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A Combined Method for the Numerical Solution of Boundary Value Problems of second order

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Abstract. In this work, a technique based on combination between collocation and spline method along with the shooting method is proposed for the solution of boundary value problem of order 2. Numerical results show that the method is simple and effective.

Keywords: Boundary value problem, Collocation method, Shooting method.

1 Introduction

It is not in general easy to find a closed-form solution of differential equations [1,2]. A part from numerical integration methods, several analytical ones, such as Adomian decomposition method [3-6] and the variational iteration method (VIM) [7-12] are proposed to find approximate, and if possible in closed form, solutions of differential equations.

In this paper, we discuss the numerical solution of boundary value problem of the form

$$[a_2D^2 + a_1D + a_0D^0]y(t) = f(t), \quad 0 < t < T, \quad (1)$$

subject to

$$y(0) = \alpha \quad y(T) = \beta. \quad (2)$$

where a_0, a_1, a_2, α and β are constants with $a_2 \neq 0$ and $y \in L_1[0, T]$.

The method of solution is based on a conjugating collocation and spline analysis combined with multiple shooting method. Note that similar study can be done for a general case when the boundary conditions are

$$\begin{cases} M_1 y(0) + N_1 y'(0) = \alpha, \\ M_2 y(T) + N_2 y'(T) = \beta, \end{cases}$$

where $M_1, N_1, M_2, N_2, \alpha$ and β are constants.

The use of spline approximation [13] and collocation method [14] has been effectively implemented for the solution of initial value problem. Additionally, the cubic spline collocation method to solve two classes of special boundary value problem has been used [15].

2 Method of solution

The following is a brief derivation of the algorithm used to solve problem (1) subject to (2).

3.1. Collocation method

For sake of simplicity we discuss the solution of (1) as initial value problem with

$$y(0) = \alpha, \quad y'(0) = \hat{\beta}_1, \quad (3)$$

where $\hat{\beta}_1$ is unknown constant which will be determined later. We firstly subdivide the interval $[0, T]$ into N uniform subintervals $I_n = [t_n, t_{n+1}]$ where $n = 0, \dots, N - 1$. Let $H_N = \{t_n = nh : n = 0, \dots, N - 1\}$ with $h = \frac{T}{N}$. Assume that the exact solution of Eq.(1) subject to condition (3) can be approximated by an element u in the space of piecewise polynomials of degree $m+d$ which are d -times continuously differentiable on I represented by

$$S_{m+d}^{(d)}(H_N) = \{u \in C^{(d)}(I) : u|_{\sigma_n} \text{ is a polynomial of degree } m+d\}$$

for known integers $m \geq 1$ and $d \geq 0$ (the number of the initial conditions minus one i.e. $d = 1$ in the present work). It should be noted that the integer m represents the number of collocation points in each subinterval I_n , ($n = 0, \dots, N - 1$); those points are defined as

$$X_n = \{t_{n,i} = t_n + c_i h : i = 1, \dots, m\}, \quad 0 < c_1 \leq \dots \leq c_m \leq 1.$$

Consequently, the exact solution, y , of (1) and (3) needs to be approximated on I by an element $u \in S_{m+d}^{(d)}(Z_N)$ satisfying

$$[a_2D^2 + a_1D + a_0D^0]u(t) = f(t), \quad t \in \bigcup_{n=0}^{N-1} X_n, \quad (4)$$

subject to

$$u(0) = \alpha, \quad u'(0) = \hat{\beta}_1, \quad (5)$$

On each subinterval, I_n , the spline u can be presented as a piecewise polynomials of degree $m + d$ of the form

$$u(t) = u_n(t_n + \tau h) = \sum_{s=0}^d a_s^{(n)} \tau^s + \sum_{r=1}^m b_r^{(n)} \tau^{d+r}, \quad t \in I_n \quad (6)$$

where $\tau \in [0, 1]$.

Substituting (6) into (4) one obtains

$$a_2 h^{-2} \left[\sum_{s=2}^d s(s-1) a_s^{(n)} c_i^{s-2} + \sum_{r=1}^m (d+r)(d+r-1) b_r^{(n)} c_i^{d+r-2} \right] + a_1 h^{-1} \left[\sum_{s=1}^d s a_s^{(n)} c_i^{s-1} + \sum_{r=1}^m (d+r) b_r^{(n)} c_i^{d+r-1} \right] + a_0 \left[\sum_{s=0}^d a_s^{(n)} c_i^s + \sum_{r=1}^m b_r^{(n)} c_i^{d+r} \right] = f(t_{n,i}),$$

where $i = 1, \dots, m$. It can be shown that the above equations can be expressed in the following $m \times m$ matrix form

$$\mathbf{Vb}^{(n)} = \mathbf{Ua}^{(n)} + \mathbf{F}$$

where

$$(\mathbf{V})_{i,r} = a_2(d+r)(d+r-1)c_i^{d+r-2} + a_1h(d+r-1)c_i^{d+r} + a_0h^2c_i^{d+r}, \quad i, r = 1, \dots, m,$$

$$(\mathbf{U})_{i,s} = a_2s(s-1)c_i^{s-2} + a_1hsc_i^{s-1} + a_0h^2c_i^s, \quad i = 1, \dots, m; \quad s = 0, \dots, d,$$

$$(\mathbf{F})_{i,1} = h^2f(t_{n,i}), \quad i = 1, \dots, m.$$

In general, $\mathbf{a}^{(n)} = [a_0^{(n)}, \dots, a_d^{(n)}]^t$ and $\mathbf{b}^{(n)} = [b_1^{(n)}, \dots, b_m^{(n)}]^t$, where $[.]^t$ means the transpose of the vector. Note that when $n = 0$, the vector $\mathbf{a}^{(0)}$ is known from the initial conditions, i.e. $\mathbf{a}^{(0)} = [\frac{h^s}{s!} \frac{d^s y}{dt^s}(0)]^t$; $s = 0, \dots, d$. On the other hand, when $n \geq 1$, we have to use the smoothness conditions at $t = t_n$ which leads to a relation between the unknown vector

$\mathbf{a}^{(n+1)}$ and the known vectors $\mathbf{a}^{(n)}$ and $\mathbf{b}^{(n)}$.

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & . & . & . & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & . & . & . & l-1 \\ 0 & 0 & 1 & 3 & 6 & 10 & . & . & . & \prod_{i=1}^2 \frac{(l-i)}{2!} \\ 0 & 0 & 0 & 1 & 4 & 10 & . & . & . & \prod_{i=1}^3 \frac{(l-i)}{3!} \\ 0 & 0 & 0 & 0 & 1 & 5 & . & . & . & \prod_{i=1}^4 \frac{(l-i)}{4!} \\ 0 & 0 & 0 & 0 & 0 & 1 & . & . & . & \prod_{i=1}^5 \frac{(l-i)}{5!} \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{G} = \begin{pmatrix} 1 & 1 & . & . & . & 1 \\ l & l+1 & . & . & . & l+m-1 \\ \frac{(l-1)l}{2!} & \frac{l(l+1)}{2!} & . & . & . & \frac{(l+m-2)(l+m-1)}{2!} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ l & \frac{(l+1)l(l-1)...3}{(l-1)!} & . & . & . & \frac{(l+m-1)...(l+m-(l-1))}{(l-1)!} \end{pmatrix}$$

In the present work, this relation is given by $\mathbf{a}^{(n+1)} = \mathbf{H}\mathbf{a}^{(n)} + \mathbf{G}\mathbf{b}^{(n)}$ where

$$\mathbf{H} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{G} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix}$$

In the next subsection, we discuss the implementation of the multiple shooting method in the numerical solution of problem (1-2).

3.2. Shooting method

In the first, we rewrite problem (1-2) in the form

$$L[y](t) = f(t), \quad t \in [0, T] \tag{7}$$

subject to

$$y(0) = \alpha, \quad y(T) = \beta, \tag{8}$$

where

$$\mathbf{L} = a_2 D^2 + a_1 D + a_0 D^0$$

Firstly we partition the domain $[0, T]$ as

$$0 = T_0 < T_1 < \dots < T_L = T.$$

Hence, theoretically, the solution of problem (7-8) on the time interval $[0, T]$ can be determined by solving problem (7-8) on the subintervals $[T_l, T_{l+1}]$, $(l = 0, \dots, L - 1)$. For

the purpose of implementing shooting method, we added extra conditions at the grid points $T_l, (l = 0, \dots, L - 1)$, so that we have the following set of initial value problems

$$L[u_l] = f(t), \quad t \in [T_l, T_{l+1}] \tag{9}$$

subject to

$$u_l(T_l) = \hat{\beta}_{2l}, \quad u'_l(T_l) = \hat{\beta}_{2l+1} \tag{10}$$

where $\hat{\beta}_j, (j = 1, \dots, 2L - 1)$ are unknown real parameters and $\hat{\beta}_0 = \alpha$. The solution of initial value problem (9-10) will be obtained by the method described in the previous subsection where the parameters $\hat{\beta}_l, (l = 0, \dots, 2L - 1)$, will be determined by solving the following system of algebraic equations

$$\begin{aligned} u_0(T_1, \hat{\beta}_1) &= u_1(T_1, \hat{\beta}_2, \hat{\beta}_3), \\ u'_0(T_1, \hat{\beta}_1) &= u'_1(T_1, \hat{\beta}_2, \hat{\beta}_3), \\ u_1(T_2, \hat{\beta}_2, \hat{\beta}_3) &= u_2(T_2, \hat{\beta}_4, \hat{\beta}_5), \\ u'_1(T_2, \hat{\beta}_2, \hat{\beta}_3) &= u'_2(T_2, \hat{\beta}_4, \hat{\beta}_5), \\ &\vdots \\ u_{k-2}(T_{k-1}, \hat{\beta}_{2k-4}, \hat{\beta}_{2k-3}) &= u_{k-1}(T_2, \hat{\beta}_{2k-2}, \hat{\beta}_{2k-1}), \\ u'_{k-2}(T_{k-1}, \hat{\beta}_{2k-4}, \hat{\beta}_{2k-3}) &= u'_{k-1}(T_2, \hat{\beta}_{2k-2}, \hat{\beta}_{2k-1}), \\ u_{k-1}(T, \hat{\beta}_{2k-2}, \hat{\beta}_{2k-1}) &= \beta \end{aligned}$$

which can be solved numerically using several numerical techniques such as Newtons method. In summary, this technique is called multiple shooting technique of order L .

3 Numerical results

Example1. Consider the following differential equation

$$y''(t) + \frac{1}{16}y(t) = 0, \quad 0 < t < 2\pi, \tag{11}$$

subject to

$$y(0) = 0, \quad y(2\pi) = 6.5244. \tag{12}$$

Applying the multiple shooting method of order four requires dividing the domain $[0, 2\pi]$ into four subinterval as follows $0 = T_0 < T_1 = \pi < T_2 = 2\pi$.

$$L[u_0] = 0, \quad 0 < t < 1, \quad u(0, :) = 0, \quad u'(0, :) = \hat{\beta}_1 \quad (13)$$

$$L[u_1] = 0, \quad 1 < t < 2, \quad u(1, :) = \hat{\beta}_2, \quad u'(1, :) = \hat{\beta}_3, \quad (14)$$

where

$$L[u_l] = D^2u_l(t) + Du_l(t) + u_l(t), \quad l = 0, 1.$$

Herein

$$u_0(t, :) = u_0(t, \hat{\beta}_1), \quad u_1(t, :) = u_1(t, \hat{\beta}_2, \hat{\beta}_3).$$

Each initial value problem of (13-14) will be solved using the collocation method where we use $m = 3$ with the collocation parameters

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{x_1+1}{2} \\ \frac{x_2+1}{2} \\ \frac{x_3+1}{2} \end{pmatrix}, \quad x_i = \cos\left(\frac{2i+1}{6}\right), \quad i = 0, 1, 2.$$

The x_i , $i = 0, 1, 2$, are the roots of the Tchybechev polynomial of degree 3. Note that, it is difficult to find a specific optimum choice for m .

Finally, the values for the parameters β_i , $i = 1, 2, 3$ are determined by solving the following algebraic system

$$u_0(\pi, \hat{\beta}_1) = u_1(\pi, \hat{\beta}_2, \hat{\beta}_3), \quad u'_0(\pi, \hat{\beta}_1) = u'_1(\pi, \hat{\beta}_2, \hat{\beta}_3)$$

$$u_1(2\pi, \hat{\beta}_2, \hat{\beta}_3) = 6.5422.$$

Using matlab package we get

$$\hat{\beta}_1 = 1, \quad \hat{\beta}_2 = 1.3782,$$

$$\hat{\beta}_3 = 1.7628.$$

4 Conclusions

In his paper, we apply a conjugating collocation and spline analysis combined with shooting method for solving boundary value problems of order 2 subject to boundary conditions. It may be concluded that the method is powerful and efficient technique for finding approximate solutions for wide classes of differential equations. In the example, we have proposed the numerical result by using matlab package.

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