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Existence solution for class of p-laplacian equations

Malihe bagheri
Department of mathematic, Golestan Institute of Higher Education,Golestan Province,Iran
bagherima@yahoo.com

mahnaz bagheri
Department of mathematice, Islamic Azad University, behshar Branch, Iran
m.bagheri@iaubs.ac.ir

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Abstract

We study existence of positive solution of the equation

$$-\Delta_p u = \lambda|u|^{p-2}u + f(x,u)$$

with zero Dirichlet boundary conditions in bounded domain $\Omega \subset \mathbb{R}^n$ where Δ_p denotes the p-laplacian operator defined by $-\Delta_p z = \mathrm{div}(|\nabla z|^{p-2}\nabla z); p, \lambda \in \mathbb{R}$ and $p > 1$.Our main result establishes the existence of weak solution.

Keywords: p-laplacian, weak solution, homogenous.

1. Introduction

In this paper, we are concerned with the existence of positive weak solution for the following problem

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u + f(x,u) & , \quad x \in \Omega \\ u = 0 & , \quad x \in \partial\Omega \end{cases} \tag{1.1}$$

Where $-\Delta_p z = \operatorname{div}(|\nabla z|^{p-2}\nabla z);\ p > 1$ and Ω is a bounded domain in $\mathbb{R}^n$.

This problem is studied in connection with the corresponding eigenvalue problem for the p-laplacian

$$-\Delta_p u = \lambda |u|^{p-2} u \tag{1.2}$$

With the Dirichlet condition

$$u = 0 \tag{1.3}$$

We concentrate on the existence of positive solution to (1.1) when $\lambda < \lambda_1$.

The similar equation (1.1) in the whole of $\mathbb{R}^n$ is studied in [1,2].Essentially the similar result as here we proved in [2] using a bifurcatin argument combined with a critical point theory.we study the problem (1.1) using the fibrering method introduce in [3,4].In section 2 we present some notation and preliminary result.

2. Notation and preliminary results

DIFINITION 1. *Let $\boldsymbol{\Omega}$ be a bounded domain in $\mathbb{R}^n$, $1 < p < \infty$.we will work in the Sobolev space $W = W_0^{1,P}(\Omega)$ equipped with the norm*

$$\|u\|_w = \left(\int_\Omega |\nabla u|^p dx\right)^{1/p} \tag{2.1}$$

DIFINITION 2. *we say that $u \in W$ is a weak solution of (1.1) if*

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx = \lambda \int_\Omega |u|^{p-2} uv dx + \int_\Omega v f(x,u) dx \tag{2.2}$$

For any $v \in W$.

We will denote by $(.,.)_w$ the duality pairing between W^* (the dual space) and W so that the principal part (2.2) can be written as

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx = (-\Delta_p u, v)_w .$$

DIFINITION 3. *A real number $\boldsymbol{\lambda}$ is called an eigenvalue and $\in W, u(x) \not\equiv 0$ is a corresponding eigenfunction to the problem (1.2), (1.3) if*

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx = \lambda \int_\Omega |u|^{p-2} uv dx \tag{2.3}$$

Holds for every $v \in W$.

LEMMA 1. *(See [5, 6]) there exists the first positive eigenvalue λ_1 of the problem (1.2),(1.3) which is characterize as the minimum of the Rayleigh quotient:*

$$\lambda_1 = \min_{\substack{u \in W \\ \int_\Omega |u|^p dx > 0}} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} > 0 \qquad (2.4)$$

It follows from the continuity of the Nemytskii operator [6] and the Sobolev Embedding theorem implies that:

(A_0) *The functional*

$$u \to \int_\Omega |u|^p dx$$

Is weakly continous on W.

Analogously, It follows from $F \in L^\infty(\Omega)$ that :

(A_1) *The functional*

$$u \to \int_\Omega F(x,u) dx$$

Is weakly continous on W.

Let us consider the Euler functional

$$I_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega |u|^p dx + \int_\Omega F(x,u) dx \qquad (2.5)$$

Associated with (1.1) where $F(x,u)$ is primery function f.

It is clear a ciritical point of I_λ corresponds to a weak solution of boundary-value problem (1.1).

We will assume that the function $F(x,u)$ is α-homogenous for every $u \in W$.let us split the function $u \in W$ as follows:

$$u(x) = rv(x) \qquad (2.6)$$

$r \in \mathbb{R}, u \in W$ and subsititute (2.6) into (2.5). We get

$$I_\lambda(rv) = \frac{|r|^p}{p}\int_\Omega |\nabla v|^p dx - \frac{\lambda |r|^p}{p}\int_\Omega |v|^p dx + r^\alpha \int_\Omega F(x,v) dx \qquad (2.7)$$

Let $u \in W$ be the critical point of $I_\lambda(u)$ .then

$$\frac{\partial I_\lambda(rv)}{\partial r} = 0,$$

$$|r|^{p-2} r \int_\Omega |\nabla v|^p \, dx - \lambda |r|^{p-2} r \int_\Omega |v|^p \, dx + \alpha r^\alpha \int_\Omega F(x,v) dx = 0$$

Looking for nontirivial solutions $u \not\equiv 0$, *we have to consider* $r \neq 0$ *under the assumption* $\alpha, r > 0, F(x,u) \neq 0$ *and* $\int_\Omega |\nabla v|^p \, dx - \lambda \int_\Omega |v|^p \, dx \neq 0$, *.Hence*

$$\int_\Omega |\nabla v|^p dx - \lambda \int_\Omega |v|^p dx + \alpha r^{\alpha - p} \int_\Omega F(x,v) dx = 0$$

We get from here that

$$r^{\alpha-p} = \frac{\int_\Omega |\nabla v|^p dx - \lambda \int_\Omega |v|^p dx}{\alpha \int_\Omega F(x,v) dx} > 0 \qquad (2.8)$$

If we calculate r from (2.8) and substitute it into (2.7), we get

$$\tilde{I}_\lambda(v) = I_\lambda(r(v)v)$$

$$= \left(\frac{1}{p} - 1\right) . \frac{\left|\int_\Omega |\nabla v|^p dx - \lambda \int_\Omega |v|^p dx\right|^{\frac{\alpha}{\alpha-p}}}{\left|\alpha \int_\Omega F(x,u) dx\right|^{\frac{p}{\alpha-p}}} \operatorname{sgn}\left(\alpha \int_\Omega F(x,v) dx\right) \qquad (2.9)$$

REMARK 1. we would like to point out that $r = r(v)$ is well defined (and consequently the function $r \to I_\lambda(rv)$ has a unique turning point) provided either

(i) $\int_\Omega F(x,v) dx > 0$ and $\int_\Omega |\nabla v|^p dx - \lambda \int_\Omega |v|^p dx > 0$

or

(ii) $\int_\Omega F(x,v) dx < 0$ and $\int_\Omega |\nabla v|^p dx - \lambda \int_\Omega |v|^p dx < 0$

LEMMA 2. Let us consider the constraint H (v) =c, where the function $H: W \to \mathbb{R}$ satisfies the following condition:

$$(H'(v), v)_w \neq 0 \ \text{ if } H(v) = c \qquad (2.10)$$

then every conditional critical point of the problem

$$\text{crit}\, \{\tilde{I}_\lambda(v) \, ; H(v) = c\} \qquad (2.11)$$

LEMMA 3. Every critical point $v_c \neq 0$ of $\tilde{I}_\lambda$satisfying (i) or (ii) generates a critical point $u_c \in W$ $u_c \neq 0$ of I_λ by the formula

$$u_c = r_c v_c(x) \qquad (2.12)$$

Where $r_c > 0$ is define by (2.8).

3. Main result

Let us consider the conditional variational problem:

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u + f(x,u) & , \quad x \in \Omega \\ u = 0 & , \quad x \in \partial\Omega \end{cases} \qquad (3.1)$$

Let $\lambda_1 > 0$ be the first positive eigenvalue of (1.2) with the Dirichlet condition $u = 0$ on $\partial\Omega$ and $u_1 = u_1(x)$ be the corresponding positive eigenfunction.

We will assume $0 \leq \lambda < \lambda_1$ and function $F(x,u)$ is α-homogenous for every $u \in W$ .Set

$$H_\lambda(v) = \int_\Omega |\nabla v|^p dx - \lambda \int_\Omega |v|^p dx \qquad (3.2)$$

It follows from Lemma 1that $H_\lambda(v) \geq 0$ for any $v \in W$ and it follows from (3.2) that

$$(H_\lambda'(v), v)_w = pH_\lambda(v)\ , v \in W\ .$$

There fore (2.10) is fulfilled if we assume

$$H_\lambda(v) = 1$$

As a contraint and we have to consider the critical point of $\tilde{I}_\lambda(v)$ satisfying $\int_\Omega F(x,u)dx > 0$ .due to the case (i) from Remark 1.

Let us consider the conditional variational problem:

(p_λ) find a maximiser $v_c \in W$ of the problem

$$0 < M_\lambda = \operatorname{Sup}\{\int_\Omega F(x,v)dx\ ;\ \int_\Omega F(x,v)dx > 0, H_\lambda(v) = 1\}$$

Then v_c is a solution to (p_λ) if and only if v_c is a minimiser of $\tilde{I}_\lambda$ subject to the constraint $H_\lambda(v) = 1$ due Lemma 2.

Proof.Let us consider the set

$$W_\lambda = \{v \in W\ ; H_\lambda(v) = 1\ \}$$

From the variational characterisation of λ_1 and $0 \le \lambda < \lambda_1$, it follows that $W_\lambda \neq \emptyset$. Next we prove that this set is bounded in w. Due to the variational characterisation (2.4) of λ_1, we get for any $v \in W_\lambda$:

$$\int_\Omega |\nabla v|^p dx - \lambda \int_\Omega |v|^p dx + 1 \le \frac{\lambda}{\lambda_1} \int_\Omega |\nabla v|^p dx + 1 \qquad (3.3)$$

It follows from (3.3) that for $0 \le \lambda < \lambda_1$, $v \in W_\lambda$:

$$\int_\Omega |\nabla v|^p dx \le \frac{\lambda_1}{\lambda_1 - \lambda}.$$

Hence the maximising sequence $\{v_n\}_{n=1}^{\infty}$ for (p_λ) is bounded in W. Consequently, we may assume that

$$v_n \to v_c \quad \text{in } W$$

Due to (A_1), we have

$$\int_\Omega F(x, u_n)dx \to \int_\Omega F(x, u)dx = M_\lambda > 0 \qquad (3.4)$$

Moreover, we have $H_\lambda(v_n) = 1$ and due to the weak lower semicontinuity of the norm $\|.\|_w$ and (A_0) we get

$$\int_\Omega |\nabla v_c|^p dx \le \lim\inf_{n\to\infty} \int_\Omega |\nabla v_n|^p dx,$$

$$\int_\Omega F(x, v_c)dx = \lim_{n\to\infty} \int_\Omega F(x, v_n)dx$$

Hence

$$H_\lambda(v_c) = \int_\Omega |\nabla v_c|^p dx - \lambda \int_\Omega |v_c|^p dx \le 1 \qquad (3.5)$$

it follows from (3.4) that $v_c \neq 0$ and we may also assume $v_c \ge 0$.we prove that, in fact, equality holds in (3.5) .assume that this is not true,

$$H_\lambda(v_c) < 1,$$

We find $k_c > 1$ such that

$$H_\lambda(k_c v_c) = 1$$

Then $\tilde{v}_c = k_c v_c \in W_\lambda$ and

$$\int_\Omega F(x, \tilde{u}_c)dx = {k_c}^\alpha \int_\Omega F(x, v_c)dx = {k_c}^\alpha M_\lambda > 0$$

Which is a contradiction. Hence $v_c \in W_\lambda$is the solution of (p_λ).

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