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Multiple solution to (p,q)-Laplacian systems with concave nonlinearities

G.A. Afrouzi and M. Bai
Department of mathematics, Faculty of Mathematical Sciences,
University of Mazandaran, Babolsar, Iran
E-mail: afrouzi@umz.ac.ir
bai_marjan@yahoo.com

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Abstract

In this paper we study the (p,q)-Laplacian systems with concave nonlinearities. Using some asymptotic behavior f at zero and infinity, three nontrivial solutions are established.

Keywords: Nonlinear boundary value problem, Concave nonlinearity, (p,q)-Laplacian systems, Variational method, Multiple solutions.

AMS Subject classification: 35j65

1 Introduction

In this paper, we consider problems

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + f_u(x,u,v) & in\ \Omega \\ -\Delta_q v = \lambda |v|^{q-2}v + f_v(x,u,v) & in\ \Omega \\ u = v = 0 & in\ \partial\Omega \end{cases} \quad (1.1)$$

Where $\Omega \subset R^N, (N \geq 1)$ is a bounded with smooth domain and $F \in C^1(\overline{\Omega} \times R^2, R)$.

The functional corresponding to problems (1.1) is

$$J_\lambda(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p\, dx + \frac{1}{q}\int_\Omega |\nabla v|^q\, dx - \frac{\lambda}{p}\int_\Omega |u|^p dx - \frac{\lambda}{q}\int_\Omega |v|^q dx - \int_\Omega F(x,u,v)\, dx$$

Let $W = W_0^{1,P}(\Omega) \times W_0^{1,q}(\Omega)$ with the norm

$$\|(u,v)\| = \|\nabla u\|_p + \|\nabla v\|_q = (\int_\Omega |\nabla u|^p\, dx)^{\frac{1}{p}} + (\int_\Omega |\nabla v|^q\, dx)^{\frac{1}{q}}.$$

It is well known operator -Δ has a sequence of eigenvalues $\{\lambda_k\}$ satisfying

$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \to +\infty$. For general $(p,q) \in (1,+\infty), (-\Delta_p, -\Delta_q)$ has a smallest eigenvalue, i.e., the principle value, λ_1 , which is positive, isolated, simple (see[2]) and admit the following variational characterization

$$\lambda_1 = \inf_{0 \neq u,v \in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)} \frac{\int_\Omega |\nabla u|^p dx + \int_\Omega |\nabla v|^q dx}{\int_\Omega |u|^p dx + \int_\Omega |v|^q dx} \qquad (1.2)$$

Furthermore, the $\lambda_1 -$ eigenfunctions do not change in Ω , and by the maximum principle we may suppose that $\phi_1 > 0$ is a $\lambda_1 -$ eigenfunction. There are many paper concerned with the resonance problem. In [7] L. Shi proved that there exists $\lambda^* > 0$ such that p-Laplacian multiple solutions for a class of (p,q)-Laplacian systems (1.1). Consider the following conditions hold:

(i) $f(0,0) = 0$.
(ii) $f \in C^1(\Omega \times R^2, R)$ and $f'(0,0) > \lambda_1$.
(iii) For some positive integer $k \geq 1$,

$$\lim_{|(s,t)| \to -\infty} \frac{f(x,s,t)}{|s|^p + |t|^q} < \lambda_1 \leq \lambda_k < \lim_{|(s,t)| \to +\infty} \frac{f(x,s,t)}{|s|^p + |t|^q} < \lambda.$$

In this paper we extend this result to the case $1 < p, q < +\infty$; Furtheremore, here the $\lim_{|(s,t)|\to+\infty} \frac{f(x,s,t)}{|s|^p+|t|^q} \in (\lambda_k, \lambda_{k+1})$ relaxed to

$$\mu_2 \leq \liminf_{|(s,t)| \to \infty} \frac{f(x,s,t)}{|s|^{p-2}s + |t|^{q-2}t} \leq \limsup_{|(s,t)| \to \infty} \frac{f(x,s,t)}{|s|^{p-2}s + |t|^{q-2}t} \leq \mu_3,$$

Where $\mu_2, \mu_3 \in (\lambda_1, +\infty)$.

Our main result is as follows.

Theorem 1.1. Assume that $f \in (\overline{\Omega} \times R^2, R)$ and $f(x,0,0) = 0\ a.e.$ If the following conditions hold

(i) There exists constant $\mu_0 > \lambda_1$ such that

$$\liminf_{|(s,t)|\to\infty} \frac{f(x,s,t)}{|s|^{p-2}s+|t|^{q-2}t} \geq \mu_0 \qquad \text{Uniformly for } a.e.\, x \in \Omega; \qquad (1.3)$$

(ii) There exist constants $\mu_1, \mu_2, \mu_3 > 0$ with $\mu_1 < \lambda_1 < \mu_2$ such that

$$\limsup_{|(s,t)|\to\infty} \frac{f(x,s,t)}{|s|^{p-2}s+|t|^{q-2}t} \leq \mu_1, \qquad (1.4)$$

$$\mu_2 \le \liminf_{|(s,t)|\to+\infty} \frac{f(x,s,t)}{|s|^{p-2}s+|t|^{q-2}t} \le \limsup_{|(s,t)|\to+\infty} \frac{f(x,s,t)}{|s|^{p-2}s+|t|^{q-2}t} \le \mu_3,$$

Hold uniformly for $a.e.\, x \in \Omega$, then there exist such that problem (1.1) admits at least three nontrivial solutions for $\lambda \in (0, \lambda^*)$.

2 proof of the main result

Define the functional $J_\lambda : W_0^{1,p}(\Omega) \times W_0^{1,q}(\boldsymbol{\Omega}) \to \mathrm{R}$ by

$$J_\lambda(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{q}\int_\Omega |\nabla v|^q\,dx - \frac{\lambda}{p}\int_\Omega |\mathrm{u}|^p\,dx - \frac{\lambda}{q}\int_\Omega |v|^q\,dx - \int_\Omega F(x,u,v)dx$$

Clearly, $J_\lambda \in C^1(W_0^{1,p}(\Omega) \times W_0^{1,q}(\boldsymbol{\Omega}),\mathrm{R})$. It is obviouse that the critical points of correspond to the weak solutions of problem (1.1).

Lemma 2.1. Assume that the assumptions of theorem 1.1 hold. Then the functional $J_\lambda(u,v)$ satisfies the (PS) condition.

Proof: Assume that $\{(u_n, v_n) = z_n\}_{n\in N} \subset W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is a (PS) sequence, i.e., for some $M > 0$,

$$|J_\lambda(u_n,v_n)| \le M, \quad \nabla J_\lambda(u_n,v_n) \to 0 \;\; as \;\; n \to \infty. \tag{2.1}$$

It suffices to prove that $\{(u_n, v_n) = z_n\}$ is bounded in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. In fact, if not, we may assume by contradiction that there exist a sequence of $\{(u_n, v_n) = z_n\}$ with $\|(u_n, v_n)\| \to +\infty$ and $\{\varepsilon\}$ with $\varepsilon_n \to 0$ in

$W_0^{-1,p'}(\Omega) \times W_0^{-1,q'}(\Omega)$ such that

$$-\Delta_p u_n = -\lambda|u_n|^{p-2}u + f_u(x,u_n,v_n) \quad \text{in} \quad W_0^{-1,p'}(\Omega) \tag{2.2}$$

Taking $-u_n^-$ as test function in (2.2), we obtain

$$\|\nabla u_n^-\|_p^p = \int_\Omega \lambda|u_n^-|^p\,dx - \int_\Omega f_u\,(x,u_n,v_n)u_n^-\,dx - \int_\Omega \varepsilon_n\,u_n^- dx.$$

Similarly,

$$\|\nabla v_n^-\|_q^q = \int_\Omega \lambda|v_n^-|^q\,dx - \int_\Omega f_v\,(x,u_n,v_n)v_n^-\,dx - \int_\Omega \varepsilon_n v_n^-\,dx.$$

In view of (4.1), for any $\varepsilon \in (0, \lambda_1 - \mu_1)$, there exists $C = C(\varepsilon) > 0$ such that

$$f(x,s,t) \geq (\mu_1 + \varepsilon)(|s|^{p-2}s + |t|^{q-2}t) - C \quad \forall s,t < 0 \ \ a.e, x \in \Omega \quad . \tag{2.3}$$

Then by the sobolev embedding and Poincare inequality there exist $C_1, C_2 > 0$

Such that

$$\|\nabla u_n^-\|_p^p + \|\nabla v_n^-\|_q^q \leq$$

$$\int_\Omega \lambda(|u_n^-|^p + |v_n^-|^q)dx + \int_\Omega (\mu_1 + \varepsilon)(|u_n^-|^p + |v_n^-|^q)dx - \int_\Omega (C - \varepsilon_n)(u_n^- + v_n^-)dx$$

$$\leq C_1 \int_\Omega \lambda(\|u_n^-\|^p + \|v_n^-\|^q)dx + \int_\Omega \frac{\mu_{1+\varepsilon}}{\lambda_1}(\|u_n^-\|^p + \|v_n^-\|^q)dx$$

$$+ C_2 \int_\Omega (\|u_n^-\| + \|v_n^-\|)\ dx.$$

Hence by $\mu_1 + \varepsilon < \lambda_1$, it follows that $\{(u_n^-, v_n^-) = z_n^-\} \subset W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is bounded. For any n, we take $\psi_{n,k} = -((u_n + v_n)k)^-$ with $k > 0$ as test function in (2.2), using again (2.3), we get

$$\int_\Omega |\nabla \psi_{n,k}|^p\ dx \leq$$

$$-\int_\Omega \lambda(|u_n^-|^{p-2} + |v_n^-|^{q-2})\psi_{n,k}dx + \int_\Omega (\mu_1 + \varepsilon)(|u_n^-|^{p-2} + |v_n^-|^{q-2})\psi_{n,k}dx$$

$$-\int_\Omega (C - \varepsilon_n)\psi_{n,k}\ dx.$$

We can obtain that $\{\|(u_n^-, v_n^-)\|_\infty\}$ is bounded. By the standard regularity theory (see [4]), it follow that there exists $C_3 > 0$ such that, for every $n, u_n \in C^{1,\sigma}(\overline{\Omega})$ for some $\sigma > 0$ and

$$\|(\nabla u_{n+} \nabla v_n)\|_\infty \leq C_3(1 + \|(u_n + v_n)\|_\infty). \tag{2.4}$$

Thus by $\|(u_n, v_n)\|_\infty \to +\infty$ it follows that $\|(u_n^+, v_n^+)\|_\infty \to +\infty$. (2.5)

We may assume that $z_n = \|(u_n, v_n)\| \to \infty\ as\ n \to \infty$. Define $\hat{u}_n = \frac{u_n}{z_n}, \hat{v}_n = \frac{v_n}{z_n}$.

Denote $g(x,s,t) = -\lambda(|s|^{p-2}s + |t|^{q-2}t) + f(x,s,t)$. In view of (2.1), for all $\phi \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, we have

$$\int_\Omega \left[|\nabla \hat{u}_n|^{p-2}\nabla \hat{u}_n \nabla \phi - \frac{g_{u(x,u_n,v_n)}}{z_n^{p-1}}dt\right] \to 0 \qquad (2.6)$$

Since g is countinous and $\|(u_n^-, v_n^-)\|_\infty$ is uniformly bounded, using (1.3)-(1.5) and (2.5), there exist constant $C_4, C_5 > 0$ and $\varepsilon \in (0, \mu_2 - \lambda_1)$ such that

$$(\mu_2 - \varepsilon)|\hat{u}_n(x)|^{p-1} - \frac{C_4}{\|z_n\|_\infty^{p-1}} \le \frac{g(x,u_n,v_n)}{\|z_n\|_\infty^{p-1}} \le (\mu_3 + \varepsilon)|\hat{u}_n|^{p-1} + \frac{C_5}{\|z_n\|_\infty^{p-1}}$$

Hold uniformly for $a.e.\, x \in \Omega$. In a similarly way, we get $\hat{u}_n \to \hat{v}_0$. By the regularity theory (see [4]), there exists a constant $M_2 > 0$ such that, for every n, $\|(\hat{u}_n, \hat{v}_n)\|_{C^{1,\sigma}} \le M_2$, set $w_n = \frac{z_n}{\|z_n\|_\infty}$. Then by the compact imbedding of $C^{1,\sigma}(\overline{\Omega})$

into $C^1(\overline{\Omega})$, passing to a subsequence if possible, we have

$$w_n \to w_0 \qquad \text{in} \quad C^1(\overline{\Omega}) \qquad (2.8)$$

With $\|\hat{u}_0\|_\infty = 1$, then $(\hat{u}_n, \hat{v}_n)$ is bounded Which $\|\hat{u}_n\|_{1,p} + \|\hat{v}_n\|_{1,q} = 1$.

Using again that $\|(\hat{u}_n, \hat{v}_n)\|$ is bounded and $\hat{u}_n = \frac{u_n^+ - u_n^-}{\|z_n\|_\infty}$, we can see that $\hat{u}_0 \ge 0$ and $\hat{u}_0 \not\equiv 0$, similarly for $\hat{v}_n = \frac{v_n^+ - v_n^-}{\|z_n\|_\infty}$ and we can see that $\hat{u}_0 \ge 0$ and $\hat{u}_0 \not\equiv 0$.

Denote $\alpha_n(x) = \frac{g(x,u_n,v_n)}{\|z_n\|_\infty^{p-1}}$. By (2.7) and (2.8) if follows that there exists $\alpha \in L^\infty(\Omega)$ satisfying

$$\mu_2 - \varepsilon \le \alpha(x) \le \mu_3 + \varepsilon \qquad (2.9)$$

Such that

$$\alpha_n \to \alpha(|\hat{u}_0|^{p-2}u_0 + |\hat{v}_0|^{p-2}v_0) \quad \text{weakly} \quad \text{in} \quad L^\infty(\Omega) \qquad (2.10)$$

By (2.6), (2.8), (2.10) we obtain

$$\int_\Omega |\nabla \hat{u}_0|^{p-2}\nabla u_0 \nabla \phi dx = \int_\Omega [\alpha(x)|\hat{u}_0|^{q-2}u_0]\,\phi dx$$

For every $\phi \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. Consequently, similarly

$$\int_\Omega |\nabla \hat{v}_0|^{p-2}\nabla v_0 \nabla \phi dx = \int_\Omega [\alpha(x)|\hat{v}_0|^{q-2}v_0]\,\phi dx$$

$(\hat{u}_0, \hat{v}_0)$ is a nontrivial solution of

$$\begin{cases} -\Delta_p w_0 = \alpha(x) w_0^{p-1} & in\ \Omega \\ -\Delta_q w_0 = \alpha(x) w_0^{q-1} & in\ \Omega \\ w_0 = 0 & on\ \partial\Omega \end{cases} \tag{2.11}$$

By the maximum principle of vazquez's [9], it follows that $w_0(x) > 0$ for $x \in \Omega$.

Furthermore, there is a positive constant $\delta > 0$ and $\varphi = (\varphi_1, \varphi_2)$ such that

$$\delta\varphi \le w_0 \qquad on\ \partial\Omega \tag{2.12}$$

By (2.9),(2.11) and $\mu_2 > \lambda_1$, for any $\varepsilon \in (0, \frac{\mu_2 - \lambda_1}{2})$, we get

$$-\Delta_p w_0 > (\lambda_1, \lambda_1 + \varepsilon) w_0^{p-1} \tag{2.13}$$

And

$$-\Delta_q w_0 > (\lambda_1, \lambda_1 + \varepsilon) w_0^{q-1}.$$

Take $\psi = (\psi_1, \psi_2)$ and $\psi = \delta\varphi$ and $\mu \in (\lambda_1, \lambda_1 + \varepsilon)$. Then we have

$$-\Delta_p \psi = \lambda_1 \psi^{p-1} \le \mu \psi^{p-1}$$

and

$$-\Delta_q \psi = \lambda_1 \psi^{q-1} \le \mu \psi^{q-1}.$$

By (2.12) and (2.13), by the method of sub and supersolution, for any $\varepsilon > 0$ small enough, we can obtain a solution $(\bar{u}, \bar{v}) \in [\psi, w_0]$ of the following problems

$$\begin{cases} -\Delta_p u = \mu u^{p-1} & in\Omega \\ -\Delta_q v = \mu v^{q-1} & in\Omega \\ u = v = 0 & on\ \partial\Omega \end{cases}$$

However, this is contrary to this fact that λ_1 is isolated. Hence $\{\|(u_n^+, v_n^+)\|\}$ is also uniformly bounded. Thus by (2.4) we can see that the sequence $\{\|(u_n, v_n\|\}$ is uniformly bounded. Then using standard arguments we can see that J_λ satisfies the (PS) condition. This completes the proof.

Define

$$f_+(x,s,t) = \begin{cases} f(x,s,t) & t,s \geq 0 \\ 0 & t,s \leq 0 \end{cases}$$

Define the corresponding functional $J^+_{\lambda(u,v)}: W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \to R$ as follows.

$$J^+_\lambda(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{q}\int_\Omega |\nabla v|^q\,dx - \frac{\lambda}{p}\int_\Omega |u^+|^p\,dx - \frac{\lambda}{q}\int_\Omega |v^+|^q\,dx - \int_\Omega F_+(x,u,v)\,dx,$$

Where $\nabla F = (f_u, f_v)$. Obviously, $J^+_\lambda \in C^1(W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega), R)$. Similarly, define

$$f_-(x,s,t) = \begin{cases} f(x,s,t) & t,s \leq 0 \\ 0 & t,s \geq 0 \end{cases}$$

Define the corresponding functional $J^-_{\lambda(u,v)}: W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \to R$ as follows.

$$J^-_\lambda(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{q}\int_\Omega |\nabla v|^q\,dx - \frac{\lambda}{p}\int_\Omega |u^-|^p\,dx - \frac{\lambda}{q}\int_\Omega |v^-|^q\,dx - \int_\Omega F_-(x,u,v)\,dx,$$

Where $\nabla F = (f_u, f_v)$. It is easily seen that $J^-_\lambda \in C^1(W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega), R)$.

Using similar arguments as in the proof of lemma 2.1, we obtain the following result.

Lemma 2.2. The functional $J^\pm_\lambda$ satisfies the (PS) condition.

To prove Theorem 1.1, we prove some preliminary results as follows.

Lemma 2.3. If $(u^\pm, v^\pm)$ is a local minimizer of $J^\pm_\lambda$, then it is also a local minimizer of J_λ.

Proof: By Theorem 1.1 of Garcia Azorero, Peral Alonso and Manfredi[5], we just need to show that $(u^\pm, v^\pm)$ is a local minimazer of J_λ in the C^1 topology. By the assumption it follow that $(u^\pm, v^\pm)$ is a $C_0^1(\overline{\Omega})$-local minimize of $J^\pm_\lambda$ i.e., there exists $\rho_1 > 0$ such that

$$J^\pm_\lambda(u^\pm, v^\pm) \leq J^\pm_\lambda(u, v), \qquad \forall u \in B_{\rho_1}(u^\pm, v^\pm)$$

Where $B_{\rho_1}(u^\pm, v^\pm) = \{(u,v) \in C_0^1(\overline{\Omega}): \|(u,v) - (u^\pm, v^\pm)\|_{C^1} < \rho_1\}$. By (1.4), (1.5), we can see that f is of p-linear growth [5]. Then, for $(u,v) \in B_{\rho_1}(u^\pm, v^\pm)$, we obtain

$$\begin{aligned}
J_\lambda(u,v) - J_\lambda(u^\pm,v^\pm) &= J_\lambda(u,v) - J_\lambda^\pm(u^\pm,v^\pm)\\
&\geq \frac{\lambda}{p}\int_\Omega [|(u,v)|^p - |(u^\pm,v^\pm)|^p]dx\\
&+\frac{\lambda}{q}\int_\Omega [|(u,v)|^q - |(u^\pm,v^\pm)|^q]dx - \int_\Omega [F(x,u,v) - F_\pm(x,u,v)]\,dx\\
&= \frac{\lambda}{p}\int_\Omega \left|(u^\mp,v^\mp)\right|^p dx + \frac{\lambda}{q}\int_\Omega \left|(u^\mp,v^\mp)\right|^q dx - \int_\Omega F_\mp(x,u,v)dx\\
&\geq \frac{\lambda}{p}\int_\Omega \left|(u^\mp,v^\mp)\right|^p dx + \frac{\lambda}{q}\int_\Omega \left|(u^\mp,v^\mp)\right|^q dx\\
&- C\left(\int_\Omega \left|(u^\mp,v^\mp)\right|^P dx + \int_\Omega \left|(u^\mp,v^\mp)\right|^q dx\right)\\
&\geq \left[\frac{\lambda}{q} - C\left\|(u^\mp,v^\mp)\right\|_\infty^{p-q}\right]\int_\Omega \left|(u^\mp,v^\mp)\right|^q dx\\
&+\left[\frac{\lambda}{p} - C\left\|(u^\mp,v^\mp)\right\|_\infty^{q-p}\right]\int_\Omega \left|(u^\mp,v^\mp)\right|^p dx
\end{aligned}$$

Note $\rho_1 \to 0$ implies$\|(u^-,v^-)\|_\infty \to 0$, together with$1 < p,q < +\infty$, we can see that there exists $\rho_2 > 0$ small enough such that

$$J_\lambda(u^\pm,v^\pm) \leq J_\lambda(u,v), \qquad \forall (u,v) \in B_{\rho_2}(u^\pm,v^\pm),$$

Where $B_{\rho_2}(u^\pm,v^\pm) = \{(u,v) \in C_0^1(\overline{\Omega}) : \|(u,v) - (u^\pm,v^\pm)\|_{C^1} < \rho_2\}$. This completes the proof.

Lemma 2.4. 0 is a local minimize of $J_\lambda^\pm$ and J_λ for $\lambda>0$.

Proof: we just consider the case of J_λ. The other cases can be treated similarly. As shown in the proof of lemma 2.3, it suffices to prove that 0 is a local minimizer of J_λ in the C^1 topology. In fact, for $(u,v) \in C_0^1(\overline{\Omega})$, we have

$$J_\lambda^{\pm}(u^{\pm},v^{\pm}) \geq \frac{\lambda}{p}\int_\Omega |(u,v)|^p dx + \frac{\lambda}{q}\int_\Omega |(u,v)|^q dx - \int_\Omega F(x,u,v)\, dx$$

$$\geq \frac{\lambda}{p}\int_\Omega |(u,v)|^p\, dx + \frac{\lambda}{q}\int_\Omega |(u,v)|^q\, dx - C\left(\int_\Omega |u|^p\, dx + \int_\Omega |v|^q dx\right)$$

$$\geq \left[\frac{\lambda}{p} - C\|u\|_\infty^{q-p}\right]\int_\Omega |u|^p dx + \left[\frac{\lambda}{q} - C\|u\|_\infty^{p-q}\right]\int_\Omega |v|^q dx$$

If we define $B_{\rho_3}(0,0) = \{(u,v) \in C_0^1(\overline{\Omega}) : \|(u,v)\|_{C^1} < \rho_3\}$, where $\rho_3 \in (0, \left(\frac{\lambda}{C_q}\right)^{\frac{1}{p-q}}, \left(\frac{\lambda}{C_p}\right)^{\frac{1}{q-p}})$, then it follows that

$$J_\lambda(u,v) \geq 0, \qquad \forall (u,v) \in B_{\rho_3}(0,0)\,.$$

The proof is complete.

Lemma 2.5. There exist $\lambda^*, t_1, t_2 > 0$ such that, for $\lambda \in (0,\lambda^*)$

$$J_\lambda(\pm t_1\phi_1, \pm t_2\phi_2) < 0 \qquad (2.14)$$

Proof: By (1.3)-(1.5), for any given $\varepsilon > 0$ and $r \in \left(p, \frac{pn}{n-p}\right)$ if $n > p$; $r \in (p, +\infty)$

If $1 \leq n \leq p$, there exist $C > 0$ such that

$$|pF(x,z) - \mu_3|z|^p| \leq \varepsilon|z|^p + C|z|^r.$$

Then, taking $\varepsilon < \mu_3 - \lambda_1$, we have
$J_\lambda(t_1\phi_1, t_2\phi_2) =$
$\frac{|t_1|^p}{p}\|\phi_1\|^p + \frac{|t_2|^q}{q}\|\phi_2\|^q + \frac{|t_1|^p}{p}\lambda\int_\Omega \phi_1^p dx + \frac{|t_2|^q}{q}\lambda\int_\Omega \phi_2^q dx - \int_\Omega F(t_1\phi_1, t_2\phi_2)\, dx \leq$
$\frac{|t_1|^p}{p}\|\phi_1\|^p + \frac{|t_2|^q}{q}\|\phi_2\|^q + \frac{|t_2|^q}{q}\lambda\int_\Omega \phi_2^q dx - \frac{|t_1|^p}{p}\mu_3\int_\Omega \phi_1^p\, dx + \frac{|t_1|^p}{p}\varepsilon\int_\Omega \phi_1^p\, dx +$
$\frac{|t_1|^r}{p}C\int_\Omega \phi_1^r\, dx = [\lambda_1 - \mu_3 + \varepsilon]\frac{|t_1|^p}{p}\int_\Omega \phi_1^p\, dx + \frac{|t_2|^q}{q}\int_\Omega \phi_2^q\, dx + \frac{|t_1|^r}{p}C\int_\Omega \phi_1^r\, dx \leq$
$-\left(\frac{\mu_3-\varepsilon}{\lambda_1} - 1\right)\frac{|t_1|^p}{p}\|\phi_1\|^p + C(\lambda|t_1|^{q-p} + |t_1|^{r-p})\|\phi_1\|^p$

Define $\varphi(z) = \lambda z^{q-p} + z^{r-p}$ *for* $z \geq 0$, where $\delta \equiv \frac{1}{p}\left(\frac{\mu_3-\varepsilon}{\lambda_1} - 1\right) > 0.$

Then $\varphi'(z) = \lambda(q-p)z^{q-p-1} + (r-p-1)z^{r-p-1}.$

It is easily seen that $\varphi'(z_0) = 0$ if $z_0 = (\frac{\lambda(p-q)}{r-p-1})^{\frac{1}{r-q}}$. denote $\delta_0 = \frac{p-q}{r-p-1}$. Then we have$\varphi(z_0) = \left[\delta_0^{\frac{q-p}{r-q}} + \delta_0^{\frac{r-p}{r-q}}\right]\lambda^{\frac{r-p}{r-q}}$.

Hence if taking $|t|$=z_0, there exists $\lambda^* > 0$ such that if $\lambda < \lambda^*$ then

$$C\varphi(z_0) < \frac{1}{p}\left(\frac{\mu_3-\varepsilon}{\lambda_1} - 1\right).$$

Thus we can see that (2.14) hold for $\lambda \in (0, \lambda^*)$ if we take $t_1 = z_0$.

Proof of theorem 1.1. By lemma 2.4, 0 is a local minimizer of $J_\lambda^\pm$ and J_λ with $J_\lambda^\pm(0,0) = J_\lambda(0,0) = 0$. In view of lemma 2.5, there exist $t_1, t_2, \lambda^* > 0$ such that, for

$$\lambda \in (0, \lambda^*), \inf_{W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)} J_\lambda^\pm(u, v) \leq J_\lambda(\pm t_1\phi_1, \pm t_2\phi_2) < 0 \qquad (2.15)$$

Then $J_\lambda^\pm$ has a nontrivial critical point $(u^\pm, v^\pm)$ of mountain pass type with $J_\lambda^\pm(u^\pm, v^\pm) > 0$, which implies that $(u^\pm, v^\pm)$ is a weak solution of the following (p,q)-Laplacian

$$\begin{cases} -\Delta_p u = \lambda|u^\pm|^{p-2} + f_u^\pm(x, u, v) & in\ \Omega \\ -\Delta_q v = \lambda|v^\pm|^{q-2} + f_v^\pm(x, u, v) & in\ \Omega \\ u = v = 0 & on\ \partial\Omega \end{cases} \qquad (2.16)$$

By the weak maximum principle we can see that $\left(\pm u_1^\pm, \pm v_1^\pm\right) \geq 0$ in Ω, which implies that $\left(u_1^\pm, v_1^\pm\right)$ is also a solution of system (1.1) and

$J_\lambda(u^\pm, v^\pm) = J_\lambda^\pm(u^\pm, v^\pm)$. In addition, by the fatter of (1.4) it follows that the functional J_λ^- is coercive on $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and hence bounded below. Combing with (2.15) implies that J_λ^- has a nontrivial global minimizer

(u_2^-, v_2^-) with $J_\lambda^-(u_2^-, v_2^-) < 0$. Then by lemma 2.3 we can see that (u_2^-, v_2^-) is a local minimizer J_λ. Thus (1.1) has at least nontrivial solutions $(u_1^-, v_1^-), (u_2^-, v_2^-), (u_1^+, v_1^+)$.

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