Banach space.

We need the following lemmas which are essential in the proof of the main results.

Lemma 1.1 ([15], Lemma1) *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \forall n \geq 1.$$

If $\sum_{n=1}^\infty \delta_n < \infty$ and $\sum_{n=1}^\infty b_n < \infty$, then

- (i) $\lim_{n \rightarrow \infty} a_n$ exists,
- (ii) $\lim_{n \rightarrow \infty} a_n = 0$ whenever $\liminf_{n \rightarrow \infty} a_n = 0$.

Lemma 1.2 [12] :*Suppose that X is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers $n \geq 1$. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq r, \limsup_{n \rightarrow \infty} \|y_n\| \leq r$ and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

Lemma 1.3 [1] : *Suppose a Banach space X has the uniform τ -opial property, C is a norm bounded, sequentially τ -compact subset of X and $T : C \rightarrow C$ is asymptotically nonexpansive in the weak sense. If $\{y_n\}$ is a sequence in C such that $\lim_{n \rightarrow \infty} \|y_n - z\|$ exists for each fixed point z of T and if $\{y_n - T^m y_n\}$ is τ -convergent to 0 for each $m \in N$, then $\{y_n\}$ is τ -convergent to a fixed point of T .*

Lemma 1.4 ([3], Theorem 5.3) : Let X be a uniformly convex Banach space such that X^* has the Kadec-Klee property and let C be a nonempty bounded closed convex subset of X . Suppose $T : C \rightarrow C$ is asymptotically nonexpansive mappings in the intermediate sense and $\{x_n\}$ is an almost orbit of T . Then $\{x_n\}$ is weakly convergent to a fixed point of T if and only if $w - \lim_{n \rightarrow \infty} (x_n - x_{n+1}) = 0$.

Now we recall some well-known definitions:

Condition (I): A mapping $T : C \rightarrow C$ with nonempty fixed point set $F(T)$ in C satisfying Condition (I) [14] if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$f(d(x, F(T))) \leq \|x - Tx\| \text{ for all } x \in C$$

A finite family of mappings $T_i : C \rightarrow C$, for all $i = 1, 2, 3, \dots, k$ with nonempty fixed point set $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$ satisfies

Condition (A)[2] if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$f(d(x, F)) \leq \frac{1}{k} \left(\sum_{i=1}^k \|x - T_i x\| \right) \text{ for all } x \in C$$

Condition (B)[2] if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$f(d(x, F)) \leq \max_{1 \leq i \leq k} \{ \|x - T_i x\| \} \text{ for all } x \in C$$

Condition (C)[2] if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that at least one of the T_i 's satisfies condition (I) (i.e. $f(d(x, F(T))) \leq \|x - T_i x\|$ for at least one $T_i, i = 1, 2, \dots, k$)

Clearly if $T_i = T$, for all $i = 1, 2, 3, \dots, k$, then Condition (A) reduces to Condition (I). Also Condition (B) reduces to Condition (I) if all but one of T_i 's are identities. Also it contains Condition (A). Furthermore Condition (C) and Condition (B) are equivalent. It is well known that every continuous and demicompact mapping must satisfy Condition (I) [14]. Since every completely continuous mapping is continuous and demicompact so it must satisfy Condition (I). Therefore to study the strong convergence of the iterative sequence $\{x_n\}$ be defined by (1.3) we use Condition (B) instead of the complete continuity of the mappings $\{T_1, T_2, \dots, T_k\}$.

2 Main Results

In this section we begin with the following lemmas. Throughout this section we denote $\{1, 2, \dots, k\}$ by I .

Lemma 2.1 Let X be a real normed space and C be a nonempty closed convex subset of X . Let $T_i : C \rightarrow C (i \in I)$ be given asymptotically nonexpansive mappings in the intermediate sense. Let

$$d_n = \max \{ \max_{1 \leq i \leq k} \sup_{x, y \in C} (\|T_i^n x - T_i^n y\| - \|x - y\|), 0 \}, \forall n \geq 1 \tag{2.1}$$

so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{\alpha_{in}\}$ be a sequence in $[\epsilon, 1 - \epsilon], \epsilon \in (0, 1)$, for each $i \in I$. Let $\{x_n\}$ be defined by (1.3) with $\sum_{n=1}^{\infty} \beta_{in} < \infty$, for all $i \in I$ and $n \geq 1$. If $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$ then $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists for all $q \in F$.

Proof: Let $q \in F$. Let $M = \sup_{1 \leq i \leq k} \{ \|u_{in} - q\| : n \in N \}$. Then for each $n \geq 1$, we have

$$\|y_{1n} - q\| = \|(1 - \alpha_{1n} - \beta_{1n})y_{0n} + \alpha_{1n}T_1^n y_{0n} + \beta_{1n}u_{1n} - q\|$$

$$\begin{aligned}
 &\leq (1 - \alpha_{1n} - \beta_{1n})\|y_{0n} - q\| + \alpha_{1n}\|T_1^n y_{0n} - q\| + \beta_{1n}\|u_{1n} - q\| \\
 &\leq (1 - \alpha_{1n} - \beta_{1n})\|y_{0n} - q\| + \alpha_{1n}(\|y_{0n} - q\| + d_n) + \beta_{1n}M \\
 &\leq \|y_{0n} - q\| + d_n + \beta_{1n}M \\
 &= \|x_n - q\| + G_n^1
 \end{aligned} \tag{2.2}$$

where $G_n^1 = d_n + \beta_{1n}M$. So we have $\sum_{n=1}^{\infty} G_n^1 < \infty$.

$$\begin{aligned}
 \|y_{2n} - q\| &= \|(1 - \alpha_{2n} - \beta_{2n})y_{1n} + \alpha_{2n}T_2^n y_{1n} + \beta_{2n}u_{2n} - q\| \\
 &\leq (1 - \alpha_{2n} - \beta_{2n})\|y_{1n} - q\| + \alpha_{2n}\|T_2^n y_{1n} - q\| \\
 &\quad + \beta_{2n}\|u_{2n} - q\| \\
 &\leq (1 - \alpha_{2n} - \beta_{2n})\|y_{1n} - q\| + \alpha_{2n}(\|y_{1n} - q\| + d_n) \\
 &\quad + \beta_{2n}\|u_{2n} - q\| \\
 &\leq \|y_{1n} - q\| + d_n + \beta_{2n}M \\
 &\leq \|x_n - q\| + 2d_n + (\beta_{1n} + \beta_{2n})M \\
 &= \|x_n - q\| + G_n^2
 \end{aligned} \tag{2.3}$$

where $G_n^2 = 2d_n + (\beta_{1n} + \beta_{2n})M$. So we have $\sum_{n=1}^{\infty} G_n^2 < \infty$. Again

$$\begin{aligned}
 \|y_{3n} - q\| &= \|(1 - \alpha_{3n} - \beta_{3n})y_{2n} + \alpha_{3n}T_3^n y_{2n} + \beta_{3n}u_{3n} - q\| \\
 &\leq (1 - \alpha_{3n} - \beta_{3n})\|y_{2n} - q\| + \alpha_{3n}\|T_3^n y_{2n} - q\| \\
 &\quad + \beta_{3n}\|u_{3n} - q\| \\
 &\leq (1 - \alpha_{3n} - \beta_{3n})\|y_{2n} - q\| + \alpha_{3n}(\|y_{2n} - q\| + d_n) \\
 &\quad + \beta_{3n}\|u_{3n} - q\| \\
 &\leq \|y_{2n} - q\| + d_n + \beta_{3n}M \\
 &\leq \|x_n - q\| + 2d_n + (\beta_{1n} + \beta_{2n})M + d_n + \beta_{3n}M \\
 &\leq \|x_n - q\| + 3d_n + (\beta_{1n} + \beta_{2n} + \beta_{3n})M \\
 &\leq \|x_n - q\| + G_n^3
 \end{aligned} \tag{2.4}$$

where $G_n^3 = 3d_n + (\beta_{1n} + \beta_{2n} + \beta_{3n})M$. So we have $\sum_{n=1}^{\infty} G_n^3 < \infty$. Let

$$\|y_{jn} - q\| \leq \|x_n - q\| + G_n^j$$

where $\{G_n^j\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty} G_n^j < \infty$ for $j = 1, 2, \dots, k - 2$. Now

$$\begin{aligned}
 \|y_{(j+1)n} - q\| &\leq (1 - \alpha_{(j+1)n} - \beta_{(j+1)n})\|y_{jn} - q\| + \alpha_{(j+1)n}\|T_{j+1}^n y_{jn} - q\| \\
 &\quad + \beta_{(j+1)n}\|u_{(j+1)n} - q\| \\
 &\leq (1 - \alpha_{(j+1)n} - \beta_{(j+1)n})\|y_{jn} - q\| + \alpha_{(j+1)n}(\|y_{jn} - q\| + d_n) \\
 &\quad + \beta_{(j+1)n}\|u_{(j+1)n} - q\|
 \end{aligned}$$

$$\begin{aligned}
 &\leq \|y_{jn} - q\| + d_n + \beta_{(j+1)n}M \\
 &\leq \|x_n - q\| + G_n^j + d_n + \beta_{(j+1)n}M \\
 &= \|x_n - q\| + G_n^{j+1}
 \end{aligned} \tag{2.5}$$

So by induction we have

$$\|y_{in} - q\| \leq \|x_n - q\| + G_n^i \tag{2.6}$$

where $\{G_n^i\}$ is a nonnegative real sequence such that $\sum_{n=1}^\infty G_n^i < \infty$ for $i = 1, 2, \dots, k - 1$. Now

$$\begin{aligned}
 \|x_{n+1} - q\| &= \|(1 - \alpha_{kn} - \beta_{kn})y_{(k-1)n} + \alpha_{kn}T_k^n y_{(k-1)n} + \beta_{kn}u_{kn} - q\| \\
 &\leq (1 - \alpha_{kn} - \beta_{kn})\|y_{(k-1)n} - q\| + \alpha_{kn}\|T_k^n y_{(k-1)n} - q\| \\
 &\quad + \beta_{kn}\|u_{kn} - q\| \\
 &\leq (1 - \alpha_{kn} - \beta_{kn})\|y_{(k-1)n} - q\| + \alpha_{kn}(\|y_{(k-1)n} - q\| + d_n) \\
 &\quad + \beta_{kn}M \\
 &\leq \|y_{(k-1)n} - q\| + d_n + \beta_{kn}M \\
 &\leq \|x_n - q\| + G_n^{k-1} + d_n + \beta_{kn}M \\
 &= \|x_n - q\| + G_n^k
 \end{aligned} \tag{2.7}$$

where $G_n^k = G_n^{k-1} + d_n + \beta_{kn}M$ so it follows that $\{G_n^k\}$ is a nonnegative real sequence such that $\sum_{n=1}^\infty G_n^k < \infty$. By applying Lemma 1.1 we get that $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists for all $q \in F$.

Lemma 2.2 *Let X be a real uniformly convex Banach space and C be a nonempty closed convex subset of X . Let $T_i : C \rightarrow C (i \in I)$ be given asymptotically nonexpansive mappings in the intermediate sense. Let $\{d_n\}$ be defined by (2.1). Let $\{x_n\}$ be defined by (1.3) with $\{\alpha_{in}\}$ be a sequence in $[\epsilon, 1 - \epsilon], \epsilon \in (0, 1)$ and $\sum_{n=1}^\infty \beta_{in} < \infty$, for all $i \in I$ and $n \geq 1$. Let $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$. Then $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$ for all $i \in I$.*

Proof: By Lemma 2.1 we have that $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists for all $q \in F$.

Let $\lim_{n \rightarrow \infty} \|x_n - q\| = d$, for some $d \geq 0$. From (2.6) we get

$$\|y_{in} - q\| \leq \|x_n - q\| + G_n^i$$

where $\{G_n^i\}$ is a nonnegative real sequence such that $\sum_{n=1}^\infty G_n^i < \infty$ for $i = 1, 2, \dots, k - 1$. Taking limsup on the both sides of above we get

$$\limsup_{n \rightarrow \infty} \|y_{in} - q\| \leq \limsup_{n \rightarrow \infty} (\|x_n - q\| + G_n^i) = d \tag{2.8}$$

for $i = 1, 2, \dots, k - 1$. Now for $2 \leq i \leq k$,

$$\|T_i^n y_{(i-1)n} - q\| \leq \|y_{(i-1)n} - q\| + d_n$$

so for $2 \leq i \leq k$,

$$\limsup_{n \rightarrow \infty} \|T_i^n y_{(i-1)n} - q\| \leq \limsup_{n \rightarrow \infty} (\|y_{(i-1)n} - q\| + d_n) \leq d \tag{2.9}$$

Now

$$\begin{aligned}
 d &= \lim_{n \rightarrow \infty} \|x_{n+1} - q\| \\
 &= \lim_{n \rightarrow \infty} \|(1 - \alpha_{kn})(y_{(k-1)n} - q + \beta_{kn}(u_{kn} - y_{(k-1)n})) + \\
 &\quad \alpha_{kn}(T_k^n y_{(k-1)n} - q + \beta_{kn}(u_{kn} - y_{(k-1)n}))\| \tag{2.10}
 \end{aligned}$$

Since $\{x_n\}$ is bounded and $\{u_{in}\}$'s are bounded for $i \in I$ so by (2.6) it follows that $\{y_{in}\}$'s are also bounded for $i = 1, 2, \dots, k - 1$ and hence $\{u_{in} - y_{(i-1)n}\}$ is bounded for $i \in I$. Now

$$\|y_{(k-1)n} - q + \beta_{kn}(u_{kn} - y_{(k-1)n})\| \leq \|y_{(k-1)n} - q\| + \beta_{kn}\|u_{kn} - y_{(k-1)n}\| \tag{2.11}$$

Taking limsup on both sides of (2.11) we get

$$\begin{aligned}
 &\limsup_{n \rightarrow \infty} \|y_{(k-1)n} - q + \beta_{kn}(u_{kn} - y_{(k-1)n})\| \\
 &\leq \limsup_{n \rightarrow \infty} (\|y_{(k-1)n} - q\| + \beta_{kn}\|u_{kn} - y_{(k-1)n}\|) \leq d \tag{2.12}
 \end{aligned}$$

Also

$$\begin{aligned}
 &\|T_k^n y_{(k-1)n} - q + \beta_{kn}(u_{kn} - y_{(k-1)n})\| \\
 &\leq \|T_k^n y_{(k-1)n} - q\| + \beta_{kn}\|u_{kn} - y_{(k-1)n}\| \leq \|y_{(k-1)n} - q\| + d_n + \beta_{kn}\|u_{kn} - y_{(k-1)n}\| \tag{2.13}
 \end{aligned}$$

Taking limsup on the both sides of (2.13) we get

$$\begin{aligned}
 &\limsup_{n \rightarrow \infty} \|T_k^n y_{(k-1)n} - q + \beta_{kn}(u_{kn} - y_{(k-1)n})\| \\
 &\leq \limsup_{n \rightarrow \infty} (\|y_{(k-1)n} - q\| + d_n + \beta_{kn}\|u_{kn} - y_{(k-1)n}\|) \leq d \tag{2.14}
 \end{aligned}$$

From Lemma 1.2 by using (2.10), (2.12), (2.14) we get

$$\lim_{n \rightarrow \infty} \|T_k^n y_{(k-1)n} - y_{(k-1)n}\| = 0 \tag{2.15}$$

Now

$$\begin{aligned}
 \|x_{n+1} - q\| &\leq (1 - \alpha_{kn} - \beta_{kn})\|y_{(k-1)n} - q\| + \alpha_{kn}\|T_k^n y_{(k-1)n} - q\| \\
 &\quad + \beta_{kn}\|u_{kn} - q\| \\
 &\leq (1 - \alpha_{kn} - \beta_{kn})\|y_{(k-1)n} - q\| + \alpha_{kn}(\|y_{(k-1)n} - q\| + d_n) \\
 &\quad + \beta_{kn}M \\
 &\leq \|y_{(k-1)n} - q\| + d_n + \beta_{kn}M
 \end{aligned}$$

which implies that

$$d \leq \liminf_{n \rightarrow \infty} \|y_{(k-1)n} - q\| \tag{2.16}$$

So from (2.8) and (2.16) we have

$$\begin{aligned}
 d &= \lim_{n \rightarrow \infty} \|y_{(k-1)n} - q\| \\
 &= \lim_{n \rightarrow \infty} \|(1 - \alpha_{(k-1)n})(y_{(k-2)n} - q + \beta_{(k-1)n}(u_{(k-1)n} - y_{(k-2)n})) + \\
 &\quad \alpha_{(k-1)n}(T_{k-1}^n y_{(k-2)n} - q + \beta_{(k-1)n}(u_{(k-1)n} - y_{(k-2)n}))\| \tag{2.17}
 \end{aligned}$$

Now

$$\|y_{(k-2)n} - q + \beta_{(k-1)n}(u_{(k-1)n} - y_{(k-2)n})\| \leq \|y_{(k-2)n} - q\| + \beta_{(k-1)n}\|u_{(k-1)n} - y_{(k-2)n}\| \tag{2.18}$$

Taking limsup on the both sides of (2.18) we get

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \|y_{(k-2)n} - q + \beta_{(k-1)n}(u_{(k-1)n} - y_{(k-2)n})\| \\ & \leq \limsup_{n \rightarrow \infty} (\|y_{(k-2)n} - q\| + \beta_{(k-1)n}\|u_{(k-1)n} - y_{(k-2)n}\|) \leq d \end{aligned} \tag{2.19}$$

Also

$$\begin{aligned} & \|T_{k-1}^n y_{(k-2)n} - q + \beta_{(k-1)n}(u_{(k-1)n} - y_{(k-2)n})\| \\ & \leq \|T_{k-1}^n y_{(k-2)n} - q\| + \beta_{(k-1)n}\|u_{(k-1)n} - y_{(k-2)n}\| \\ & \leq \|y_{(k-2)n} - q\| + d_n + \beta_{(k-1)n}\|u_{(k-1)n} - y_{(k-2)n}\| \end{aligned} \tag{2.20}$$

Taking limsup on the both sides of (2.20) we get

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \|T_{k-1}^n y_{(k-2)n} - q + \beta_{(k-1)n}(u_{(k-1)n} - y_{(k-2)n})\| \\ & \leq \limsup_{n \rightarrow \infty} (\|y_{(k-2)n} - q\| + d_n + \beta_{(k-1)n}\|u_{(k-1)n} - y_{(k-2)n}\|) \leq d \end{aligned} \tag{2.21}$$

From Lemma 1.2 by using (2.17), (2.19), (2.21) we get

$$\lim_{n \rightarrow \infty} \|T_{k-1}^n y_{(k-2)n} - y_{(k-2)n}\| = 0 \tag{2.22}$$

Now

$$\begin{aligned} \|y_{(k-1)n} - q\| & \leq (1 - \alpha_{(k-1)n} - \beta_{(k-1)n})\|y_{(k-2)n} - q\| + \alpha_{(k-1)n}\|T_{k-1}^n y_{(k-2)n} - q\| \\ & \quad + \beta_{(k-1)n}\|u_{(k-1)n} - q\| \\ & \leq (1 - \alpha_{(k-1)n} - \beta_{(k-1)n})\|y_{(k-2)n} - q\| + \alpha_{(k-1)n}(\|y_{(k-2)n} - q\| + d_n) \\ & \quad + \beta_{(k-1)n}M \\ & \leq \|y_{(k-2)n} - q\| + d_n + \beta_{(k-1)n}M \end{aligned}$$

which implies that

$$d \leq \liminf_{n \rightarrow \infty} \|y_{(k-2)n} - q\| \tag{2.23}$$

So from (2.8) and (2.23) we have

$$\begin{aligned} d & = \lim_{n \rightarrow \infty} \|y_{(k-2)n} - q\| \\ & = \lim_{n \rightarrow \infty} \|(1 - \alpha_{(k-2)n})(y_{(k-3)n} - q + \beta_{(k-2)n}(u_{(k-2)n} - y_{(k-3)n})) + \\ & \quad \alpha_{(k-2)n}(T_{k-2}^n y_{(k-3)n} - q + \beta_{(k-2)n}(u_{(k-2)n} - y_{(k-3)n}))\| \end{aligned} \tag{2.24}$$

Then proceeding as above we get

$$\lim_{n \rightarrow \infty} \|T_{k-2}^n y_{(k-3)n} - y_{(k-3)n}\| = 0 \tag{2.25}$$

Continuing the above process we can get

$$\lim_{n \rightarrow \infty} \|T_3^n y_{2n} - y_{2n}\| = 0 \tag{2.26}$$

$$\lim_{n \rightarrow \infty} \|T_2^n y_{1n} - y_{1n}\| = 0 \tag{2.27}$$

$$\lim_{n \rightarrow \infty} \|T_1^n x_n - x_n\| = 0 \tag{2.28}$$

Now

$$\begin{aligned} \|y_{in} - y_{(i-1)n}\| &= \|(1 - \alpha_{in} - \beta_{in})y_{(i-1)n} + \alpha_{in}T_i^n y_{(i-1)n} + \beta_{in}u_{in} - y_{(i-1)n}\| \\ &\leq \alpha_{in}\|T_i^n y_{(i-1)n} - y_{(i-1)n}\| + \beta_{in}\|u_{in} - y_{(i-1)n}\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned} \tag{2.29}$$

$$\|x_n - y_{in}\| \leq \|x_n - y_{1n}\| + \|y_{1n} - y_{2n}\| + \dots + \|y_{(i-1)n} - y_{in}\| \rightarrow 0 \text{ as } n \rightarrow \infty \tag{2.30}$$

for $i = 1, 2, \dots, k - 1$. Now for $2 \leq i \leq k$,

$$\begin{aligned} \|x_n - T_i^n x_n\| &\leq \|x_n - y_{(i-1)n}\| + \|y_{(i-1)n} - T_i^n y_{(i-1)n}\| + \|T_i^n y_{(i-1)n} - T_i^n x_n\| \\ &\leq 2\|x_n - y_{(i-1)n}\| + \|y_{(i-1)n} - T_i^n y_{(i-1)n}\| + d_n \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned} \tag{2.31}$$

Again

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1 - \alpha_{kn})\|y_{(k-1)n} - x_n\| + \alpha_{kn}\|T_k^n y_{(i-1)n} - x_n\| + \beta_{kn}\|u_{kn} - y_{(k-1)n}\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned} \tag{2.32}$$

Since T_i 's are uniformly continuous and $\lim_{n \rightarrow \infty} \|x_n - T_i^n x_n\| = 0$ so we have

$$\lim_{n \rightarrow \infty} \|T_i x_n - T_i^{n+1} x_n\| = 0 \tag{2.33}$$

$$\begin{aligned} \|x_n - T_i x_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T_i^{n+1} x_{n+1}\| \\ &\quad + \|T_i^{n+1} x_{n+1} - T_i^{n+1} x_n\| + \|T_i^{n+1} x_n - T_i x_n\| \\ &\leq 2\|x_{n+1} - x_n\| + \|x_{n+1} - T_i^{n+1} x_{n+1}\| + d_{n+1} + \|T_i^{n+1} x_n - T_i x_n\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \text{ (by (2.31), (2.32), (2.33))} \end{aligned}$$

Theorem 2.1 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X . Let $T_i : C \rightarrow C (i \in I)$ be given asymptotically nonexpansive mappings in the intermediate sense. Let $\{d_n\}$ be defined by (2.1). Let $\{x_n\}$ be defined by (1.3) with $\{\alpha_{in}\}$ be a sequence in $[\epsilon, 1 - \epsilon], \epsilon \in (0, 1)$ and $\sum_{n=1}^{\infty} \beta_{in} < \infty$, for all $i \in I$. Let $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$. If $\{T_i : i \in I\}$ satisfies Condition (\bar{B}) , then $\{x_n\}$ converges strongly to some common fixed point of $\{T_1, T_2, \dots, T_k\}$.*

Proof: By Lemma 2.1 we have that $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists for all $q \in F$. Let $\lim_{n \rightarrow \infty} \|x_n - q\| = d$, for some $d \geq 0$. If $d = 0$, then there is nothing to prove. Let us assume that $d > 0$. From (2.7) we get

$$\|x_{n+1} - q\| \leq \|x_n - q\| + G_n^k$$

where $\{G_n^k\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty} G_n^k < \infty$. This gives that

$$d(x_{n+1}, F) \leq d(x_n, F) + G_n^k$$

By Lemma 1.1 we have that $\lim_{n \rightarrow \infty} d(x_n, F)$ exists. Also by Lemma 2.2 we get that $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$ for all $i \in I$. Again since $\{T_1, T_2, \dots, T_k\}$ satisfy Condition (\bar{B}) we have that $\lim_{n \rightarrow \infty} f(d(x_n, F)) = 0$. Since $f : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function with $f(0) = 0$ and $\lim_{n \rightarrow \infty} d(x_n, F)$ exists so we conclude that $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. Now we will show that $\{x_n\}$ is a Cauchy sequence. From (2.7) we get

$$\|x_{n+1} - q\| \leq \|x_n - q\| + G_n^k$$

where $\{G_n^k\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty} G_n^k < \infty$. Thus for any $p > 1$ we have that

$$\begin{aligned} \|x_{n+p} - q\| &\leq \|x_{n+p-1} - q\| + G_{n+p-1}^k \\ &\leq \|x_{n+p-2} - q\| + G_{n+p-2}^k + G_{n+p-1}^k \\ &\dots\dots\dots \\ &\leq \|x_n - q\| + \sum_{j=n}^{n+p-1} G_j^k \end{aligned}$$

Since $\sum_{n=1}^{\infty} G_n^k < \infty$ and $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ so there exists $N_1 \in N$ such that for all $n \geq N_1$ we have $d(x_n, F) < \frac{\epsilon}{3}$ and $\sum_{n=N_1}^{\infty} G_n^k < \frac{\epsilon}{6}$. Therefore there exists $\bar{x} \in F$ such that $\|x_{N_1} - \bar{x}\| = d(x_{N_1}, \bar{x}) < \frac{\epsilon}{3}$. Therefore we have

$$\begin{aligned} \|x_{n+p} - x_n\| &\leq \|x_{n+p} - \bar{x}\| + \|x_n - \bar{x}\| \\ &< \|x_{N_1} - \bar{x}\| + \sum_{j=N_1}^{n+p-1} G_j^k + \|x_{N_1} - \bar{x}\| + \sum_{j=N_1}^{n-1} G_j^k \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{3} + \frac{\epsilon}{6} = \epsilon \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence. Since C is closed subset of X so C is complete. Hence there exists $x^* \in C$ such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. Now note that

$$|d(x^*, F) - d(x_n, F)| \leq \|x^* - x_n\| \rightarrow 0 \text{ for all } n \tag{2.34}$$

Since $\lim_{n \rightarrow \infty} x_n = x^*$ and $\lim_{n \rightarrow \infty} d(x_n, F) = 0$ therefore from (2.34) it follows that $d(x^*, F) = 0$ that is $x^* \in F$. Thus $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \dots, T_k\}$.

Theorem 2.2 *Let X be a uniformly convex Banach space satisfies Opial's property and C be a nonempty closed convex subset of X . Let $T_i : C \rightarrow C (i \in I)$ be given asymptotically nonexpansive mappings in the intermediate sense. Let $\{d_n\}$ be defined by (2.1). Let $\{x_n\}$ be defined by (1.3) with $\{\alpha_{in}\}$ be a sequence in $[\epsilon, 1 - \epsilon], \epsilon \in (0, 1)$ and $\sum_{n=1}^{\infty} \beta_{in} < \infty$, for all $i \in I$. Let $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$. Then $\{x_n\}$ converges weakly to some common fixed point of $\{T_1, T_2, \dots, T_k\}$.*

Proof: By Lemma 2.2 we get $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$, for all $i \in I$. So by the uniform continuity of T_1 we get $\lim_{n \rightarrow \infty} \|x_n - T_1^m x_n\| = 0$ for all $m \in N$. Then by applying Lemma 1.3 with the τ -topology taken as weak topology and we get the conclusion as follows: By Lemma 1.3 there exist $z_1 \in F(T_1)$ such that $x_n \rightharpoonup z_1$ ($x_n \rightarrow z_1$ weakly) as $n \rightarrow \infty$. Similarly by Lemma 1.3 there exist $z_2 \in F(T_2)$ such that $x_n \rightharpoonup z_2$ as $n \rightarrow \infty$ and $z_3 \in F(T_3)$ such that $x_n \rightharpoonup z_3$ as $n \rightarrow \infty$ and $z_k \in F(T_k)$ such that $x_n \rightharpoonup z_k$ as $n \rightarrow \infty$. Since weak limit is unique so we must have $z_1 = z_2 = z_3 = \dots = z_k \in \bigcap_{i=1}^k F(T_i) = F$. Thus $\{x_n\}$ converges weakly to a common fixed point of $\{T_1, T_2, \dots, T_k\}$. This completes the proof.

Theorem 2.3 *Let X be a uniformly convex Banach space such that X^* has the Kadec-Klee property and C be a nonempty closed convex subset of X . Let $T_i : C \rightarrow C$ ($i \in I$) be given asymptotically nonexpansive mappings in the intermediate sense. Let $\{d_n\}$ be defined by (2.1). Let $\{x_n\}$ be defined by (1.3) with $\{\alpha_{in}\}$ be a sequence in $[\epsilon, 1 - \epsilon]$, $\epsilon \in (0, 1)$ and $\sum_{n=1}^{\infty} \beta_{in} < \infty$, for all $i \in I$. Let $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$. If $\{T_1, T_2, \dots, T_k\}$ satisfy Condition (\bar{B}) , then $\{x_n\}$ converges weakly to some common fixed point of $\{T_1, T_2, \dots, T_k\}$.*

Proof: By Lemma 2.2 we get $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0$, for all $i \in I$. So by the uniform continuity of T_i we get

$$\lim_{n \rightarrow \infty} \|x_n - T_i^m x_n\| = 0 \text{ for any } m \geq 1 \quad (2.35)$$

Now by Lemma 2.1 we have that $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists for all $q \in F$. Again as in the proof of Theorem 2.1 we can prove that $\{x_n\}$ is a Cauchy sequence. So for any $m \in N$ we have

$$\|x_{n+m} - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty \quad (2.36)$$

From (2.35) and (2.36) we get

$$\|x_{n+m} - T_i^m x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

which in other word implies that

$$\lim_{n \rightarrow \infty} [\sup_{m \geq 0} \|x_{n+m} - T_i^m x_n\|] = 0 \quad (2.37)$$

So from (2.37) it follows that $\{x_n\}$ is almost orbit of T_i for all $i \in I$. Also from (2.32) we have that $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. So $\{x_{n+1} - x_n\}$ is strongly convergent to 0, therefore $\{x_{n+1} - x_n\}$ is weakly convergent to 0. Thus by Lemma 1.4 we conclude that $\{x_n\}$ is weakly convergent to a fixed point of T_i . Since weak limit is unique so we must have that $\{x_n\}$ converges weakly to a common fixed point of $\{T_1, T_2, \dots, T_k\}$. This completes the proof.

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