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On critical exponent for the existence and multiplicity of positive weak solutions for a class of (p, q)-Laplacian nonlinear system

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Abstract

In this paper, we prove the existence of positive weak solution for the nonlinear elliptic system

$$\begin{cases} -\Delta_p u = \lambda_1 u^a + \mu_1 v^b, & x \in \Omega, \\ -\Delta_q v = \lambda_2 u^c + \mu_2 v^d, & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases}$$

where $\Delta_s z = \operatorname{div}(|\nabla z|^{s-2}\nabla z)$, $s > 1$, λ_1, λ_2, μ_1 and μ_2 are positive parameters, and Ω is a bounded domain in $\mathbb{R}^N$, $a + c < p - 1$ and $b + d < q - 1$. We also discuss a multiplicity result when $0 < \lambda_1, \lambda_2, \mu_1, \mu_2 < \lambda^*$ for some λ^*. We obtain our results via the method of sub - and super solutions.

Keywords: Positive weak solution; p-Laplacian; Sub - and super solutions.
AMS Subject Classification: 35J55, 35J65.

1 Introduction

Consider the system

$$\begin{cases} -\Delta_p u = \lambda_1 u^a + \mu_1 v^b, & x \in \Omega, \\ -\Delta_q v = \lambda_2 u^c + \mu_2 v^d, & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_s z = \operatorname{div}(|\nabla z|^{s-2}\nabla z)$, $s > 1$, $\lambda_1, \lambda_2, \mu_1$ and μ_2 are positive parameters, and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. In addition, we assume that $1 < p, q < N$.

Chen in [3] discussed the system $-\Delta_p u = \lambda u^a v^b, -\Delta_q v = \lambda u^c v^d$ with zero Dirichlet boundary condition. Our goal is somewhat modest: to find an analogue result for problem (1.1). The boundary value problem

$$\begin{cases} -\Delta_p u = \lambda_1 f_1(u) + \mu_1 g_1(v), & x \in \Omega, \\ -\Delta_q v = \lambda_2 f_2(v) + \mu_2 g_2(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

have been studied extensively in recent years (see [1, 6]). Ali and Shivaji [1] studied the existence of positive solution for the problem (1.2) with $\lambda_1 + \mu_1$ and $\lambda_2 + \mu_2$ large when

$$\lim_{x \to 0} \frac{g_1(M[g_2(x)]^{1/q-1})}{x^{p-1}} = 0$$

for every $M > 0$, $\lim_{x\to 0} \frac{f_1(x)}{x^{p-1}} = 0$ and $\lim_{x\to 0} \frac{f_2(x)}{x^{q-1}} = 0$. Also we refer to [6] for results on systems related to (1.2) in the case $p = q$ and nonlinearities with falling zeroes.

These problems arise in some physical models and are interesting in applications at combustion, mathematical biology, chemical reactions.

The above assumption for the problem (1.1) implies $bc < (p-1)(q-1)$. In this work, we first prove the existence of positive solution with each positive parameters $\lambda_1, \lambda_2, \mu_1, \mu_2$, and next establish the existence of at least two positive solution when $0 < \lambda_1, \lambda_2, \mu_1, \mu_2 < \lambda^*$ for some λ^*. Our approach is based on the method of sub- and supersolutions (see [4, 5, 7]).

The main results of this paper are Theorems 1.1 and 1.3.

Theorem 1.1. *Suppose that $a, d \geq 0$, $b, c > 0$, $a + c < p - 1$ and $b + d < q - 1$. Then problem* (1.1) *has a positive weak solution for each positive parameters $\lambda_1, \lambda_2, \mu_1$ and μ_2.*

Remark 1.2. *By Theorem* 1.1 *in* [1], *there exists a positive solution of the problem* (1.1) *for $\lambda_1 + \mu_1$ and $\lambda_2 + \mu_2$ large. But in our paper,* (1.1) *has a positive solution for each positive parameters $\lambda_1, \lambda_2, \mu_1$ and μ_2.*

Theorem 1.3. *Suppose that $a, d \geq 0$, $b, c > 0$, $a + c < p - 1$ and $b + d < q - 1$. Then there exists $\lambda^* > 0$ such that for $0 < \lambda_1, \lambda_2, \mu_1, \mu_2 < \lambda^*$,* (1.1) *has at least two positive weak solutions.*

2 Proof of Theorem 1.1

Proof. We shall establish Theorem 1.1 by constructing a positive weak subsolution $(\psi_1, \psi_2) \in {W_0}^{1,p} \times {W_0}^{1,q}$ and a supersolution $(z_1, z_2) \in {W_0}^{1,p} \times {W_0}^{1,q}$ of (1.1) such that $\psi_i \leq z_i$ for $i = 1, 2$.

That is , ψ_i, z_i satisfies $(\psi_1, \psi_2) = (0,0) = (z_1, z_2)$ on $\partial\Omega$,

$$\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1.\nabla f_1 dx \leq \int_\Omega \left(\lambda_1\psi_1^a + \mu_1\psi_2^b\right) f_1 dx,$$
$$\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2.\nabla f_2 dx \leq \int_\Omega \left(\lambda_2\psi_1^c + \mu_2\psi_2^d\right) f_2 dx,$$
$$\int_\Omega |\nabla z_1|^{p-2}\nabla z_1.\nabla f_1 dx \geq \int_\Omega \left(\lambda_1 z_1^a + \mu_1 z_2^b\right) f_1 dx,$$
$$\int_\Omega |\nabla z_2|^{q-2}\nabla z_2.\nabla f_2 dx \geq \int_\Omega \left(\lambda_2 z_1^c + \mu_2 z_2^d\right) f_2 dx,$$

for all test functions $f_1 \in {W_0}^{1,p}$ and $f_2 \in {W_0}^{1,q}$ with $f_1, f_2 \geq 0$.

Let λ_p and λ_q be the first eigenvalue of the problems, respectively,

$$\begin{aligned} -\Delta_p\phi_p &= \lambda_p\phi_p^{p-1}, x \in \Omega, \quad \phi_p = 0, x \in \partial\Omega, \\ -\Delta_q\phi_q &= \lambda_q\phi_q^{q-1}, x \in \Omega, \quad \phi_q = 0, x \in \partial\Omega, \end{aligned} \tag{2.1}$$

where ϕ_p and ϕ_q denote the corresponding eigenfunctions, respectively, satisfying $\phi_p, \phi_q > 0$ in Ω and $|\nabla\phi_p| > 0, |\nabla\phi_q| > 0$ on $\partial\Omega$. Without loss of generality, we let $\|\phi_p\|_p = \|\phi_q\|_q = 1$.

Since $bc < (p-1-a)(q-1-d)$, we can take k such that

$$\frac{c}{q-1-d} < k < \frac{p-1-a}{b}. \tag{2.2}$$

We shall verify that $(\psi_1(x), \psi_2(x)) = (\sigma\phi_p^{p'}, \sigma^k\phi_q^{q'})$ is a subsolution of (1.1), where $p' = p/(p-1), q' = q/(q-1)$ and $\sigma > 0$ is small and specified later. Let the test function $f_1(x) \in {W_0}^{1,p}$ with $f_1(x) \geq 0$. Then it follows from (2.1) that

$$\begin{aligned} \int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1.\nabla f_1 dx &= (ap')^{p-1}\int_\Omega \phi_p\, |\nabla\phi_p|^{p-2}\nabla\phi_p.\nabla f_1 dx \\ &= (ap')^{p-1}\int_\Omega \Big[|\nabla\phi_p|^{p-2}\nabla\phi_p.\nabla(\phi_p f_1) - |\nabla\phi_p|^p f_1\Big] dx \\ &= (ap')^{p-1}\int_\Omega (\lambda_p{\phi_p}^p - |\nabla\phi_p|^p) f_1 dx. \end{aligned}$$

Similarly,

$$\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2.\nabla f_2 dx = (a^k q')^{q-1}\int_\Omega (\lambda_q{\phi_q}^q - |\nabla\phi_q|^q) f_2 dx,$$

for all $f_2(x) \in {W_0}^{1,q}$ with $f_2(x) \geq 0$. Let $\eta > 0, \mu > 0$ be such that

$$\lambda_p{\phi_p}^p - |\nabla\phi_p|^p \leq 0, \quad \lambda_q{\phi_q}^q - |\nabla\phi_q|^q \leq 0, \qquad x \in \bar{\Omega}_\eta, \tag{2.3}$$

and $\mu \leq \phi_p, \phi_q \leq 1$ on $\Omega_0 = \Omega \setminus \bar{\Omega}_\eta$, where $\bar{\Omega}_\eta = \{x \in \Omega \ : \ d(x, \partial\Omega) \leq \eta\}$. (This is possible since $\phi_p = \phi_q = 0$ and $|\nabla\phi_p|, |\nabla\phi_q| > 0$ on $\partial\Omega$.)

We have from (2.3) that

$$(ap')^{p-1}\int_{\bar{\Omega}_\eta} (\lambda_p{\phi_p}^p - |\nabla\phi_p|^p) f_1 dx \leq 0 \leq \int_{\bar{\Omega}_\eta} \left(\lambda_1\psi_1^a + \mu_1\psi_2^b\right) f_1 dx,$$

and

$$(a^k q')^{q-1}\int_{\bar{\Omega}_\eta}(\lambda_q {\phi_q}^q - |\nabla\phi_q|^q) f_2 dx \le 0 \le \int_{\bar{\Omega}_\eta}\Big(\lambda_2\psi_1^c + \mu_2\psi_2^d\Big) f_2 dx.$$

On the other hand, in Ω_0 let $r_1 = (p-1-a)/c,\ r_2 = (p-1-a)/(p-1-a-c),\ s_1 = (q-1-d)/b,\ s_2 = (q-1-d)/(q-1-d-b)$; Note that $(1/r_1)+(1/r_2) = 1, (1/s_1)+(1/s_2) = 1$. We have from (2.2) that

$$p-1-\frac{a}{r_1}-\frac{kb}{r_2} \ge p-1-a-kb > 0,$$
$$k(q-1-\frac{d}{s_2})-\frac{c}{s_1} \ge k(q-1-d)-c > 0.$$

Thus we choose $\sigma > 0$ such that

$$\sigma^{p-1-\frac{a}{r_1}-\frac{kb}{r_2}} p'^{p-1}\lambda_p {\phi_p}^p \le {\lambda_1}^{\frac{1}{r_1}} {\mu_1}^{\frac{1}{r_2}} \mu^{a\delta+q}, \quad x\in\Omega_0,$$
$$\sigma^{k(q-1-\frac{d}{s_2})-\frac{c}{s_1}} q'^{q-1}\lambda_q {\phi_q}^q \le {\lambda_2}^{\frac{1}{s_1}} {\mu_2}^{\frac{1}{s_2}} \mu^{d\gamma+p}, \quad x\in\Omega_0,$$

where $\delta = p/(p-1-a), \gamma = q/(q-1-d)$. Furthermore

$$a\delta r_1 = \frac{ap}{p-1-a-c} \ge \frac{pa}{p-1} = p'a,$$
$$d\gamma s_2 = \frac{dq}{q-1-d-b} \ge \frac{qd}{q-1} = q'd,$$

and

$$ps_1 = p(\frac{q-1-d}{b}) > p(\frac{c}{p-1-a}) \ge \frac{pc}{p-1} = p'c,$$
$$qr_2 = q(\frac{p-1-a}{c}) > q(\frac{b}{q-1-d}) \ge \frac{qb}{q-1} = q'b.$$

These relations and Young inequality show that

$$\begin{aligned}(ap')^{p-1}\int_{\Omega_0}(\lambda_p {\phi_p}^p - |\nabla\phi_p|^p) f_1 dx &\le (ap')^{p-1}\int_{\Omega_0}\lambda_p {\phi_p}^p f_1 dx\\
&\le \int_{\Omega_0}({\lambda_1}^{\frac{1}{r_1}}\sigma^{\frac{a}{r_1}}\mu^{a\delta})({\mu_1}^{\frac{1}{r_2}}\sigma^{\frac{kb}{r_2}}\mu^{q}) f_1 dx\\
&\le \int_{\Omega_0}\left[\frac{({\lambda_1}^{\frac{1}{r_1}}\sigma^{\frac{a}{r_1}}\mu^{a\delta})^{r_1}}{r_1} + \frac{({\mu_1}^{\frac{1}{r_2}}\sigma^{\frac{kb}{r_2}}\mu^{q})^{r_2}}{r_2}\right] f_1 dx\\
&\le \int_{\Omega_0}\left[({\lambda_1}^{\frac{1}{r_1}}\sigma^{\frac{a}{r_1}}\mu^{a\delta})^{r_1} + ({\mu_1}^{\frac{1}{r_2}}\sigma^{\frac{kb}{r_2}}\mu^{q})^{r_2}\right] f_1 dx\\
&= \int_{\Omega_0}(\lambda_1\sigma^a\mu^{a\delta r_1} + \mu_1\sigma^{kb}\mu^{qr_2}) f_1 dx\\
&\le \int_{\Omega_0}(\lambda_1\sigma^a {\phi_p}^{p'a} + \mu_1\sigma^{kb} {\phi_q}^{q'b}) f_1 dx\\
&= \int_{\Omega_0}(\lambda_1\psi_1^a + \mu_1\psi_2^b) f_1 dx,\end{aligned}$$

and

$$
\begin{aligned}
(a^k q')^{q-1} \int_{\Omega_0} (\lambda_q {\phi_q}^q - |\nabla \phi_q|^q) f_2 dx &\leq (a^k q')^{q-1} \int_{\Omega_0} \lambda_q {\phi_q}^q f_2 dx \\
&\leq \int_{\Omega_0} (\lambda_2{}^{\frac{1}{s_1}} \sigma^{\frac{c}{s_1}} \mu^p)(\mu_2{}^{\frac{1}{s_2}} \sigma^{\frac{kd}{s_2}} \mu^{d\gamma}) f_2 dx \\
&\leq \int_{\Omega_0} \left[(\lambda_2{}^{\frac{1}{s_1}} \sigma^{\frac{c}{s_1}} \mu^p)^{s_1} + (\mu_2{}^{\frac{1}{s_2}} \sigma^{\frac{kd}{s_2}} \mu^d)^{s_2} \right] f_2 dx \\
&= \int_{\Omega_0} (\lambda_2 \sigma^c \mu^{ps_1} + \mu_2 \sigma^{kd} \mu^{ds_2}) f_2 dx \\
&\leq \int_{\Omega_0} (\lambda_2 \sigma^c {\phi_p}^{p'c} + \mu_2 \sigma^{kd} {\phi_q}^{q'd}) f_2 dx \\
&= \int_{\Omega_0} (\lambda_2 \psi_1^c + \mu_2 \psi_2^d) f_2 dx.
\end{aligned}
$$

Thus the (ψ_1, ψ_2) is a subsolution of (1.1).

Next, we construct a supersolution of (1.1). Let $e_p(x), e_q(x)$ be the positive solutions, respectively, of the problems

$$
\begin{aligned}
-\Delta_p e_p = 1, x \in \Omega, \quad & e_p = 0, x \in \partial\Omega, \\
-\Delta_q e_q = 1, x \in \Omega, \quad & e_q = 0, x \in \partial\Omega.
\end{aligned} \tag{2.4}
$$

Let $(z_1, z_2) := (Ae_p, Be_q)$, where the constants $A, B > 0$ are large and to be chosen later. Then we have from (2.4) that

$$
\begin{aligned}
\int_\Omega |\nabla z_1|^{p-2} \nabla z_1 . \nabla f_1 dx &= A^{p-1} \int_\Omega f_1 dx, \\
\int_\Omega |\nabla z_2|^{q-2} \nabla z_2 . \nabla f_2 dx &= B^{q-1} \int_\Omega f_2 dx.
\end{aligned} \tag{2.5}
$$

Let $l = \|e_p\|_\infty, L = \|e_q\|_\infty$. Since $p, q > 1, a < p-1, d < q-1$, so that

$$
\lim_{x\to\infty} \frac{x^a}{x^{p-1}} = 0 = \lim_{x\to\infty} \frac{x^d}{x^{q-1}}, \lim_{x\to\infty} x^{p-1} = \infty = \lim_{x\to\infty} x^{q-1}.
$$

These imply that there exist positive large constants A,B such that

$$
A^{p-1} \geq \lambda_1 (Al)^a + \mu_1 (BL)^b, B^{q-1} \geq \lambda_2 (Al)^c + \mu_2 (BL)^d.
$$

Thus

$$
\begin{aligned}
A^{p-1} \int_\Omega f_1 dx &\geq \int_\Omega (\lambda_1 {z_1}^a + \mu_1 {z_2}^b) f_1 dx, \\
B^{q-1} \int_\Omega f_2 dx &\geq \int_\Omega (\lambda_2 {z_1}^c + \mu_2 {z_2}^d) f_2 dx,
\end{aligned}
$$

i.e., (z_1, z_2) is a supersolution of (1.1) with $z_i \leq \psi_i$ for A, B larg, $i = 1, 2$. Thus, there exists a solution (u, v) of (1.1) with $\psi_1 \leq u \leq z_1, \psi_2 \leq v \leq z_2$. This completes the proof of theorem 1.1. □

3 Proof of Theorem 1.3

Proof. To prove Theorem 1.3, we will construct a solution (ψ_1, ψ_2), a strict supersolution (ζ_1, ζ_2), a strict subsolution (ω_1, ω_2), and a supersolution (z_1, z_2) for (1.1) such that $(\psi_1, \psi_2) \leq (\zeta_1, \zeta_2) \leq (z_1, z_2)$, $(\psi_1, \psi_2) \leq (\omega_1, \omega_2) \leq (z_1, z_2)$, and $(\omega_1, \omega_2) \nleq (\zeta_1, \zeta_2)$. Then by the three-solution theorem [2, 7], (1.1) has a least three distinct solutions (u_i, v_i), $i = 1, 2, 3$, such that

$$(u_1, v_1) \in [(\psi_1, \psi_2), (\zeta_1, \zeta_2)], (u_2, v_2) \in [(\omega_1, \omega_2), (z_1, z_2)], \tag{3.1}$$

and

$$(u_3, v_3) \in [(\psi_1, \psi_2), (z_1, z_2)] \setminus ([(\psi_1, \psi_2), (\zeta_1, \zeta_2)] \cup [(\omega_1, \omega_2), (z_1, z_2)]). \tag{3.2}$$

We first note that $(\psi_1, \psi_2) = (0, 0)$ is a solution (hence a subsolution). As in section 2, we can always construct a large supersolution (z_1, z_2). By the Theorem 1.1 the problem

$$\begin{cases} -\Delta_p \tilde{\omega_1} = \lambda_1 \tilde{\omega_1}^a + \mu_1 \tilde{\omega_2}^b, & x \in \Omega, \\ -\Delta_q \tilde{\omega_2} = \lambda_2 \tilde{\omega_1}^c + \mu_2 \tilde{\omega_2}^d, & x \in \Omega, \\ \tilde{\omega_1} = 0 = \tilde{\omega_2}, & x \in \partial\Omega, \end{cases}$$

has a positive weak solution $(\tilde{\omega_1}, \tilde{\omega_2})$. Let $\vartheta, \alpha > 0$ be such that

$$\frac{c}{q-1} < \vartheta < \frac{p-1}{b},$$

and

$$\max\{\alpha^{p-1-a}, \alpha^{p-1-\vartheta b}, \alpha^{\vartheta(q-1)-c}, \alpha^{\vartheta(q-1-d)}\} < 1.$$

(This is possible since $bc < (p-1)(q-1)$). Let $\omega_1 = \alpha\tilde{\omega_1}$ and $\omega_2 = \alpha^{\vartheta}\tilde{\omega_2}$. Then for $x \in \Omega$ we have

$$\begin{aligned} \int_\Omega |\nabla\omega_1|^{p-2}\nabla\omega_1.\nabla f_1 dx &= \alpha^{p-1}\int_\Omega |\nabla\tilde{\omega_1}|^{p-2}\nabla\tilde{\omega_1}.\nabla f_1 dx \\ &= \alpha^{p-1}\int_\Omega (\lambda_1\tilde{\omega_1}^a + \mu_1\tilde{\omega_2}^b) f_1 dx \\ &= \int_\Omega (\alpha^{p-1-a}\lambda_1\omega_1{}^a + \alpha^{p-1-\vartheta b}\mu_1\omega_2{}^b) f_1 dx \\ &< \int_\Omega (\lambda_1\omega_1{}^a + \mu_1\omega_2{}^b) f_1 dx, \end{aligned}$$

and

$$\begin{aligned} \int_\Omega |\nabla\omega_2|^{q-2}\nabla\omega_2.\nabla f_2 dx &= \alpha^{\vartheta(q-1)}\int_\Omega |\nabla\tilde{\omega_2}|^{q-2}\nabla\tilde{\omega_2}.\nabla f_2 dx \\ &= \alpha^{\vartheta(q-1)}\int_\Omega (\lambda_2\tilde{\omega_1}^c + \mu_2\tilde{\omega_2}^d) f_2 dx \\ &= \int_\Omega (\alpha^{\vartheta(q-1)-c}\lambda_2\omega_1{}^c + \alpha^{\vartheta(q-1-d)}\mu_2\omega_2{}^d) f_2 dx \\ &< \int_\Omega (\lambda_2\omega_1{}^c + \mu_2\omega_2{}^d) f_2 dx. \end{aligned}$$

Thus (ω_1, ω_2) is a strict subsolution of (1.1). Finally we construct the strict supersolution (ζ_1, ζ_2). Let e_p, e_q, l, L be as described in section 2, and

$$(\zeta_1, \zeta_2) = (\epsilon \frac{e_p}{l}, \epsilon \frac{e_q}{L}), \quad \lambda^* = \min\{\frac{\epsilon^{p-1}}{\beta l^{p-1}}, \frac{\epsilon^{p-1}}{\beta' l^{p-1}}, \frac{\epsilon^{q-1}}{\beta L^{q-1}}, \frac{\epsilon^{q-1}}{\beta' L^{q-1}}\}$$

where $0 < \epsilon < 1, \beta > 1$ and $\beta' = \beta/(\beta - 1)$. Note that $\|\zeta_1\|_\infty = \|\zeta_2\|_\infty = \epsilon$. For $0 < \lambda_1, \lambda_2, \mu_1, \mu_2 < \lambda^*$ observe that

$$\begin{aligned} \frac{\epsilon^{p-1}}{l^{p-1}} &= \frac{\epsilon^{p-1}}{l^{p-1}}(\frac{1}{\beta} + \frac{1}{\beta'}) \\ &= \frac{\epsilon^{p-1}}{\beta l^{p-1}} + \frac{\epsilon^{p-1}}{\beta' l^{p-1}} \\ &> \lambda_1 + \mu_1 \geq \lambda_1 {\zeta_1}^a + \mu_1 {\zeta_2}^b, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \frac{\epsilon^{q-1}}{L^{q-1}} &= \frac{\epsilon^{q-1}}{L^{q-1}}(\frac{1}{\beta} + \frac{1}{\beta'}) \\ &= \frac{\epsilon^{q-1}}{\beta L^{q-1}} + \frac{\epsilon^{q-1}}{\beta' L^{q-1}} \\ &> \lambda_2 + \mu_2 \geq \lambda_2 {\zeta_1}^c + \mu_2 {\zeta_2}^d. \end{aligned} \tag{3.4}$$

Using the inequalities (3.3) and (3.4), we have

$$\int_\Omega |\nabla \zeta_1|^{p-2} \nabla \zeta_1 . \nabla f_1 dx = \frac{\epsilon^{p-1}}{l^{p-1}} \int_\Omega f_1 dx > \int_\Omega (\lambda_1 {\zeta_1}^a + \mu_1 {\zeta_2}^b) f_1 dx,$$
$$\int_\Omega |\nabla \zeta_2|^{q-2} \nabla \zeta_2 . \nabla f_2 dx = \frac{\epsilon^{q-1}}{L^{q-1}} \int_\Omega f_2 dx > \int_\Omega (\lambda_2 {\zeta_1}^c + \mu_2 {\zeta_2}^d) f_2 dx,$$

for all $f_1 \in W_0^{1,p}, f_2 \in W_0^{1,q}$ with $f_1, f_2 > 0$. Thus (ζ_1, ζ_2) is strict supersolution of (1.1). Here we can choose ϵ small so that $(\omega_1, \omega_2) \not\leq (\zeta_1, \zeta_2)$. We note that the proof of Theorem 1.1 we can choose (z_1, z_2) large enough that $(\zeta_1, \zeta_2) \leq (z_1, z_2)$. Thus there exist three solutions (u_i, v_i), $i = 1, 2, 3$, such that satisfies (3.1) and (3.2). Since $(\psi_1, \psi_2) = (0, 0)$ is a solution it may turn out that $(u_1, v_1) \equiv (\psi_1, \psi_2) \equiv (0, 0)$. In any case we have two positive solutions (u_2, v_2) and (u_3, v_3) and Theorem 1.3 is proven. $\square$

References

[1] Jaffar Ali, R. Shivaji, *Positive solutions for a class of p-Laplacian systems with multiple parameters*, J. Math. Anall. Appl. **335** (2007), 1013-1019.

[2] Herbert Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (5) (1976) 620-709.

[3] C. Chen, *On positive weak solutions for a class of quasilinear elliptic systems*, Nonlin. Anal. **62** (2005) 751-756

[4] P. Drábek, J. Hernandez, *Existence and uniqueness of positive solutions for some quasilinear elliptic problem*, Nonlin. Anal. TMA, **44** (2001), 189-204.

[5] P. Drábek, P. Krejčí., P. Takáč., *Nonlinear Differential Equations*, Chapman & Hall/CRC, 1999.

[6] Eun Kyoung lee, R. Shivaji, Jinglong Ye, *Positive solutions for elliptic equations involving nonlinearities with fallig zeroes*, Appl. Math. Letters **22** (2009), 846-851.

[7] R. Shivaji, *A remark on the existence of three solutions via sub-super solutions*, Nonlin. Anal. and Appl., Lecture notes in pure and applied mathematics. **109** (1987), 561-566.