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Bernstein polynomials for solving Abel's integral equation

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Abstract

This paper presents a numerical method for solving Abel's integral equation as singular Volterra integral equations. In the proposed method, the functions in Abel's integral equation are approximated based on Bernstein polynomials (BPs) and therefore, the solving of Abel's integral equation is reduced to the solving of linear algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Keywords: Abel's integral equations, Singular Volterra integral equations, Bernstein polynomials.

1. Introduction

Abel integral equation was derived by Abel when he was generalizing and solving the Tautochrone problem. It allows users to compute the total time required for a particle to fall along a given curve. This integral equation has two forms [1].

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First Kind:

$$\int_0^t \frac{y(x)}{\sqrt{t-x}}\,dx = g(t), \tag{1}$$

Second Kind:

$$y(t) = \int_0^t \frac{y(x)}{\sqrt{t-x}}\,dx + g(t), \tag{2}$$

where $g(t)$ is a given function and $y(t)$ is an unknown function. The Abel problem is to find a path $y(t)$ that with a specified total time of descent from a given initial height, $g(t)$, a particle will follow if it moves without initial velocity and only under the influence of gravity.

Abel's equation is one of the integral equations derived directly from a concrete problem of mechanics or physics (without passing through a differential equation). Historically, Abel's problem is the first one to lead to the study of integral equations.

The generalized Abel's integral equations on a finite segment appeared for the first time in the paper of Zeilon [2].

Several numerical methods for approximating the solution of integral equations are known. For Fredholm–Hammerstein integral equations, the classical method of successive approximations was introduced in [3]. A variation of the Nystrom method was presented in [4]. A collocation type method was developed in [5]. In [6], Brunner applied a collocation-type method to nonlinear Volterra–Hammerstein integral equations and integro-differential equations, and discussed its connection with the iterated collocation method. Guoqiang [7] introduced and discussed the asymptotic error expansion of a collocation-type method for Volterra–Hammerstein integral equations. The methods in [5, 7] transform a given integral equation into a system of nonlinear equations, which has to be solved with some kind of iterative method. A numerical solution of weakly singular Volterra integral equations was introduced in [8]. However, very few references have been found in technical literature dealing with integral equations.

The rest of this paper is as follows. In Section 2, BPs are introduced, therefore we approximate functions by using BPs and also we discuss best approximation and convergence analysis. Also, we get a new operational matrix for Abel's integral equation by BPs in Section 3. In Section 4, we apply BPs for solving the first and second of Abel's integral equation. Finally, section 5 concludes our work in this paper.

2. Bernstein polynomials and their properties

2.1. Definition of Bernstein polynomials

The Bernstein polynomials (BPs) of mth-degree are defined on the interval $[0,1]$ as follows:

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \quad i = 0,1,\cdots,m \tag{3}$$

Set $\{B_{0,m}(x), B_{1,m}(x), \cdots, B_{m,m}(x)\}$ in Hilbert space $L^2[0,1]$ is a complete basis. Therefore, any polynomial of degree m can be expanded in terms of linear combination of $B_{i,m}(x)$ $(i = 0,1,\cdots,m)$ as follows

$$P(x) = \sum_{i=0}^{m} c_i \, B_{i,m}(x). \tag{4}$$

By using binomial expansion of $(1-x)^{m-i}$, we have

$$\begin{aligned} B_{i,m}(x) &= \binom{m}{i} x^i \, (1-x)^{m-i} = \binom{m}{i} x^i \left(\sum_{k=0}^{m-i} (-1)^k \binom{m}{i} \binom{m-i}{k} x^k \right) \\ &= \sum_{k=0}^{m-i} (-1)^k \binom{m}{i} \binom{m-i}{k} x^{i+k} \;, \end{aligned}$$

for $i = 0,1,\ldots,m.$ Now, we define

$$\begin{aligned} \Phi_m(x) &= [B_{0,m}(x), B_{1,m}(x), \ldots, B_{m,m}(x)]^T, \\ T_m(x) &= [1, x, \ldots, x^m]^T. \end{aligned} \tag{5}$$

Therefore we can write

$$\Phi_m(x) = A \, T_m(x), \tag{6}$$

where

$$A_{i+1,\, j+1} = \begin{cases} (-1)^{j-i} \binom{m}{i} \binom{m-i}{j-i} & i \le j, \\ 0 & i > j, \end{cases} \qquad i, j = 0,1,\ldots,m. \tag{7}$$

Matrix A is a $(m+1)\times(m+1)$ upper triangular matrix and $\det(A)=\prod_{i=0}^{m}\binom{m}{i}\neq 0$, therefore A is an invertible matrix.

2.2. Approximation of functions by using BPs

Suppose that $H=L^2[0,1]$ is a Hilbert space with the inner product that is defined by $<f,g>=\int_0^1 f(x)g(x)\,dx$ and $\{B_{0,m},B_{1,m},\cdots,B_{m,m}\}\subset H$ be the set of BPs of m th-degree. Let $S_m=Span\{B_{0,m},B_{1,m},\cdots,B_{m,m}\}$ and f be an arbitrary element in H. Since S_m is a finite dimensional and closed subspace, therefore S_m is a complete subset of H. So, f has the unique best approximation out of S_m such as $s_0\in S_m$. Therefore, exist the unique coefficients c_i, $i=0,1,\cdots,m$ such that [9]

$$f(x)\approx s_0(x)=\sum_{i=0}^{m}c_i\,B_{i,m}(x)=c^T\Phi_m(x), \tag{8}$$

where $c^T=[c_1,c_2,\cdots,c_m]$ can be obtained by $c^T<\Phi_m,\Phi_m>=<f,\Phi_m>$, such that

$$<f,\Phi_m>=\int_0^1 f(x)\Phi_m(x)^T\,dx=\left[<f,B_{0,m}>,<f,B_{1,m}>,\cdots,<f,B_{m,m}>\right]. \tag{9}$$

We define $Q=<\Phi_m,\Phi_m>$ that is a $(m+1)\times(m+1)$ matrix and is said dual matrix of $\Phi_m(x)$, and we can obtain

$$Q_{i+1,\,j+1}=\int_0^1 B_{i,m}(x)B_{j,m}(x)\,dx=\frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}},\quad i,j=0,1,\cdots m. \tag{10}$$

Lemma 2.1. Suppose that the function $f:[0,1]\to R$ is $m+1$ times continuously differentiable $\left(i.e.\ f\in C^{m+1}([0,1])\right)$, and $S_m=Span\{B_{0,m},B_{1,m},\cdots,B_{m,m}\}$. If c^TB be the best approximation f out of S then

$$\left\|f-c^TB\right\|_{L^2[0,1]}\leq\frac{\hat{K}}{(m+1)!\sqrt{2m+3}}, \tag{11}$$

where $\hat{K}=\max_{x\in[0,1]}\left|f^{(m+1)}(x)\right|$.

Proof. We know that Set $\{1,x,\ldots,x^m\}$ is a basis for polynomials space of degree m. Therefore we define $y_1(x)=f(0)+x\,f'(0)+\frac{x^2}{2!}f''(0)+\cdots+\frac{x^m}{m!}f^{(m)}(0)$. From Taylor expansion we have

$$\left| f(x) - y_1(x) \right| = \left| f^{(m+1)}(\xi_x) \frac{x^{m+1}}{(m+1)!} \right|, \tag{12}$$

where $\xi_x \in (0,1)$. Since $c^T B$ is the best approximation f out of S , $y_1 \in S_m$ and from (12) we have

$$\left\| f - c^T B \right\|^2_{L^2[0,1]} \le \left\| f - y_1 \right\|^2_{L^2[0,1]} = \int_0^1 \left| f(x) - y_1(x) \right|^2 dx = \int_0^1 \left| f^{(m+1)}(\xi_x) \right|^2 \left(\frac{x^{m+1}}{(m+1)!} \right)^2 dx$$

$$\le \frac{\hat{K}^2}{(m+1)!^2} \int_0^1 x^{2m+2} dx = \frac{\hat{K}^2}{(m+1)!^2 \, (2m+3)}.$$

Then by taking square roots, the proof is complete. □

The previous Lemma shows that the error vanishes as $m \to \infty$.

3. Solution of Abel integral equation

In this section, Abel integral equations (1) and (2) are solved by using BPs. At first, the functions $g(t)$ and $y(t)$ are approximated by (8) as follows

$$y(t) \approx C^T \Phi_m(t), \tag{13}$$

$$g(t) \approx G^T \Phi_m(t). \tag{14}$$

Substituting (13) and (14) into (1) and (2), the integral equations are transformed as

First Kind: $$\int_0^t \frac{C^T \Phi_m(x)}{\sqrt{t-x}} dx = G^T \Phi_m(t), \tag{15}$$

Second Kind: $$C^T \Phi_m(t) = \int_0^t \frac{C^T \Phi_m(x)}{\sqrt{t-x}} dx + G^T \Phi_m(t). \tag{16}$$

Now we need to obtain the operational matrix $F_{(m+1)\times(m+1)}$ for $\int_0^t \frac{\Phi_m(x)}{\sqrt{t-x}} dx$ where

$$\int_0^t \frac{\Phi_m(x)}{\sqrt{t-x}} dx \approx F \, \Phi_m(t). \tag{17}$$

Therefore, we can write

$$\int_0^t \frac{\Phi_m(x)}{\sqrt{t-x}}dx = t^{-\frac{1}{2}} * \Phi_m(t), \quad 0 \le t \le 1, \tag{18}$$

where $*$ denotes the convolution product and

$$t^{-\frac{1}{2}} * \Phi_m(t) = \left[t^{-\frac{1}{2}} * B_{0,m}(t),\, t^{-\frac{1}{2}} * B_{1,m}(t),\, \ldots, t^{-\frac{1}{2}} * B_{m,m}(t) \right]^T. \tag{19}$$

From (6) we have

$$t^{-\frac{1}{2}} * \Phi_m(t) = t^{-\frac{1}{2}} * \left(AT_m(t)\right) = A\left(t^{-\frac{1}{2}} * T_m(t) \right). \tag{20}$$

Since $\int_0^t \frac{x^i}{\sqrt{t-x}}dx = \frac{\Gamma\left(\frac{1}{2}\right) i!}{\Gamma\left(\frac{3}{2}+i\right)} t^{\frac{1}{2}+i}$ $(i = 0, 1, \ldots, m)$, we get

$$t^{-\frac{1}{2}} * T_m(t) = \left[t^{-\frac{1}{2}} * 1,\, t^{-\frac{1}{2}} * t, \ldots, t^{-\frac{1}{2}} * t^m \right]^T$$

$$= \Gamma\left(\frac{1}{2}\right) \left[\frac{0!}{\Gamma\left(\frac{3}{2}\right)} t^{\frac{1}{2}},\, \frac{1!}{\Gamma\left(\frac{5}{2}\right)} t^{\frac{3}{2}},\, \ldots,\, \frac{m!}{\Gamma\left(\frac{3}{2}+m\right)} t^{\frac{1}{2}+m} \right]^T = \Gamma\left(\frac{1}{2}\right) D\, \overline{T}, \tag{21}$$

where $D_{(m+1)\times(m+1)}$ and $\overline{T}_{(m+1)\times 1}$ are as follows:

$$D_{i,j} = \begin{cases} \dfrac{i!}{\Gamma\left(\frac{1}{2}+i+1\right)} & i = j, \\ 0 & i \ne j, \end{cases} \quad i, j = 0, 1, \ldots, m, \qquad \overline{T} = \left[t^{\frac{1}{2}}, t^{\frac{1}{2}+1}, \ldots, t^{\frac{1}{2}+m} \right]^T. \tag{22}$$

Now, we need to approximate $t^{\frac{1}{2}+i}$ $(i = 0, 1, \ldots, m)$ with respect to BPs by using (8). Therefore, we continue our work as follows:

$$t^{\frac{1}{2}+i} \approx E_i^T \Phi_m(t), \tag{23}$$

where $E_i = [E_{i,0}, E_{i,1}, \ldots, E_{i,m}]^T$, $(i = 0, 1, \ldots, m)$. Therefore we have

$$E_i = D^{-1}\left(\int_0^1 t^{\frac{1}{2}+i}\,\Phi_m(t)\,dt\right)$$

$$= D^{-1}\left[\int_0^1 t^{\frac{1}{2}+i} B_{0,m}(t)\,dt, \int_0^1 t^{\frac{1}{2}+i} B_{1,m}(t)\,dt, \ldots, \int_0^1 t^{\frac{1}{2}+i} B_{m,m}(t)\,dt\right]^T = D^{-1}\overline{E}_i, \qquad (24)$$

where

$$\overline{E}_i = [\overline{E}_{i,0}, \overline{E}_{i,1}, \ldots, \overline{E}_{i,m}]^T, \qquad (25)$$

$$\overline{E}_{i,j} = \int_0^1 t^{\frac{1}{2}+i} B_{j,m}(t)\,dt = \frac{m!\,\Gamma\left(i+j+\frac{3}{2}\right)}{j!\,\Gamma\left(i+m+\frac{5}{2}\right)}, \quad i, j = 0,1,\ldots,m. \qquad (26)$$

Now, we suppose E is an $(m+1)\times(m+1)$ matrix that has vector $D^{-1}\,\overline{E}_i$ $(i = 0,1,\ldots,m)$ for ith column's.

Finally, from (18)-(26), we obtain

$$\int_0^t \frac{\Phi_m(x)}{\sqrt{t-x}}\,dx \approx A\,D\,E^T\ \Phi_m(t). \qquad (27)$$

Therefore

$$F = A\,D\,E^T. \qquad (28)$$

Then from (13) , (14) and (17) the Eqs. (1) and (2) reduce to

First Kind:

$$C^T F\,\Phi_m(t) = G^T \Phi_m(t), \qquad (29)$$

Second Kind:

$$C^T \Phi_m(t) = C^T F\,\Phi_m(t) + G^T \Phi_m(t). \qquad (30)$$

Then we get the following systems

First Kind:

$$C^T F = G^T \,, \tag{31}$$

Second Kind:

$$C^T = C^T F + G^T \,. \tag{32}$$

Eqs. (31) and (32) are a linear systems in terms of C and the solution is

First Kind:

$$C^T = G^T F^{-1}, \tag{33}$$

Second Kind:

$$C^T = G^T \left(I - F\right)^{-1}. \tag{34}$$

Thus $y_m(t) = C^T \Phi_m(t)$ is the approximate solution of the Eqs. (1) and (2).

4. Numerical examples

We applied the method presented in this paper and solved two examples given in [10]. This method differs from the collocation method given in [5, 6] and method of [8] and thus could be used as a basis for comparison.

Example 4.1. Consider the second kind of Abel's integral equation:

$$y(t) = t^2 + \frac{16}{15} t^{\frac{5}{2}} - \int_0^t \frac{y(x)}{\sqrt{t-x}} dx,$$

which has the exact solution $y(t) = t^2$. We applied the proposed method with $m = 3$ and we obtain

$$G = \left[\frac{128}{45045} \,,\, -\frac{256}{12285} \,,\, \frac{67829}{135135} \,,\, \frac{93173}{45045} \right]^T,$$

$$F = \begin{bmatrix} \frac{1904}{6435} & \frac{608}{585} & \frac{224}{2145} & \frac{2176}{6435} \\ -\frac{928}{15015} & \frac{27712}{45045} & \frac{29888}{45045} & \frac{1536}{5005} \\ \frac{128}{5005} & -\frac{7424}{45045} & \frac{7936}{9009} & \frac{1024}{2145} \\ -\frac{256}{45045} & \frac{512}{15015} & -\frac{512}{4095} & \frac{8192}{9009} \end{bmatrix},$$

$$C = \left[0\ ,\ 0\ ,\ \frac{1}{3}\ ,\ 1\right]^T.$$

Therefore $y_3(t) = C^T \Phi_3(t) = t^2$ which is the exact solution.

Example 4.2. Consider the first kind of Abel's integral equation:

$$\frac{2}{105}\sqrt{t}\left(48t^3 - 56t^2 + 105\right) = \int_0^t \frac{y(x)}{\sqrt{t-x}}\,dx,$$

which has the exact solution $y(t) = t^3 - t^2 + 1$. We applied the proposed method with $m = 3$ and we get

$$G = \left[\frac{11056}{45045}\ ,\ \frac{213344}{135135}\ ,\ \frac{33248}{27027}\ ,\ \frac{84352}{45045}\right]^T,$$

$$C = \left[1,1,\frac{2}{3},1\right].$$

Also, F is similar to the previous example.

Therefore $y_3(t) = C^T \Phi_3(t) = t^3 - t^2 + 1$ which is the exact solution.

Conclusion

The aim of present work is to develop an efficient and accurate method for solving singular Volterra integral equations by BPs. the original integral equations are transformed to a system of linear algebraic equations. In this method we get good approximation with low terms of basis. Illustrative examples are included to demonstrate the validity and applicability of the technique.

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