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Solving of Nonlinear System of Fredholm-Volterra Integro-Differential Equations by using Discrete Collocation Method

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Abstract

In this paper, we solve nonlinear system of Fredholm-Volterra integro-differential equations by using discrete collocation method. These types of systems of integral equations are important and they can be used in engineering and some of the applied sciences such as population dynamics, reaction-diffusion in small cells and models of epidemic diffusion. Also these equations with convolution kernel can be solved by discrete collocation method. By the above mentioned method we approximate solution of equation by no smooth piecewise polynomials, for validity and ability the method we solve some examples with high accuracy.

1. Introduction

At first we introduce some of methods for solving integral equations and integro-differential equations. In [1, 6] authors used from numerical methods for solving Fredholm integral equations also in [8] Fredholm integro-differential was solved by wavelet Petrov-Galerkin method. One application of the above equation is in chemical absorption kinetics, see [5]. Also in [3, 7] some of results about solving Volterra integral equations are presented. Semi orthogonal spline wavelets and spline are used for solving integro-differential equation respectively in [2, 4]. Consider the system of nonlinear Fredholm-Volterra integro-differential equations with convolution kernel,

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$$\begin{cases} \sum_{r=0}^{m-1} p_r(t)\, y_1^{(r)}(t) = f_1(t) + \int_0^t k_1(t,\tau) G_1(y_1(\tau))\, d\tau + \int_0^1 k_2(t,\tau) G_2(y_2(\tau))\, d\tau, \\ \sum_{r=0}^{m-1} q_r(t)\, y_2^{(r)}(t) = f_2(t) + \int_0^t k_3(t,\tau) G_3(y_1(\tau))\, d\tau + \int_0^1 k_4(t,\tau) G_4(y_2(\tau))\, d\tau, \\ y_1^{(r)}(0) = \alpha_r,\ y_2^{(r)}(0) = \beta_r,\ r = 0,1,\dots,(m-2), \quad t \in [0, \mathrm{T}] \end{cases} \tag{1}$$

where $p_r(t),\, q_r(t)$ for $r = 0,1,\ldots,m-1,$ and $k_i(t,\tau)$ for $i = 1,\ldots,4$ and $f_s(t)$ for $s = 1,2$ are known functions and $y_1(\tau), y_2(\tau)$ are unknown that must be determined. $G_i(y_i(\tau))$ for $i = 1,\ldots,4$ are nonlinear respect to $y_1(\tau)$, $y_2(\tau)$. For approximating of $y_s(t),\ s = 1,2$ we use Lagrange polynomial interpolation obtained by no smooth piecewise polynomial space, so we introduce mesh I_h on $[0,\mathrm{T}]$,

$I_h = \{ t_n = nh \ ; \quad n = 0,1,\dots,N\,, \qquad N\,h = \mathrm{T} \}.$

We define $u_s(\tau),\ s = 1,2$ as an element of no smooth piecewise polynomial with degree less than m as follows,

$$y_s(\tau) \approx u_s(\tau) = \begin{cases} u_{s,0}(\tau), & t_0 < \tau \le t_1, \\ u_{s,1}(\tau), & t_1 < \tau \le t_2, \\ \vdots & \\ u_{s,N-1}(\tau), & t_{N-1} < \tau \le t_N, \\ s = 1,2. & \end{cases} \tag{2}$$

In fact $u_s(\tau) \in S_{m-1}^{(-1)}(\mathrm{I}_h)$ where,

$$S_{m-1}^{(-1)}(\mathrm{I}_h) = \{u_s(\tau) |\ u_s(\tau) = u_{s,n}(t) \in \Pi_{m-1}, \quad \tau \in (t_n, t_{n+1}], \quad s=1,2\,,\ n=0,\dots N-1\}, \tag{3}$$

$t_0 = 0, \quad t_1 = \frac{T}{N}, \dots, T_N,$

to simplicity we choose T=1 and we begin to use Lagrange interpolation with $(c_i, y_{s,n,i})$ points for $n = 0,1,\dots,\ (N-1),\ s = 1,2$ and $i = 0,1,\ldots m$, to find $u_{s,n}(t)$ in (2) where $y_{s,n,i}$ is an approximation $y_s(t_{n,i})$ and $c_i, \quad i = 1,\dots,m$ are m-points Gauss as collocation parameters that can be obtained by roots of Legendre polynomial on $[0,1]$, we give Gauss's points in the case of $m = 4$,

$$p(s) = \sqrt{7}(20s^3 - 30s^2 + 12s - 1) = 0,$$

$$c_1 = \frac{5-\sqrt{15}}{10}, \qquad c_2 = \frac{1}{2}, \qquad c_3 = \frac{5+\sqrt{15}}{10}, \tag{4}$$

and also, $c_4 = 1$ can be introduced as an end- point of [0, 1], also mesh points follow that,

$t_{n,i} = t_n + c_i h\,, \quad i = 1,\dots,m\,, \quad n = 0,1,\dots,(N-1).$

By considering of interpolation we can write

$$u_s(t_{n,i}) = y_{s,n,i}\,, \quad i=1,\dots,m\,, \quad n=0,1,\dots,(N-1),\; s=1,2. \tag{5}$$

and,

$$u_{s,n}(t_n+\theta h) = \sum_{i=1}^{m} L_i(\theta) y_{s,n,i}\,, \quad \theta\in[0,1], \quad s=1,2, \qquad n=0,1,\dots(N-1), \tag{6}$$

where $L_i(\theta),\, i=1,2,\dots,m$ are Legendre polynomials.

Assume $\tau = t_n+\theta h$, so $\tau\in(t_n,t_{n+1}]$, in this way $u_s(\tau),\, s=1,2$ will be restricted to subintervals such as $(t_n,t_{n+1}]$.

According the above process now, nonlinear system (1) can be converted the following form:

$$\begin{cases}\sum_{r=0}^{m-1} p_r(t) y_1^{(r)}(t) = f_1(t) + \int_0^{t_n} k_1(t,\tau) G_1(y_1(\tau))\,d\tau + \int_{t_n}^{t} k_1(t,\tau) G_1(y_1(\tau))\,d\tau + \\ \int_0^{t_n} k_2(t,\tau) G_2(y_2(\tau))\,d\tau + \int_{t_n}^{1} k_2(t,\tau) G_2(y_2(\tau))\,d\tau, \\ \sum_{r=0}^{m-1} q_r(t) y_2^{(r)}(t) = f_2(t) + \int_0^{t_n} k_3(t,\tau) G_3(y_1(\tau)\,d\tau + \int_{t_n}^{t} k_3(t,\tau) G_3(y_1(\tau))\,d\tau + \\ \int_0^{t_n} k_4(t,\tau) G_4(y_2(\tau))\,d\tau + \int_{t_n}^{1} k_4(t,\tau) G_4(y_2(\tau))\,d\tau. \end{cases} \tag{7}$$

Also the discrete form of system (1), is given by,

$$F_{n,1}(t) = f_1(t) + \int_0^{t_n} k_1(t,\tau) G_1(y_1(\tau))\,d\tau + \int_0^{t_n} k_2(t,\tau) G_2(y_2(\tau))\,d\tau, \tag{8}$$

$$\sum_{r=0}^{m-1} p_r(t) y_1^{(r)}(t) = F_{n,1}(t) + \int_{t_n}^{t} k_1(t,\tau) G_1(y_1(\tau))\,d\tau + \int_{t_n}^{1} k_2(t,\tau) G_2(y_2(\tau))\,d\tau. \tag{9}$$

$$F_{n,2}(t) = f_2(t) + \int_0^{t_n} k_3(t,\tau) G_3(y_1(\tau))\,d\tau + \int_0^{t_n} k_4(t,\tau) G_4(y_2(\tau))\,d\tau, \tag{10}$$

$$\sum_{r=0}^{m-1} q_r(t) y_2^{(r)}(t) = F_{n,2}(t) + \int_{t_n}^{t} k_3(t,\tau) G_3(y_1(\tau))\,d\tau + \int_{t_n}^{1} k_4(t,\tau) G_4(y_2(\tau))\,d\tau. \tag{11}$$

We use Eqs.(2-7) for obtaining of Eqs.(8-11), so we have

$$\begin{cases} F_{n,1}(t_{n,i}) = f_1(_{n,i}) + \int_0^{t_n} k_1(t_{n,i},\tau) G_1(u_1(\tau))\,d\tau + \int_0^{t_n} k_2(t_{n,i},\tau) G_2(u_2(\tau))\,d\tau, \\ F_{n,2}(t_{n,i}) = f_2(t_{n,i}) + \int_0^{t_n} k_3(t_{n,i},\tau) G_3(u_s(\tau))\,d\tau + \int_0^{t_n} k_4(t_{n,i},\tau) G_4(u_s(\tau))\,d\tau, \\ n=0,1,\dots,(N-1), \quad i=1,\dots,m, \quad s=1,2. \end{cases} \tag{12}$$

and,

$$\begin{cases} \sum_{i=1}^{m} L_i^{(r)}(t_{n,i})\, y_{1,n,i} = F_{n,1}(t_{n,i}) + \int_{t_n}^{t_{n,i}} k_1(t_{n,i},\tau) G_1(u_s(\tau))\, d\tau + \int_{t_n}^{1} k_2(t_{n,i},\tau) G_2(u_s(\tau))\, d\tau, \\ \sum_{i=1}^{m} L_i^{(r)}(t_{n,i})\, y_{2,n,i} = F_{n,2}(t_{n,i}) + \int_{t_n}^{t_{n,i}} k_3(t_{n,i},\tau) G_3(u_s(\tau))\, d\tau + \int_{t_n}^{1} k_4(t_{n,i},\tau) G_4(u_s(\tau))\, d\tau, \\ n = 0,1,\ldots,(N-1), \quad i = 1,\ldots,m, \quad r = 0,1,\ldots,(m-1), \quad s = 1,2. \end{cases} \tag{13}$$

where $u_s(\tau)$, $s=1,2$ is given in (3). We compute $F_{n,1}(t_{n,i})$ and $F_{n,2}(t_{n,i})$; $n = 0,1,\ldots,\ (N-1)$, and $i = 1,2,\ldots,m$ by (12) such that they can use for obtaining $y_{s,n,i}$'S in system (13).
In (12) for $n = j$ we have:

$$\begin{cases} F_{j,1}(t_{j,i}) = f_1(t_{j,i}) + \int_0^{t_j} k_1(t_{j,i},\tau) G_1(u_s(\tau))\, d\tau + \int_0^{t_j} k_2(t_{j,i},\tau) G_2(u_s(\tau))\, d\tau, \\ F_{j,2}(t_{j,i}) = f_2(t_{j,i}) + \int_0^{t_j} k_3(t_{j,i},\tau) G_3(u_s(\tau))\, d\tau + \int_0^{t_j} k_4(t_{j,i},\tau) G_4(u_s(\tau))\, d\tau, \\ j = 0,1,\ldots,(N-1), \ s = 1,2. \end{cases} \tag{14}$$

From (3), (7) and (14) we can write

$$\begin{cases} F_{0,1}(t_{0,i}) = f_1(t_{0,i}), \quad i = 1,\ldots,m, \\ F_{j,1}(t_{j,i}) = f_1(t_{j,i}) + \sum_{p=0}^{(j-1)} \left[\int_{t_p}^{t_{p+1}} k_1(t_{p,i},\tau) G_1(u_{1,p}(\tau))\, d\tau + \int_{t_p}^{t_{p+1}} k_2(t_{p,i},\tau) G_2(u_{2,p}(\tau))\, d\tau \right], \\ j = 1,\ldots,(N-1), \end{cases} \tag{15}$$

and

$$\begin{cases} F_{0,2}(t_{0,i}) = f_2(t_{0,i}), \quad i = 1,\ldots,m, \\ F_{j,2}(t_{j,i}) = f_2(t_{j,i}) + \sum_{p=0}^{(j-1)} \left[\int_{t_p}^{t_{p+1}} k_3(t_{p,i},\tau) G_3(u_{1,p}(\tau))\, d\tau + \int_{t_p}^{t_{p+1}} k_4(t_{p,i},\tau) G_4(u_{2,p}(\tau))\, d\tau \right], \\ j = 1,\ldots,(N-1). \end{cases} \tag{16}$$

By substituting $F_{n,1}(t_{n,i})$ and $F_{n,2}(t_{n,i})$ for $n = 0,1,\ldots(N-1)$, $i = 1,2,\ldots,m$ in (14) we obtain $y_{s,n,i}$'S by the following system:

$$\begin{cases} \sum_{i=1}^{m} L_i^{(r)}(t_{n,i})\, y_{1,n,i} = F_{n,1}(t_{n,i}) + \int_{t_n}^{t_{n,i}} k_1(t_{n,i},\tau)\, G_1(u_{1,n}(\tau))\, d\tau + \sum_{p=n}^{(N-1)} \int_{t_p}^{t_{p+1}} k_2(t_{n,i},\tau)\, G_2(u_{2,p}(\tau))\, d\tau, \\ \sum_{i=1}^{m} L_i^{(r)}(t_{n,i})\, y_{2,n,i} = F_{n,2}(t_{n,i}) + \int_{t_n}^{t_{n,i}} k_3(t_{n,i},\tau)\, G_3(u_{1,n}(\tau))\, d\tau + \sum_{p=n}^{(N-1)} \int_{t_p}^{t_{p+1}} k_4(t_{n,i},\tau)\, G_4(u_{2,p}(\tau))\, d\tau \\ n = 1,\dots,(N-1), \quad i = 1,\dots,m,\ r = 0,\dots,(m-1). \end{cases} \qquad (17)$$

Here Lagrange polynomials introduce by $L_q(\theta) = \prod_{\substack{i=1 \\ i\neq q}}^{m} \frac{(\theta - c_i)}{(c_q - c_i)}, \tau = t_p + \theta h$, where $\theta = \frac{\tau - t_p}{h}$ and

$$\text{also}, u_{s,p}(\tau) = \sum_{q=1}^{m} L_q(\theta) y_{s,n,q} = \sum_{q=1}^{m} L_q\left(\frac{\tau - t_p}{h}\right) y_{s,n,q}, \qquad p = 0,1,\dots,(N-1), \quad s = 1,2. \qquad (18)$$

Considering of the above formulations, at first by solving algebraic system (17), we find $y_{s,n,i}$'S, $i = 1,2,\dots,m$, $n = 0,1,\dots,(N-1)$, $s = 1,2$, then by (18), $u_{s,p}(\tau)$, , $p = 1,\dots,(N-1)$, $s = 1,2$ can be found and finally $u_s(t)$, $s = 1,2$ is give by (2) as an approximation of $y_s(t)$, $s = 1,2$.

3. Numerical results

In this section, for validity and ability of proposed method we solve two examples of nonlinear system of Fredholm-Volterra integro-differential equations

Example 1.

Consider nonlinear system of Fredholm-Volterra integro-differential equations system with convolution kernel as follows,

$$\begin{cases} y_1^{(3)}(t) + t\, y_1'(t) + y_1(t) = f_1(t) + \int_0^t e^{t-\tau}\, y_1(\tau)\, d\tau + \int_0^1 (t-\tau)^3\, y_2^{\,3}(\tau)\, d\tau, \\ (t^2+1)\, y_2''(t) + y_2'(t) + y_2(t) = f_2(t) + \int_0^t \sin(t-\tau)\, y_1(\tau)\, d\tau + \int_0^1 (t-\tau)^4\, y^2{}_2(\tau)\, d\tau, \\ y_1(0) = 1,\ y_2(0) = 0,\ y_2'(0) = 0, \end{cases}$$

with $f_1(t) = \frac{66}{13} - 4e^t + \frac{19}{4}t + \frac{47}{11}t^2 - \frac{t^3}{10}$ and $f_2(t) = \frac{10}{11} + \frac{27}{5}t + \frac{4}{3}t^2 + \frac{15}{2}t^3 - \frac{t^4}{7} - \cos t + \sin t$ and exact solutions are $y_1(t) = t^2 + t + 1$ and $y_2(t) = t^3$.

For $m = 4, h = \frac{1}{2}, T = 1,$ we use Eqs.(15-18) in the discrete collocation method that the interpolation is interdicted the following form,

$$u_1(t)=\begin{cases}1+t+t^2-2.98996\times10^{-11}t^3, & 0<t\le\frac{1}{2},\\ 1+t+t^2+1.61435\times10^{-11}t^3, & \frac{1}{2}<t\le1,\end{cases}$$

and

$$u_2(t)=\begin{cases}-3.99694\times10^{-11}+5.276\times10^{-11}t-1.46381\times10^{-11}t^2+t^3, & 0<t\le\frac{1}{2},\\ -1.38254\times10^{-10}+5.12586\times10^{-11}t+3.30758\times10^{-11}t^2+t^3, & \frac{1}{2}<t\le1,\end{cases}$$

Absolute error in some points is show in table1.

Table1. Numerical results for example 1.

t_i	$\|u_1(t)-y_1(t)\|_2$	$\|u_2(t)-y_2(t)\|_2$
0	2.0×10^{-10}	3.9×10^{-11}
0.1	2.0×10^{-10}	3.4×10^{-11}
0.2	2.1×10^{-10}	2.9×10^{-11}
0.3	2.2×10^{-10}	2.5×10^{-11}
0.4	2.3×10^{-10}	2.0×10^{-11}
0.5	2.5×10^{-10}	1.6×10^{-11}
0.6	2.0×10^{-11}	9.7×10^{-11}
0.7	2.8×10^{-11}	8.8×10^{-11}
0.8	3.6×10^{-11}	8.0×10^{-11}
0.9	4.4×10^{-11}	7.1×10^{-11}
1	5.4×10^{-11}	6.1×10^{-11}

From the numerical results are shown in the table 1, we conclude, that the proposed method has a high accuracy.

Example 2.

Consider the following nonlinear system of Fredholm-Volterra integro-differential equations,

$$\begin{cases}t\,y_1'(t)+y_1(t)=f_1(t)+\int_0^t(t-\tau)^2e^{y_1(\tau)}d\tau+\int_0^1\cos(t-\tau)\,y_2(\tau)\,d\tau,\\ y_2''(t)+y_2(t)=f_2(t)+\int_0^t\sin(t-\tau)\cos(y_1(\tau))\,d\tau+\int_0^1e^{t-\tau}y_2(\tau)\,d\tau,\\ y_1(0)=0,\ y_2(0)=-2,\ y_2'(0)=0,\end{cases}$$

with $f_1(t)=2-2e^t+4t+t^2-2\cos(1-t)+3\sin(1-t)+4\sin t$ and $f_2(t)=3e^{-1+t}+t^2-\frac{1}{2}t\sin t$ and exact solutions are $y_1(t)=t$ and $y_2(t)=t^2-2$. In the case of $m=4,\ h=\frac{1}{2},\ T=1$ in the discrete collocation method and by using $(15-18)$, the interpolation $u_1(t)$ and $u_2(t)$ can be introduced by,

$$u_1(t)=\begin{cases} -1.31077\times10^{-11}+t+8.17124\times10^{-14}t^2+1.89004\times10^{-12}t^3, & 0<t\le\frac{1}{2}, \\ 8.22595\times10^{-11}+t+1.66985\times10^{-10}t^2-5.33191\times10^{-11}t^3, & \frac{1}{2}<t\le1, \end{cases}$$

and

$$u_2(t)=\begin{cases} -2+3.31916\times10^{-10}t+t^2-5.83213\times10^{-11}t^3, & 0<t\le\frac{1}{2}, \\ -2+3.96184\times10^{-10}t+t^2-9.86802\times10^{-11}t^3, & \frac{1}{2}<t\le1, \end{cases}$$

absolute errors in some points are given in table 2 .

Table 2 . Numerical results for example 2

t_i	$\|u_1(t)-y_1(t)\|_2$	$\|u_2(t)-y_2(t)\|_2$
0	1.3×10^{-11}	7.2×10^{-11}
0.1	1.3×10^{-11}	3.9×10^{-11}
0.2	1.3×10^{-11}	5.4×10^{-12}
0.3	1.3×10^{-11}	2.8×10^{-11}
0.4	1.3×10^{-11}	6.1×10^{-11}
0.5	1.3×10^{-11}	9.3×10^{-11}
0.6	1.5×10^{-11}	1.2×10^{-10}
0.7	1.0×10^{-11}	6.6×10^{-11}
0.8	7.6×10^{-12}	9.2×10^{-12}
0.9	5.2×10^{-12}	4.7×10^{-11}
1	3.2×10^{-12}	1.0×10^{-10}

By considering numerical results in the table 2, we have concluded that the discrete collocation method has a high accuracy and it can be used for other nonlinear problems.

4. Conclusion

In this paper, we have used discrete collocation method for solving system of non-linear Fredholm-Volterra integro-differential equation. By using this method we have reached high accuracy that numerical results have shown that our claim is right.

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